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Justification of the Lugiato-Lefever Model from a Damped Driven ϕ^4 Equation

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Abstract: The Lugiato-Lefever equation is a damped and driven version of the well-known nonlinear Schrödinger equation. It is a mathematical model describing complex phenomena in dissipative and nonlinear optical cavities. Within the last two decades, the equation has gained much attention as it has become the basic model describing microresonator (Kerr) frequency combs. Recent works derive the Lugiato-Lefever equation from a class of damped driven ϕ^4 equations closed to resonance. In this paper, we provide a justification of the envelope approximation. From the analysis point of view, the result is novel and non-trivial as the drive yields a perturbation term that is not square integrable. The main approach proposed in this work is to decompose the solutions into a combination of the background and the integrable component. This paper is the first part of a two-manuscript series.

Keywords: Lugiato-Lefever equation; nonlinear Schrödinger equation; ϕ^4 equation; small-amplitude approximation

1. Introduction

The Lugiato-Lefever equation is given by [1]

$$\mathrm{i}A_\tau = -A_{\xi\xi} - \frac{\mathrm{i}\alpha}{2}A - \frac{3\lambda}{2\omega}|A|^2A + F, \quad \xi \in \mathbb{R}, \tau \geq 0, \tag{1}$$

with real valued parameters, which is nothing else but a damped driven nonlinear Schrödinger equation. It models spatiotemporal pattern formation in dissipative, diffractive and nonlinear optical cavities submitted to a continuous laser pump [2–6]. The same model was shown rather immediately to also appear in dispersive optical ring cavities [7]. Lugiato-Lefever equation has raised a wide interest particularly following its recent successful experimental application in the study of broadband microresonator-based optical frequency combs [8,9], that has opened applicative avenues (see References [10,11] for reviews on the subject and References [12,13] on the application of the Lugiato-Lefever equation to describe the nonlinear dynamics of passive optical cavities and the generation of frequency combs).

Recently, Ferré et al. [14] showed that the dynamics of the Lugiato-Lefever equation can also be obtained from a driven dissipative sine-Gordon model. The former equation is a single envelope approximation, that is, modulation equation, of the latter. Even in the region far from the conservative limit, where the approximation is expectedly no longer valid, they were reported to still exhibit qualitatively similar dynamical behaviors.

Herein, instead of the sine-Gordon equation, we consider a nonlinear damped driven ϕ^4 model

$$u_{tt} + \epsilon^2\alpha u_t - \beta u_{xx} + \gamma u - \lambda u^3 = \epsilon^3 h\left(e^{\mathrm{i}\Omega t} + e^{-\mathrm{i}\Omega t}\right), \tag{2}$$

where $\alpha, \beta, \gamma > 0$, $h \in \mathbb{R}$ and ϵ is a small positive parameter. The nonlinearity is considered to be 'softening', that is, $\lambda < 0$. The 'hardening' case $\lambda > 0$ will be discussed in the second part of this paper series, whose results can be extended to the sine-Gordon equation. This equation belongs to the class of nonlinear Klein-Gordon models that have broad applications [15].

Introducing the slow time and space τ and ξ defined as $\tau = \epsilon^2 t$ and $\xi = \epsilon\sqrt{\frac{2\omega^3}{\gamma\beta}}\,(x - vt)$, where $v = d\omega/dk = \beta k/\omega$ is the group velocity of the linear traveling wave and k and ω satisfy the dispersion relation $\omega^2 = \beta k^2 + \gamma$, we define the slowly modulated ansatz function as

$$X(t,x) = \epsilon A(\tau,\xi)e^{\mathrm{i}(kx-\omega t)} + \frac{\lambda\epsilon^3}{9\beta k^2 - 9\omega^2 + \gamma}A(\tau,\xi)^3 e^{3\mathrm{i}(kx-\omega t)} + \text{c.c}\,. \tag{3}$$

The modulation amplitude A is a complex valued function satisfying Equation (1), where $F(\tau,\xi) = -\frac{h}{2\omega}e^{-\mathrm{i}(\kappa\xi-\nu\tau)}$ with $\kappa = k/\epsilon$ and $\Omega = \gamma/\omega - \epsilon^2\nu$. Inserting the ansatz function (3) into (2), we get the residual terms

$$\text{Res}(t,x) = \mathcal{O}(\epsilon^4). \tag{4}$$

The same modulation equation has been derived in early reports, for example, in References [16–18], to describe matters driven by an external ac field. Analytical studies of various solutions of the damped, driven continuous nonlinear Schrödinger equation have also been reported in References [19,20]. Nevertheless, despite the long history of the problem, a rigorous justification of the approximation is interestingly still lacking. The main challenge is due to the external drive F that is not integrable. Our work presented in this paper is to provide an answer to the missing piece.

