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Double Fuzzy Sumudu Transform to Solve Partial Volterra Fuzzy Integro-Differential Equations

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Abstract: In this paper, the double fuzzy Sumudu transform (DFST) method was used to find the solution to partial Volterra fuzzy integro-differential equations (PVFIDE) with convolution kernel under Hukuhara differentiability. Fundamental results of the double fuzzy Sumudu transform for double fuzzy convolution and fuzzy partial derivatives of the n -th order are provided. By using these results the solution of PVFIDE is constructed. It is shown that DFST method is a simple and reliable approach for solving such equations analytically. Finally, the method is demonstrated with examples to show the capability of the proposed method.

Keywords: double fuzzy Sumudu transform; partial Volterra fuzzy integro-differential equations; n -th order fuzzy partial H -derivative

1. Introduction

Modeling of different physical systems gives us different differential, integral and integro-differential equations. We are not always sure that the models obtained are perfect. The fuzzy set theory is one of the most popular theories for describing this situation. The fuzzy logic is introduced with the proposal of fuzzy set theory by Zadeh and is applied when the observational parameters are imprecise or unclear. The neutrosophic logic is considered as the extension of the fuzzy logic and the measure of indeterminacy is added to the measures of truthiness and falseness. The theory of neutrosophic statistics can be applied when to observation indeterminate, imprecise, vague, and incomplete parameters. For more details [1–4].

The concept of fuzzy sets, fuzzy numbers and arithmetic operations firstly introduced by Zadeh [5]. In [6] Seikkala defined fuzzy derivatives. The concept of fuzzy integration was given by Dubois and Prade [7]. One of the first applications of fuzzy integration was given by Wu and Ma who investigated the fuzzy Fredholm integral equation of the second kind. The idea of fuzzy partial differential equations was introduced by Buckley in [8]. Allahveranloo proposed the difference method for solving this equations in [9].

In recent years, many mathematicians have studied the solution of fuzzy differential equations [10–12], fuzzy integral equations [13–17], and fuzzy integro-differential equations [18–21], which play a key role in engineering [22,23]. These equations in a fuzzy setting are a natural way to model the ambiguity of dynamic systems in different scientific fields such as physics, geography, medicine, and biology [24,25].

The fundamental tool in operational calculus are integral transforms. They are used in solving many practical problems in applied mathematics, physics and engineering. The integral transforms be very useful in solving partial differential equations. They convert the original function to a function that is simpler to solve. The Fourier transform is the precursor of the integral transforms. This transform is used to express functions in a finite interval. Similar integral transforms are Laplace, Mellin and Hankel transforms. In the 1990s Watugala [26,27] has introduced a new integral transform called the Sumudu transform. Later, Weerakon [28] used the Sumudu transform for solving partial differential

equations. Some fundamental theorems and properties for Sumudu transform can be seen in [8,29]. Furthermore, in [30] the Sumudu transform is applied for Bessel functions and equations.

One of the recent methods in handling problems modelled under fuzzy environment is fuzzy Sumudu transform proposed by Ahmad and Abdul Rahman [31]. This transform is used for solving of fuzzy differential equations, fuzzy integral equations and fuzzy integro-differential equations as the problem is reduced to problem which is much simpler to be solved.

In [29] Ahmad and Abdul Rahman proposed the idea of the fuzzy method of transformation of Sumudu to solve fuzzy partial differential equations. The technique of the fuzzy Sumudu transform method for solving a fuzzy convolution Volterra integral equations and the fuzzy integro-differential equation was developed in [32,33]. The studies are followed by the application of fuzzy Sumudu transform on fuzzy fractional differential equations and fuzzy Volterra integral equations in [34]. In [35] is introduced double fuzzy Sumudu transform (DFST) method and is applied to solve fuzzy convolution Volterra integral equation of two variable.

In [36] the solution of classical partial integro-differential equations was discussed using classical double Elzaki transform method. In the present paper we investigate the solution of PVFIDE with convolution kernel under Hukuhara differentiability using DFST method. The main difficulties overcome in solving the this problem are related to the application of the DFST for fuzzy partial H -derivative of the n -th order. So, we obtain a new results on the Sumudu transform for fuzzy partial H -derivative of the n -th order. After, the studied equation we convert to a nonlinear system of partial Volterra integro-differential equations in a crisp case. To be find the lower and upper functions of the solution we us DFST and we convert this system to system of algebraic equations.