An early work justifying the modulation equation without damping and drive was due to References [21,22]. The presence of external drives would not bring any problem should one consider nonlinear systems that correspond to a parabolic linear operator [21,23–26]. In the context of Equation (2), this corresponds to $\alpha \to \infty$, in which case the modulation equation would be a Ginzburg-Landau-type equation, that is, there is no factor i on the left hand side of (1).

Recently we have considered the reduction of a Klein-Gordon equation with external damping and drive into a damped driven discrete nonlinear Schrödinger equation [27]. To overcome the nonintegrability of the solutions, we worked in a periodic domain. The present report extends our result in Reference [27] by proposing a method that also works in $L^2(\mathbb{R})$.

This paper is organized as follows. To provide a rigorous proof justifying the modulation equation, we formulate our method in Section 2 by decomposing the solutions into the background and particular parts. Using a small-amplitude approximation, we derive the Lugiato-Lefever equation and Section 3 presents the local and global existence of homogeneous solutions of the amplitude equation. The main result on the error bound of the approximation as time evolves is presented in Section 4. In Section 5, we summarise the results and discuss their signification.

2. Solution Decomposition

Since the external drive term $F(\tau,\xi)$ is not integrable in the spatial variable, that is, $F(\tau,\xi) \notin L^p(\mathbb{R})$ for any integer $1 \le p < \infty$, in general $A(\tau,\xi)$ is also not integrable. Let $A_p(\tau,\xi)$ be a particular solution of Equation (1) which can be written as

$$A_p(\tau,\xi) = R\, e^{-\mathrm{i}(\kappa\xi-\nu\tau)}\,, \tag{5}$$

where R is a complex constant such that

$$R = -\frac{h}{2\omega}\,\frac{1}{\frac{3\lambda}{2\omega}|R|^2 - (\kappa^2+\nu) + \frac{\mathrm{i}\alpha}{2}}\,, \tag{6}$$

and $|R|^2$ satisfies the cubic equation,

$$\frac{9\lambda^2}{4\omega^2}|R|^6 - \frac{6\lambda}{2\omega}(\kappa^2+\nu)|R|^4 + \left[\frac{\alpha^2}{4} + (\kappa^2+\nu)^2\right]|R|^2 - \frac{h^2}{4\omega^2} = 0\,.$$

Since $\lambda < 0$, the cubic equation has only one real solution. Furthermore, we have

$$\|A_p(\tau,\xi)\|_{L^\infty(\mathbb{R})} = \sup_{\xi\in\mathbb{R}} |A_p(\tau,\xi)| = |R| < +\infty\,. \tag{7}$$

To handle the non-integrability condition we can work on $\mathbb{T}$, rather than on $\mathbb{R}$ (see Reference [27], where a similar problem was considered in the discrete case). In this paper, we propose a different approach by introducing the decomposition,

$$\begin{aligned} A(\tau,\xi) &:= e^{\mathrm{i}\nu\tau}\phi(\tau,\xi) + A_p(\tau,\xi) \\ &= e^{\mathrm{i}\nu\tau}\left[\phi(\tau,\xi) + \eta(\xi)\right], \end{aligned} \tag{8}$$

where $\phi(\tau,\xi)$ is the integrable term and $\eta(\xi) = Re^{-\mathrm{i}\kappa\xi}$. The initial condition for our system is

$$A(0,\xi) = \varphi(\xi) + \eta(\xi)\,. \tag{9}$$

with $\varphi \in H^k(\mathbb{R})$. Here, the space $H^k(\mathbb{R})$ with a nonnegative integer k denotes the Sobolev space with norm defined as

$$\|\phi\|_{H^k(\mathbb{R})} = \left[\sum_{i=0}^{k} \|D_\xi^i\phi\|^2_{L^2(\mathbb{R})}\right]^{1/2}, \tag{10}$$

and $H^0(\mathbb{R}) = L^2(\mathbb{R})$ with

$$\|\phi\|_{L^2(\mathbb{R})} = \left[\int_{\mathbb{R}} |\phi(\xi)|^2\, d\xi\right]^{1/2}. \tag{11}$$

The differential equation for ϕ is given by

$$\mathrm{i}\phi_\tau = -\phi_{\xi\xi} - \frac{\mathrm{i}\alpha}{2}\phi - \left(\frac{3\lambda}{2\omega}|R|^2 - \nu\right)\phi + N(\phi)\,, \tag{12}$$

where the nonlinearity term is

$$N(\phi) := -\frac{3\lambda}{2\omega}\left[|\phi+\eta|^2 - |\eta|^2\right](\phi+\eta)\,. \tag{13}$$

Using (9), we have the initial condition

$$\phi(0,\cdot) = \varphi.$$

3. Existence of Solutions of the Inhomogeneous Nonlinear Schrödinger Equation

In this section, we prove the local and global existence of solutions of the inhomogeneous part of the nonlinear Schrödinger equation. For an excellent review, the interested reader can further consult, for example, References [28,29].

The local existence is stated in the following theorem.

Theorem 1. *Let $k \geq 1$ be an integer. For every $\varphi \in H^k(\mathbb{R})$, there exists a positive constant τ_m depending on the initial data and k such that the differential Equation (12) admits a unique maximal solution $\phi(\tau)$ on $[0,\tau_m)$ with*

$$\phi \in C\left([0,\tau_m), H^k(\mathbb{R})\right), \tag{14}$$

and either,

1. $\tau_m = +\infty$, *and (12) admits a global solution, or*
2. $\tau_m < +\infty$, *then* $\|\phi(\tau)\|_{H^k(\mathbb{R})} \to \infty$ *as* $\tau \to \tau_m$ *and the solution blows up at finite time* τ_m*. Moreover,* $\limsup \|\phi(\tau)\|_{L^\infty(\mathbb{R})} \to \infty$ *as* $\tau \to \tau_m$.

Proof. We prove the theorem in three steps.

Step 1. Local existence. Using Duhamel's formula, we can write the solution of the differential Equation (12) as

$$\phi(\tau) = U(\tau)\varphi + \mathrm{i}\int_0^\tau U(\tau-\tau')\left[\frac{\mathrm{i}\alpha}{2}\phi(\tau') + \left(\frac{3\lambda}{2\omega}|R|^2 - \nu\right)\phi(\tau') - N(\phi(\tau'))\right] d\tau' \,, \tag{15}$$

where $U(\tau)$ is a one-dimensional free Schrödinger time evolution operator. Define

$$\mathcal{B} = \left\{\phi \in C\left([0,\tilde{\tau}], H^k(\mathbb{R})\right) \Big| \|\phi(\tau)\|_{H^k(\mathbb{R})} \le M,\ \forall \tau \in [0,\tilde{\tau})\right\} \tag{16}$$

as a closed ball on $C\left([0,\tilde{\tau}], H^k(\mathbb{R})\right)$ equipped with norm,

$$\|\phi\|_{\mathcal{B}} = \sup_{\tau\in[0,\tilde{\tau})} \|\phi(\tau)\|_{H^k(\mathbb{R})} \,. \tag{17}$$

For $\phi \in H^k(\mathbb{R})$, we define a nonlinear operator

$$\mathcal{K}[\phi](\tau) = U(\tau)\varphi + \mathrm{i}\int_0^\tau U(\tau-\tau')\left[\frac{\mathrm{i}\alpha}{2}\phi(\tau') + \left(\frac{3\lambda}{2\omega}|R|^2 - \nu\right)\phi(\tau') - N(\phi(\tau'))\right] d\tau'. \tag{18}$$

We want to prove that the operator $\mathcal{K}$ is a contraction mapping on $\mathcal{B}$. Using the fact that the free Schrödinger operator $U(\tau)$ is a linear operator and unitary on $H^k(\mathbb{R})$, we have