The paper is organized as follows: In Section 2, some definitions and results of fuzzy numbers, fuzzy functions and fuzzy partial derivative of the n -th order is given. In Section 3, the definition of DFST is recalled, double fuzzy convolution theorem is stated. New results on DFST for fuzzy partial derivative of the n -th order are proposed. In Section 4, the DFST is applied to fuzzy partial convolution Volterra fuzzy integro-differential equation to construct the general technique. In Section 5, an example is provided to demonstrate the proposed method and finally in Section 6, conclusions are drawn.

2. Preliminaries and Notations

In this section, we give some basics definitions and theorems for fuzzy number, fuzzy-valued function and derivative of fuzzy-valued function.

Definition 1 ([37]). *A fuzzy number is defined as the mapping $u : \mathbb{R} \rightarrow [0, 1]$ satisfying the following four properties:*

- (i) u is upper semi-continuous on \mathbb{R} ;
- (ii) $u(x) = 0$ outside of some interval $[c, d]$;
- (iii) there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c, a]$, decreasing on $[b, d]$ and $u(x) = 1$ for each $x \in [a, b]$;
- (iv) $u(rx + (1 - r)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$, $r \in [0, 1]$.

needed throughout the paper such

Denote E^1 the set of all fuzzy numbers and $D = \mathbb{R}_+ \times \mathbb{R}_+$. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi(a)$ and therefore $\mathbb{R} \subset E^1$.

Definition 2 ([38]). *Let $u \in E^1$ and $r \in (0, 1]$. The r -level set of u is the crisp set*

$$[u]^r = \{x \in \mathbb{R} : u(x) \geq r\},$$

where $[u]^r$ denotes r -level set of fuzzy number u .

It can be concluded that any r -level set is bounded and closed interval and denoted by $[\underline{u}(r), \bar{u}(r)]$ for all $r \in [0, 1]$, where the functions $\underline{u}, \bar{u} : [0, 1] \rightarrow \mathbb{R}$ are the lower and upper bound of $[u]^r$, respectively.

Definition 3 ([38]). A fuzzy number in parametric form is given as an order pair of the form $u = (\underline{u}(r), \bar{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions:

- (i) $\underline{u}(r)$ is a bounded left continuous monotonic increasing function in $[0, 1]$;
- (ii) $\bar{u}(r)$ is a bounded left continuous monotonic decreasing function in $[0, 1]$;
- (iii) $\underline{u}(r) \leq \bar{u}(r)$.

For arbitrary fuzzy number $u = (\underline{u}(r), \bar{u}(r)), v = (\underline{v}(r), \bar{v}(r))$ and an arbitrary crisp number $k \in \mathbb{R}$ the addition and the scalar multiplication are defined by $[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]$ and

$$[k \odot u]^r = k \cdot [u]^r = \begin{cases} [k\underline{u}(r), k\bar{u}(r)], & k \geq 0 \\ [k\bar{u}(r), k\underline{u}(r)], & k < 0. \end{cases}$$

The neutral element with respect to \oplus in E^1 is denoted by $\tilde{0} = \chi_{\{0\}}$.

For basic algebraic properties of fuzzy numbers, please see ([37]).

Definition 4 ([39]). Let $x, y \in E^1$ and exists $z \in E^1$, such that $x = y \oplus z$. Then z is called the H -difference of x and y and is given by $x \ominus y$.

We use the Hausdorff metric as a distance between fuzzy numbers.

Definition 5 ([37]). For arbitrary fuzzy numbers $u = (\underline{u}(r), \bar{u}(r))$ and $v = (\underline{v}(r), \bar{v}(r))$ the quantity

$$d(u, v) = \sup_{r \in [0, 1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}$$

is the distance between u, v .

The metric d is a complete metric space in E^1 .

For any fuzzy-number-valued function $w : D \rightarrow E^1$ we define the functions $\underline{w}(\cdot, \cdot, r), \bar{w}(\cdot, \cdot, r) : D \rightarrow \mathbb{R}$, for all $r \in [0, 1]$. These functions are called the left and right r -level functions of w .

Definition 6 ([15]). A fuzzy-number-valued function $w : D \rightarrow E^1$ is said to be continuous at $(s_0, t_0) \in D$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $d(f(s, t), f(s_0, t_0)) < \epsilon$ whenever $|s - s_0| + |t - t_0| < \delta$. If w be continuous for each $(s, t) \in D$ then we say that w is continuous on D .

Let $R > 0$. Denote $D_R = D \cap \bar{U}(0, R)$, where

$$\bar{U}(0, R) = \{(x, y) : x^2 + y^2 \leq R^2\}$$

is the closed circle with radius R .