$$\|U(\tau)\phi\|_{H^k(\mathbb{R})} = \|\phi\|_{H^k(\mathbb{R})} \,, \tag{19}$$

for any $\phi \in H^k(\mathbb{R})$. Thus, we get

$$\begin{aligned} \|\mathcal{K}[\phi](\tau)\|_{H^k(\mathbb{R})} &\le \|U(\tau)\varphi\|_{H^k(\mathbb{R})} + \int_0^\tau \left\|U(\tau-\tau')\left[\tfrac{\mathrm{i}\alpha}{2}\phi(\tau') + \left(\tfrac{3\lambda}{2\omega}|R|^2 - \nu\right)\phi(\tau') - N(\phi(\tau'))\right]\right\|_{H^k(\mathbb{R})} d\tau' \\ &\le \|\varphi\|_{H^k(\mathbb{R})} + \int_0^\tau \left\|\left[\tfrac{\mathrm{i}\alpha}{2}\phi(\tau') + \left(\tfrac{3\lambda}{2\omega}|R|^2 - \nu\right)\phi(\tau') - N(\phi(\tau'))\right]\right\|_{H^k(\mathbb{R})} d\tau' \\ &\le \|\varphi\|_{H^k(\mathbb{R})} + \left(\tfrac{\alpha}{2} + \tfrac{3|\lambda|}{2\omega}|R|^2 + |\nu|\right)\tau M + \tau \sup_{\tau'\in[0,\tau]} \|N(\phi(\tau'))\|_{H^k(\mathbb{R})} \,. \end{aligned} \tag{20}$$

Note that

$$\left[|\phi+\eta|^2 - |\eta|^2\right](\phi+\eta) = |\phi|^2\phi + 2|\phi|^2\eta + |\eta|^2\phi + \eta^2\bar{\phi} + \phi^2\bar{\eta} \,. \tag{21}$$

Since $H^k(\mathbb{R})$ is an algebra for $k > 1/2$, and using the fact that $|D_\xi^k\, \eta(\xi)| = |\kappa|^k|R|$, thus for $\tau \in [0,\tilde{\tau})$ we get

$$\begin{aligned} \|N(\phi(\tau))\|_{H^k(\mathbb{R})} &\le \frac{3|\lambda|}{2\omega}\left(\|\phi\|^3_{H^k(\mathbb{R})} + 3|R|\|\phi\|^2_{H^k(\mathbb{R})} + 2|R|^2\|\phi\|_{H^k(\mathbb{R})}\right) \\ &\le \frac{3|\lambda|M}{2\omega}\left(M^2 + 3|R|M + 2|R|^2\right) \,. \end{aligned} \tag{22}$$

Assuming that $\|\varphi\|_{H^k(\mathbb{R})} < M/2$, hence we have,

$$\|\mathcal{K}[\phi](\tau)\|_{\mathcal{B}} \le \frac{M}{2} + \frac{\tilde{\tau}M}{2}\left[\alpha + 2|\nu| + \frac{3|\lambda|}{\omega}\left(3|R|^2 + 3|R|M + M^2\right)\right] \,. \tag{23}$$

If we pick

$$\tilde{\tau} < \frac{1}{\alpha + 2|\nu| + \frac{3|\lambda|}{\omega}\left(3|R|^2 + 3|R|M + M^2\right)}, \tag{24}$$

then $\mathcal{K}$ maps $\mathcal{B}$ to itself.

Let $\phi, \tilde{\phi} \in \mathcal{B}$, then

$$\begin{aligned} \|\mathcal{K}[\phi](\tau) - \mathcal{K}[\tilde{\phi}](\tau)\|_{H^k(\mathbb{R})} &\leq \int_0^\tau \left\| \left(\tfrac{\mathrm{i}\alpha}{2} + \tfrac{3\lambda}{2\omega}|R|^2 - \nu\right)(\phi(\tau') - \tilde{\phi}(\tau')) - (N(\phi(\tau')) - N(\tilde{\phi}(\tau')))\right\|_{H^k(\mathbb{R})} d\tau' \\ &\leq \int_0^\tau \left(\tfrac{\alpha}{2} + \tfrac{3|\lambda|}{2\omega}|R|^2 + |\nu|\right) \|(\phi(\tau') - \tilde{\phi}(\tau'))\|_{H^k(\mathbb{R})}\, d\tau' \\ &\quad + \int_0^\tau \|N(\phi(\tau')) - N(\tilde{\phi}(\tau'))\|_{H^k(\mathbb{R})}\, d\tau' \\ &\leq \tau\left(\tfrac{\alpha}{2} + \tfrac{3|\lambda|}{2\omega}|R|^2 + |\nu|\right) \sup_{\tau' \in [0,\tau]} \|(\phi(\tau') - \tilde{\phi}(\tau'))\|_{H^k(\mathbb{R})} \\ &\quad + \tau \sup_{\tau' \in [0,\tau]} \|N(\phi(\tau')) - N(\tilde{\phi}(\tau'))\|_{H^k(\mathbb{R})}. \end{aligned} \tag{25}$$