Let $w : D \rightarrow E^1$ be fuzzy-valued function with parametric form $(\underline{w}(x, y, r), \bar{w}(x, y, r))$ for all $r \in [0, 1]$.

Theorem 1. Let for all $r \in [0, 1]$

1. the functions $\underline{w}(x, y, r)$ and $\bar{w}(x, y, r)$ are Riemann-integrable on D_R ,

2. there are constants $\underline{M}(r) > 0$ and $\overline{M}(r) > 0$, such that

$$\int \int_{D_R} |\underline{w}(x, y, r)| dx dy \leq \underline{M}(r), \quad \int \int_{D_R} |\overline{w}(x, y, r)| dx dy \leq \overline{M}(r)$$

for every $R > 0$.

Then the function $w(x, y)$ is improper fuzzy Riemann-integrable on D and

$$(FR) \int_0^\infty (FR) \int_0^\infty w(x, y) dx dy = \left(\int_0^\infty \int_0^\infty \underline{w}(x, y, r) dx dy, \int_0^\infty \int_0^\infty \overline{w}(x, y, r) dx dy \right).$$

Proof. Define the function $\underline{I} : (0, \infty) \rightarrow \mathbb{R}_+$ by

$$\underline{I}(R) = \int \int_{D_R} |\underline{w}(x, y, r)| dx dy \text{ for all } r \in [0, 1].$$

From condition 2, it follows that \underline{I} is bounded and monotonically increasing. Hence, there exists

$$\lim_{R \rightarrow \infty} \underline{I}(R) = \int_0^\infty \int_0^\infty \underline{w}(x, y, r) dx dy.$$

□

For fuzzy valued functions $w = w(x, y)$ we define the n -th order partial H-derivatives with respect to x and y as given in [11].

Definition 7. Let $w : (a, b) \times (c, d) \rightarrow E^1$ be a fuzzy function. We call that w is H-differentiable of the n -th order at $x_0 \in (a, b)$, with respect to x , if there exists an element $\frac{\partial^n w(x_0, y)}{\partial x^n} \in E^1$ such that

1. for all $h > 0$ sufficiently small the H-differences

$$\frac{\partial^{n-1} w(x_0 + h, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}}, \quad \frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0 - h, y)}{\partial x^{n-1}},$$

exist and the following limits hold (in the metric d)

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\partial^{n-1} w(x_0 + h, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0 - h, y)}{\partial x^{n-1}} \right) = \frac{\partial^n w(x_0, y)}{\partial x^n} \end{aligned}$$

or

2. for all $h > 0$ sufficiently small the H-differences

$$\frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0 + h, y)}{\partial x^{n-1}}, \quad \frac{\partial^{n-1} w(x_0 - h, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}},$$

exist and the following limits hold (in the metric d)

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{-1}{h} \left(\frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0 + h, y)}{\partial x^{n-1}} \right) \\ &= \lim_{h \rightarrow 0} \frac{-1}{h} \left(\frac{\partial^{n-1} w(x_0 - h, y)}{\partial x^{n-1}} \ominus \frac{\partial^{n-1} w(x_0, y)}{\partial x^{n-1}} \right) = \frac{\partial^n w(x_0, y)}{\partial x^n}. \end{aligned}$$

Similarly,

Definition 8. Let $w : (a, b) \times (c, d) \rightarrow E^1$ be a fuzzy function. We call that w is H -differentiable of the n -th order at $y_0 \in (c, d)$, with respect to y , if there exists an element $\frac{\partial^n w(x, y_0)}{\partial y^n} \in E^1$ such that

1. For $h > 0$ sufficiently small the H -differences

$$\frac{\partial^{n-1}w(x, y_0 + h)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}}, \frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0 - h)}{\partial y^{n-1}},$$

exist and the following limits hold (in the metric d)

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\partial^{n-1}w(x, y_0 + h)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0 - h)}{\partial y^{n-1}} \right) = \frac{\partial^n w(x, y_0)}{\partial y^n} \end{aligned}$$

or

2. For $h > 0$ sufficiently small the H -differences

$$\frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0 + h)}{\partial y^{n-1}}, \frac{\partial^{n-1}w(x, y_0 - h)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}},$$

exist and the following limits hold (in the metric d)

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{-1}{h} \left(\frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0 + h)}{\partial y^{n-1}} \right) \\ &= \lim_{h \rightarrow 0} \frac{-1}{h} \left(\frac{\partial^{n-1}w(x, y_0 - h)}{\partial y^{n-1}} \ominus \frac{\partial^{n-1}w(x, y_0)}{\partial y^{n-1}} \right) = \frac{\partial^n w(x, y_0)}{\partial y^n}. \end{aligned}$$

The first type of differentiability as in Definition 7 and Definition 8 are referred as (i)-differentiable, while the second type as (ii)-differentiable.