Note that we have the following inequalities

$$\begin{aligned} |\phi^2 - \tilde{\phi}^2| &= |\phi + \tilde{\phi}||\phi - \tilde{\phi}| \leq (|\phi| + |\tilde{\phi}|)|\phi - \tilde{\phi}|, \\ ||\phi|^2 - |\tilde{\phi}|^2| &= |\phi\bar{\phi} - \tilde{\phi}\bar{\tilde{\phi}}| = |\phi(\bar{\phi} - \bar{\tilde{\phi}}) + \bar{\tilde{\phi}}(\phi - \tilde{\phi})| \leq (|\phi| + |\tilde{\phi}|)|\phi - \tilde{\phi}|, \\ ||\phi|^2\phi - |\tilde{\phi}|^2\tilde{\phi}| &= |(|\phi|^2 + |\tilde{\phi}|^2)(\phi - \tilde{\phi}) + \phi\tilde{(\phi)}(\bar{\phi} - \bar{\tilde{\phi}})| \leq \frac{3}{2}(|\phi|^2 + |\tilde{(\phi)}|^2)|\phi - \tilde{(\phi)}|. \end{aligned}$$

Thus, for $\tau \in [0, \tilde{\tau}]$ we get

$$\|N(\phi(\tau)) - N(\tilde{\phi}(\tau))\|_{H^k(\mathbb{R})} \leq \frac{3|\lambda|}{2\omega}\left(2|R|^2 + 6|R|M + 3M^2\right) \|\phi(\tau) - \tilde{\phi}(\tau)\|_{H^k(\mathbb{R})}. \tag{26}$$

Hence, we have

$$\|\mathcal{K}[\phi](\tau) - \mathcal{K}[\tilde{\phi}](\tau)\|_{\mathcal{B}} \leq \frac{\tilde{\tau}}{2}\left[\alpha + 2|\nu| + \frac{9|\lambda|}{\omega}\left(|R|^2 + 2|R|M + M^2\right)\right] \|\phi(\tau) - \tilde{\phi}(\tau)\|_{\mathcal{B}}. \tag{27}$$

If we pick

$$\tilde{\tau} < \min\left[\frac{1}{\alpha + 2|\nu| + \frac{3|\lambda|}{\omega}\left(3|R|^2 + 3|R|M + M^2\right)}, \frac{2}{\alpha + 2|\nu| + \frac{9|\lambda|}{\omega}\left(|R|^2 + 2|R|M + M^2\right)}\right], \tag{28}$$

then the nonlinear operator $\mathcal{K}$ is a contraction mapping in $\mathcal{B}$. Therefore, by the Banach fixed point theorem, there exists a fixed point of $\mathcal{K}$ which is a solution of (15), and hence of (12).

Step 2. Uniqueness Let $\phi, \tilde{\phi} \in C\left([0, \tilde{\tau}), H^k(\mathbb{R})\right)$ be solutions of (12) with the same initial condition $\varphi \in H^k(\mathbb{R})$. Let $\psi = \tilde{\phi} - \phi$. By Duhamel's formula, we have

$$\psi(\tau) = \mathrm{i}\int_0^\tau U(\tau - \tau')\left[\frac{\mathrm{i}\alpha}{2}\psi(\tau') + \left(\frac{3\lambda}{2\omega}|R|^2 - \nu\right)\psi(\tau') - \left(N(\tilde{\phi}(\tau')) - N(\phi(\tau'))\right)\right] d\tau'. \tag{29}$$

Taking the norm in $H^k(\mathbb{R})$ and using the property of the nonlinear term as in the local existence proof, we get

$$\|\psi(\tau)\|_{H^k(\mathbb{R})} \leq C(M)\int_0^\tau \|\psi(\tau')\|_{H^k(\mathbb{R})}\, d\tau'. \tag{30}$$

By Gronwall's inequality, we conclude that $\|\psi(\tau)\|_{H^k(\mathbb{R})} = 0$ for all $\tau \in [0, \tilde{\tau})$, and hence $\phi' = \phi$.