Theorem 2 ([11]). Let $w : \mathbb{R} \times \mathbb{R} \rightarrow E^1$ be a continuous fuzzy-valued function and $w(x, y) = (\underline{w}(x, y, r), \overline{w}(x, y, r))$ for all $r \in [0, 1]$. Then

1. if $w(x, y)$ is (i)-differentiable of the n -th order with respect to x , then $\underline{w}(x, y, r)$ and $\overline{w}(x, y, r)$ are differentiable of the n -th order with respect to x and

$$\frac{\partial^n w(x, y)}{\partial x^n} = \left(\frac{\partial^n \underline{w}(x, y, r)}{\partial x^n}, \frac{\partial^n \overline{w}(x, y, r)}{\partial x^n} \right), \tag{1}$$

2. if $w(x, y)$ is (ii)-differentiable of the n -th order with respect to x , then $\underline{w}(x, y, r)$ and $\overline{w}(x, y, r)$ are differentiable of the n -th order with respect to x and

$$\frac{\partial^n w(x, y)}{\partial x^n} = \left(\frac{\partial^n \overline{w}(x, y, r)}{\partial x^n}, \frac{\partial^n \underline{w}(x, y, r)}{\partial x^n} \right). \tag{2}$$

3. Two-Dimensional Fuzzy Sumudu Transform

In this part, we give DFST definition and its inverse. We introduced the concept of double fuzzy convolution and give two new results of DFST for fuzzy partial derivative of the n -th order.

Definition 9 ([35]). Let $w : \mathbb{R} \times \mathbb{R} \rightarrow E^1$ be a continuous fuzzy-valued function and the function $e^{-x-y} \odot w(ux, vy)$ is improper fuzzy Riemann-integrable on D , then

$$(FR) \int_0^\infty (FR) \int_0^\infty e^{-x-y} \odot w(ux, vy) dx dy,$$

is called DFST and is denote by

$$W(u, v) = S[w(x, y)] = (FR) \int_0^\infty (FR) \int_0^\infty e^{-x-y} \odot w(ux, vy) dx dy, \tag{3}$$

for $u \in [-\tau_1, \tau_2]$ and $v \in [-\sigma_1, \sigma_2]$, where the variables u, v are used to factor the variables x, y in the argument of the fuzzy-valued function and $\tau_1, \tau_2, \sigma_1, \sigma_2 > 0$.

The parametric form of DFST is follows

$$S[w(x, y)] = (s[\underline{w}(x, y, r)], s[\overline{w}(x, y, r)]), \tag{4}$$

where

$$s[\underline{w}(x, y, r)] = \int_0^\infty \int_0^\infty e^{-x-y} \underline{w}(ux, vy, r) dx dy, \tag{5}$$

$$s[\overline{w}(x, y, r)] = \int_0^\infty \int_0^\infty e^{-x-y} \overline{w}(ux, vy, r) dx dy. \tag{6}$$

The equation (3) we can rewrite in the form

$$W(u, v) = S[w(x, y)] = \frac{1}{uv} (FR) \int_0^\infty (FR) \int_0^\infty e^{-\left(\frac{x}{u} + \frac{y}{v}\right)} \odot w(x, y) dx dy. \tag{7}$$

Definition 10 ([35]). Double fuzzy inverse Sumudu transform can be written as the formula

$$S^{-1} [W(u, v)] = w(x, y) = \left(s^{-1}[\underline{W}(u, v, r)], s^{-1}[\overline{W}(u, v, r)] \right), \tag{8}$$

where

$$s^{-1}[\underline{W}(u, v, r)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{x}{u}} du \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{\frac{y}{v}} \underline{W}(u, v, r) dv,$$

$$s^{-1}[\overline{W}(u, v, r)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{x}{u}} du \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{\frac{y}{v}} \overline{W}(u, v, r) dv.$$

For all $r \in [0, 1]$ the functions $\underline{W}(u, v, r)$ and $\overline{W}(u, v, r)$ must be analytic functions for all u and v in the region defined by the inequalities $Reu \geq \gamma$ and $Rev \geq \delta$, where γ and δ are real constants to be chosen suitably.

In [40] classical double Sumudu transform is applied on some special functions.

1. Let $g(x, y) = 1$ for $x > 0, y > 0$, then $s[g(x, y)] = 1$.
2. Let $g(x, y) = x^m y^n$, where m, n are positive integers, then

$$s[g(x, y)] = (m!)(n!)u^m v^n. \tag{9}$$

3. Let $g(x, y) = e^{ax+by}$, where a, b are any constants, then

$$s[g(x, y)] = \frac{1}{(1 - au)(1 - bv)}. \tag{10}$$

4.

$$s[\cos(ax + by)] = \frac{(1 - abuv)}{(1 + a^2u^2)(1 + b^2v^2)}, \tag{11}$$

$$s[\sin(ax + by)] = \frac{(bv + au)}{(1 + a^2u^2)(1 + b^2v^2)}. \tag{12}$$

Theorem 3 ([35]). Let $g(x, y)$ be a continuous fuzzy-valued function. If $G(u, v)$ is the double fuzzy Sumudu transform of $g(x, y)$ and a, b are arbitrary constants, then

$$S[e^{ax+by} \odot g(x, y)] = \frac{1}{(1 - au)(1 - bv)} \odot G\left(\frac{u}{1 - au}, \frac{v}{1 - bv}\right). \tag{13}$$

In [35] DFST theorems and properties generated by DFST are given.

Definition 11 ([35]). If $k(x, y)$ and $w(x, y)$ are fuzzy Riemann integrable functions, then double fuzzy convolution of $k(x, y)$ and $w(x, y)$ is given by

$$(k ** w)(x, y) = (FR) \int_0^y (FR) \int_0^x k(x - \alpha, y - \beta)w(\alpha, \beta)d\alpha d\beta \tag{14}$$

and the symbol $**$ denotes the double convolution respect to x and y .

Theorem 4 ([35]). Let $k : D \rightarrow \mathbb{R}$ and $w(x, y)$ be fuzzy functions. Then the DFST of the double fuzzy convolution k and w , is given by

$$S[(k ** w)(x, y)] = uvS[k(x, y)] \odot S[w(x, y)]. \tag{15}$$

We introduce results of DFST for fuzzy partial derivatives.

Theorem 5. Let $w : \mathbb{R} \times \mathbb{R} \rightarrow E^1$ be a continuous fuzzy-valued function. The functions $e^{-x-y} \odot w(ux, vy)$, $e^{-x-y} \odot \frac{\partial^n w(ux, vy)}{\partial x^n}$ are improper fuzzy Riemann-integrable on D . Then

$$S\left[\frac{\partial^n w(x, y)}{\partial x^n}\right] = \frac{\partial^n}{\partial x^n} S[w(x, y)], \tag{16}$$

where $S[w(x, y)]$ denotes the DFST of the function w and $n \in \mathbb{N}$.

Proof. Let the function $w(x, y)$ is (i) -differentiable. From definition of DFST, we have

$$\begin{aligned} S\left[\frac{\partial^n w(x, y)}{\partial x^n}\right] &= (FR) \int_0^\infty (FR) \int_0^\infty e^{-x-y} \odot \frac{\partial^n w(ux, vy)}{\partial x^n} dx dy \\ &= \left(\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial^n w(ux, vy)}{\partial x^n} dx dy, \int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial^n \bar{w}(ux, vy)}{\partial x^n} dx dy \right) \end{aligned}$$

$$= \frac{\partial^n}{\partial x^n} \left[\int_0^\infty \int_0^\infty e^{-x-y} \underline{w}(ux, vy) dx dy, \int_0^\infty \int_0^\infty e^{-x-y} \overline{w}(ux, vy) dx dy \right] = \frac{\partial^n}{\partial x^n} S[w(x, y)].$$

□

Theorem 6. Let $w : \mathbb{R} \times \mathbb{R} \rightarrow E^1$ be a fuzzy-valued function. The functions $e^{-x-y} \odot w(ux, vy)$, $e^{-x-y} \odot \frac{\partial^n w(ux, vy)}{\partial x^n}$ are improper fuzzy Riemann-integrable on D . For all $x > 0$ and $n \in \mathbb{N}$ there exist to continuous partial H -derivatives to $(n - 1)$ -th order with respect to x and there exists $\frac{\partial^n w(x, y)}{\partial x^n}$. Then

1. if the function $w(x, y)$ is (i)-differentiable then

$$S \left[\frac{\partial^n w(x, y)}{\partial x^n} \right] = \left(s \left[\frac{\partial^n \underline{w}(x, y)}{\partial x^n} \right], s \left[\frac{\partial^n \overline{w}(x, y)}{\partial x^n} \right] \right),$$