Step 3. Maximal solution. We can construct the maximal solution by repeating Step 1 with the initial condition $\phi(\tilde{\tau} - \tau_0)$ for some $0 < \tau_0 < \tilde{\tau}$ and using the uniqueness condition to glue the solutions. Clearly, if $\tau_m = +\infty$, then we have a global solution and if $\tau_m < +\infty$, then $\|\phi(\tau)\|_{H^k(\mathbb{R})} \to \infty$ as $\tau \to \tau_m$.

Finally, we will show that if $\tau_m < +\infty$, then $\limsup \|\phi(\tau)\|_{L^\infty(\mathbb{R})} \to \infty$ as $\tau \to \tau_m$. Suppose that $\limsup \|\phi(\tau)\|_{L^\infty(\mathbb{R})} < \infty$ as $\tau \to \tau_m$. Let us write $[0, \tau_m) = [0, \tau_m - \epsilon] \cup (\tau_m - \epsilon, \tau_m)$ with $\epsilon > 0$. Since $\phi \in C\left([0, \tau_m), H^k(\mathbb{R})\right)$ and $H^k(\mathbb{R})$ is embedded to $L^\infty(\mathbb{R})$ for $k \geq 1$, then

$$\sup_{\tau \in [0, \tau_m - \epsilon]} \|\phi(\tau)\|_{L^\infty(\mathbb{R})} \leq \sup_{\tau \in [0, \tau_m - \epsilon]} \|\phi(\tau)\|_{H^k(\mathbb{R})} < \infty .$$

Because $\limsup \|\phi(\tau)\|_{L^\infty(\mathbb{R})} < \infty$ as $\tau \to \tau_m$ and ϵ is arbitrary positive number, then there exists a positive number M such that

$$\sup_{\tau \in [0, \tau_m)} \|\phi(\tau)\|_{L^\infty(\mathbb{R})} \leq M < \infty . \tag{31}$$

Using Duhamel's formula and using the property of the nonlinear term as in the local existence proof, we get

$$\|\phi(\tau)\|_{H^k(\mathbb{R})} \leq \|\varphi\|_{H^k(\mathbb{R})} + C(M) \int_0^\tau \|\phi(\tau')\|_{H^k(\mathbb{R})}\, d\tau' . \tag{32}$$

Applying Gronwall's inequality, then for $\tau \in [0, \tau_m)$, we have

$$\|\phi(\tau)\|_{H^k(\mathbb{R})} \leq \|\varphi\|_{H^k(\mathbb{R})} e^{C(M)\tau_m} , \tag{33}$$

which contradicts with the blow up of $\|\phi(\tau)\|_{H^k(\mathbb{R})}$ at $\tau \to \tau_m$. Hence, $\limsup \|\phi_\tau\|_{L^\infty(\mathbb{R})} \to \infty$ as $\tau \to \tau_m$. □

Due to the type of nonlinearity and presence of the damping term, differential Equation (12) does not possess any conserved quantities. However, we can define the energy function associated to Equation (12) as

$$E[\phi](\tau) = \int_{\mathbb{R}} \left\{ |\phi_\xi|^2 - \left(\frac{3\lambda}{2\omega}|R|^2 - \nu \right) |\phi|^2 - \frac{3\lambda}{4\omega} \left[|\phi + \eta|^2 - |\eta|^2 \right]^2 \right\} d\xi . \tag{34}$$

Since $\lambda < 0$, then E is non-negative. It is worth mentioning that if $\alpha = 0$ (no damping term), then this energy function E is conserved. Furthermore, we still can use this energy function to prove the global existence in $H^1(\mathbb{R})$ for small initial energy and prove that the solution in fact possesses more regularity and hence prove the global existence on $H^k(\mathbb{R})$.

Now, we prove the following lemma about energy estimate

Lemma 1. *Let ϕ be a solution of differential Equation (12) with the initial data $\varphi \in H^1(\mathbb{R})$ such that $E_0 = E[\varphi] \leq \delta$ with δ a positive real constant. There exist a positive real constant δ_0 such that for every $0 < \delta < \delta_0$ we have the following estimate,*

$$E[\phi](\tau) \leq K e^{-\alpha\tau} , \tag{35}$$

where K is a real positive constant depending on the initial data.