2. if the function $w(x, y)$ is (ii)-differentiable then

$$S \left[\frac{\partial^n w(x, y)}{\partial x^n} \right] = \left(s \left[\frac{\partial^n \overline{w}(x, y)}{\partial x^n} \right], s \left[\frac{\partial^n \underline{w}(x, y)}{\partial x^n} \right] \right),$$

where

$$s \left[\frac{\partial^n \underline{w}(x, y, r)}{\partial x^n} \right] = \frac{1}{u^n} s [\underline{w}(x, y, r)] - \sum_{j=1}^n \frac{1}{u^j} s \left[\frac{\partial^{n-j} \underline{w}(0, y, r)}{\partial x^{n-j}} \right], \tag{17}$$

$$s \left[\frac{\partial^n \overline{w}(x, y, r)}{\partial x^n} \right] = \frac{1}{u^n} s [\overline{w}(x, y, r)] - \sum_{j=1}^n \frac{1}{u^j} s \left[\frac{\partial^{n-j} \overline{w}(0, y, r)}{\partial x^{n-j}} \right]. \tag{18}$$

Proof. Let the function $w(x, y)$ is (i)-differentiable. By induction we proof the equation (17). For $n = 1$ from condition (4) we have

$$S [w'_x(x, y)] = (s[\underline{w}'_x(x, y, r)], s[\overline{w}'_x(x, y, r)]).$$

By us part integration on x and condition (4) we obtain

$$s[\underline{w}'_x(x, y, r)] = \int_0^\infty \int_0^\infty e^{-x-y} \underline{w}'_x(ux, vy, r) dx dy = \frac{1}{u} (s[\underline{w}(x, y, r)] - s[\underline{w}(0, y, r)]).$$

Let for $n = k$ the equation (17) holds. Then

$$s \left[\frac{\partial^k \underline{w}(x, y, r)}{\partial x^k} \right] = \frac{1}{u^k} s [\underline{w}(x, y, r)] - \sum_{j=1}^k \frac{1}{u^j} s \left[\frac{\partial^{k-j} \underline{w}(0, y, r)}{\partial x^{k-j}} \right].$$

Hence, for $n = k + 1$ we get

$$\begin{aligned} s \left[\frac{\partial^{k+1} \underline{w}(x, y, r)}{\partial x^{k+1}} \right] &= \frac{\partial}{\partial x} s \left[\frac{\partial^k \underline{w}(x, y, r)}{\partial x^k} \right] = \frac{\partial}{\partial x} \left(\frac{1}{u^k} s [\underline{w}(x, y, r)] \right) - \frac{\partial}{\partial x} \left(\sum_{j=1}^k \frac{1}{u^j} s \left[\frac{\partial^{k-j} \underline{w}(0, y, r)}{\partial x^{k-j}} \right] \right) \\ &= \frac{1}{u^k} s [\underline{w}'_x(x, y, r)] - \sum_{j=1}^k \frac{1}{u^j} s \left[\frac{\partial^{k+1-j} \underline{w}(0, y, r)}{\partial x^{k+1-j}} \right] \\ &= \frac{1}{u^{k+1}} (s [\underline{w}(x, y, r)] - s [\underline{w}(0, y, r)]) - \sum_{j=1}^k \frac{1}{u^j} s \left[\frac{\partial^{k+1-j} \underline{w}(0, y, r)}{\partial x^{k+1-j}} \right] \\ &= \frac{1}{u^{k+1}} s [\underline{w}(x, y, r)] - \sum_{j=1}^{k+1} \frac{1}{u^j} s \left[\frac{\partial^{k+1-j} \underline{w}(0, y, r)}{\partial x^{k+1-j}} \right]. \end{aligned}$$

□

4. DFST for Solving PVFIDE

In this section, we application of the DFST method for solving of PVFIDE. This equation is defined as

$$\sum_{i=1}^m a_i \odot \frac{\partial^i w(x,y)}{\partial x^i} \oplus \sum_{j=1}^n b_j \odot \frac{\partial^j w(x,y)}{\partial y^j} \oplus c \odot w(x,y) = g(x,y) \oplus (FR) \int_0^y (FR) \int_0^x k(x-\alpha, y-\beta) \odot w(\alpha, \beta) d\alpha d\beta, \tag{19}$$

with initial conditions

$$\frac{\partial^i w(0,y)}{\partial x^i} = \varphi_i(y), \quad i = 0, 1, \dots, m-1 \tag{20}$$

$$\frac{\partial^i w(x,0)}{\partial y^j} = \psi_j(x), \quad i = 0, 1, \dots, n-1, \tag{21}$$

where $k : [0, b] \times [0, d] \rightarrow \mathbb{R}$, is a continuous functions and $g, w : [0, b] \times [0, d] \rightarrow E^1, \varphi_i : [0, d] \rightarrow E^1, \psi_j : [0, b] \rightarrow E^1$ are continuous fuzzy functions and $a_i, i = 1, 2, \dots, m, b_j, j = 1, 2, \dots, n, c$, are constants.

Applying DFST on both side of it we get the following

$$S \left[\sum_{i=1}^m a_i \frac{\partial^i w(x,y)}{\partial x^i} \right] \oplus S \left[\sum_{j=1}^n b_j \frac{\partial^j w(x,y)}{\partial y^j} \right] \oplus S[c \odot w(x,y)] = S[g(x,y)] \oplus S \left[(FR) \int_0^y (FR) \int_0^x k(x-\alpha, y-\beta) \odot w(\alpha, \beta) d\alpha d\beta \right],$$

By using double fuzzy convolution (15) we obtain

$$\sum_{i=1}^m a_i \odot S \left[\frac{\partial^i w(x,y)}{\partial x^i} \right] \oplus \sum_{j=1}^n b_j \odot S \left[\frac{\partial^j w(x,y)}{\partial y^j} \right] \oplus c \odot S[w(x,y)] = S[g(x,y)] \oplus uvs[k(x,y)] \odot S[w(x,y)]$$

Let the constants $a_i, i = 1, \dots, m, b_j, j = 1, \dots, n, c$ be positive and the function $k(x,y) > 0$.

1. if $w(x,y)$ is (i) -differentiable, then

$$\sum_{i=1}^m a_i s \left[\frac{\partial^i \underline{w}(x,y,r)}{\partial x^i} \right] + \sum_{j=1}^n b_j s \left[\frac{\partial^j \underline{w}(x,y,r)}{\partial y^j} \right] + cs[\underline{w}(x,y,r)] = s[\underline{g}(x,y,r)] + uvs[k(x,y)]s[\underline{w}(x,y,r)]$$

and

$$\sum_{i=1}^m a_i s \left[\frac{\partial^i \overline{w}(x,y,r)}{\partial x^i} \right] + \sum_{j=1}^n b_j s \left[\frac{\partial^j \overline{w}(x,y,r)}{\partial y^j} \right] + cs[\overline{w}(x,y,r)] = s[\overline{g}(x,y,r)] + uvs[k(x,y)]s[\overline{w}(x,y,r)]$$

Then from (17) and (18) we have

$$\left(\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x,y)] \right) s[\underline{w}(x,y,r)] = s[\underline{g}(x,y,r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s \left[\frac{\partial^{i-k} \underline{w}(0,y,r)}{\partial x^{i-k}} \right] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s \left[\frac{\partial^{j-k} \underline{w}(x,0,r)}{\partial y^{j-k}} \right]$$

and

$$\left(\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x,y)] \right) s[\overline{w}(x,y,r)] = s[\overline{g}(x,y,r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s \left[\frac{\partial^{i-k} \overline{w}(0,y,r)}{\partial x^{i-k}} \right] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s \left[\frac{\partial^{j-k} \overline{w}(x,0,r)}{\partial y^{j-k}} \right]$$

Using the initial conditions (20) and (21) we get

$$\begin{aligned} & \left(\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)] \right) s[\underline{w}(x, y, r)] \\ &= s[\underline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\varphi_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\psi_{j-k}^{j-k}(x, 0, r)] \end{aligned}$$

and

$$\begin{aligned} & \left(\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)] \right) s[\overline{w}(x, y, r)] \\ &= s[\overline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\overline{\varphi}_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\overline{\psi}_{j-k}^{j-k}(x, 0, r)] \end{aligned}$$

Then

$$s[\underline{w}(x, y, r)] = \frac{s[\underline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\varphi_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\psi_{j-k}^{j-k}(x, 0, r)]}{\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)]} \tag{22}$$

and

$$s[\overline{w}(x, y, r)] = \frac{s[\overline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\overline{\varphi}_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\overline{\psi}_{j-k}^{j-k}(x, 0, r)]}{\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)]} \tag{23}$$

2. if $w(x, y)$ is (ii)-differentiable, then

$$s[\underline{w}(x, y, r)] = \frac{s[\underline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\overline{\varphi}_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\overline{\psi}_{j-k}^{j-k}(x, 0, r)]}{\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)]} \tag{24}$$

and

$$s[\overline{w}(x, y, r)] = \frac{s[\overline{g}(x, y, r)] + \sum_{i=1}^m \sum_{k=1}^i \frac{a_i}{u^k} s[\varphi_{i-k}^{i-k}(0, y, r)] + \sum_{j=1}^n \sum_{k=1}^j \frac{b_j}{v^k} s[\psi_{j-k}^{j-k}(x, 0, r)]}{\sum_{i=1}^m \frac{a_i}{u^i} + \sum_{j=1}^n \frac{b_j}{v^j} + c - uvs[k(x, y)]} \tag{25}$$

By using the inverse of DFST we obtain $w(y, y) = (\underline{w}(x, y, r), \overline{w}(x, y, r))$.

5. Examples

In this section, we find the solution of partial convolution Volterra fuzzy integro-differential equation using DFST.

Example 1. Consider the following PVFIDE

$$\begin{aligned} w''_{xx}(x, y) \oplus w''_{yy}(x, y) \oplus w(x, y) &= g(x, y) \oplus (FR) \int_0^y (FR) \int_0^x e^{x-\alpha+y-\beta} w(\alpha, \beta) d\alpha d\beta, \\ (x, y) &\in [0, 1] \times [0, 1] \quad r \in [0, 1] \end{aligned}$$

with initial conditions

$$w(x, 0) = (e^x(2 + r), e^x(4 - r)), \quad w'_y(x, 0) = (e^x(2 + r), e^x(4 - r)),$$

$$w(0, y) = (e^y(2 + r), e^y(4 - r)), \quad w'_x(0, y) = (e^y(2 + r), e^y(4 - r))$$

and

$$g(x, y) = (e^{x+y}(2 + xy)(2 + r), e^{x+y}(2 + xy)(4 - r)).$$

In this case $m = n = 2, a_1 = b_1 = 0, a_2 = b_2 = 1, c = 1,$

$k(x - \alpha, y - \beta) = e^{x-\alpha+y-\beta} > 0$ for $0 \leq \alpha \leq x \leq 1$ and $0 \leq \beta \leq y \leq 1,$

$$\psi_0(x, 0) = (e^x(2 + r), e^x(4 - r)), \quad \varphi_0(y) = (e^y(2 + r), e^y(4 - r)).$$

$$\psi_1(x, 0) = (e^x(2 + r), e^x(4 - r)), \quad \varphi_1(y) = (e^y(2 + r), e^y(4 - r)).$$

From (10), we find

$$s[k(x, y)] = s[e^{x+y}] = \frac{1}{(1 - u)(1 - v)},$$

$$S[\psi_0(x)] = S[\psi_1(x)] = (s[e^x(2 + r)], s[e^x(4 - r)]) = \left(\frac{1}{1 - u}(2 + r), \frac{1}{1 - u}(4 - r) \right),$$

$$S[\varphi_0(y)] = S[\varphi_1(y)] = (s[e^y(2 + r)], s[e^y(4 - r)]) = \left(\frac{1}{1 - v}(2 + r), \frac{1}{1 - v}(4 - r) \right)$$

From Theorem 3 we obtain

$$\begin{aligned} S[g(x, y)] &= (s[(2 + xy)e^{x+y}(2 + r)], s[(2 + xy)e^{x+y}(4 - r)]) \\ &= \left(\left(3 - \frac{uv}{(1 - u)(1 - v)} \right) \frac{(2 + r)}{(1 - u)(1 - v)}, \left(3 - \frac{uv}{(1 - u)(1 - v)} \right) \frac{(4 - r)}{(1 - u)(1 - v)} \right). \end{aligned}$$

Then, of (22) and (23) for the solution of the equation we have

$$s[\underline{w}(x, y, r)] = \frac{1}{(1 - u)(1 - v)}(2 + r).$$

and

$$s[\overline{w}(x, y, r)] = \frac{1}{(1 - u)(1 - v)}(4 - r).$$

By inverse double Sumudu transform the solution of the equation is $w(x, y) = (e^{x+y}(2 + r), e^{x+y}(4 - r)).$

6. Conclusions

In this paper, the double fuzzy Sumudu transform method for solving partial convolution Volterra fuzzy integro-differential equations have been studied. The concept of double fuzzy convolution have been introduced. New results on DFST for fuzzy partial H-derivative of the n -th order have been proposed.

By using the parametric form of fuzzy functions we convert the investigated equation to a nonlinear system of partial Volterra integro-differential equations in a crisp case. Applying DFST method for this system we obtain system of algebraic equations. Hence we find the lower and upper functions of the solution. Finally, the examples to show that the investigation method is effective in solving the equations of considered kind.

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