Proof. First, we multiply Equation (12) with $-\bar{\phi}_\tau - \frac{\alpha}{2}\bar{\phi}$, integrate over the spatial variable ξ and keep the real part. Then, we get

$$\frac{dE[\phi](\tau)}{d\tau} + \alpha E[\phi](\tau) = \frac{3\alpha\lambda}{4\omega} \int_{\mathbb{R}} \left[|\phi + \eta|^2 - |\eta|^2 \right] |\phi|^2\, d\xi. \tag{36}$$

Since $\alpha > 0$, using Cauchy-Schwartz inequality we have

$$\begin{aligned} \tfrac{3\alpha\lambda}{4\omega} \int_{\mathbb{R}} \left[|\phi + \eta|^2 - |\eta|^2 \right] |\phi|^2\, d\xi &\leq \alpha \left[\tfrac{3|\lambda|}{4\omega} \int_{\mathbb{R}} \left[|\phi + \eta|^2 - |\eta|^2 \right]^2 d\xi \right]^{\frac{1}{2}} \left[\tfrac{3|\lambda|}{4\omega} \int_{\mathbb{R}} |\phi|^4\, d\xi \right]^{\frac{1}{2}} \\ &\leq \alpha E^{\frac{1}{2}} \left\| \tfrac{3|\lambda|}{4\omega} \phi \right\|^2_{L^4(\mathbb{R})} . \end{aligned} \tag{37}$$

Using the one dimensional Gagliardo-Nirenberg-Sobolev inequality,

$$\left\|\frac{3|\lambda|}{4\omega}\phi\right\|_{L^4(\mathbb{R})} \leq C\|\phi_\xi\|^{1/4}_{L^2(\mathbb{R})}\left\|\frac{3|\lambda|}{4\omega}\phi\right\|^{3/4}_{L^2(\mathbb{R})} \tag{38}$$

and Young inequality,

$$ab \leq p\, a^{\frac{1}{p}} + (1-p)b^{\frac{1}{1-p}} \tag{39}$$

for $a, b > 0$ and $p \in (0, 1)$, we get

$$\left\|\frac{3|\lambda|}{4\omega}\phi\right\|_{L^4(\mathbb{R})} \leq C\left[\|\phi_\xi\|^2_{L^2(\mathbb{R})} + \frac{3|\lambda|}{2\omega}|R|^2\|\phi\|^2_{L^2(\mathbb{R})}\right]^{1/2} \leq C\, E^{1/2}. \tag{40}$$

Thus, we have

$$\frac{dE(\tau,\phi)}{d\tau} + \alpha E(\tau,\phi) \leq \alpha C E^{3/2}. \tag{41}$$

Integrating this inequality, we get

$$E(\tau) \leq \frac{4E_0}{\left[C\sqrt{E_0}(1-e^{\alpha\tau/2}) + 2\, e^{\alpha\tau/2}\right]^2}. \tag{42}$$

Pick $\delta_0 = 4/C^2$, then we have the desired inequality (35). □

Now, we prove the global existence in the following theorem.

Theorem 2. *Let $k \geq 1$ be an integer. For every $\varphi \in H^k(\mathbb{R})$ such that $E_0 = E[\varphi] \leq \delta$ where δ is a positive real constant. There exists a positive real constant δ_0 such that for every $0 < \delta < \delta_0$, the differential Equation (12) admits a unique global solution $\phi \in C\left([0,+\infty), H^k(\mathbb{R})\right)$.*

Proof. First consider the case of $k = 1$. Pick δ_0 as in the previous theorem, then the global existence directly follows from (35). Thus, we have

$$\phi \in C\left([0,+\infty), H^1(\mathbb{R})\right). \tag{43}$$

Now consider the case of $k > 1$. In Theorem 1, we already construct a unique local maximal solution such that

$$\phi \in C\left([0,\tau^k_m), H^k(\mathbb{R})\right). \tag{44}$$

We need to prove that $\tau^k_m = +\infty$. Consider $\tau_0 < +\infty$, then we have

$$\sup_{\tau\in[0,\tau_0]} \|\phi(\tau)\|_{H^1(\mathbb{R})} < \infty. \tag{45}$$

Since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, then (43) implies

$$\sup_{\tau\in[0,\tau_0]} \|\phi(\tau)\|_{L^\infty(\mathbb{R})} < \infty. \tag{46}$$

Applying the blow up alternative in the local existence theorem, we deduce that $\tau^k_m > \tau_0$. Since τ_0 is arbitrary, then we conclude $\tau^k_m = +\infty$ and the proof is finished. □

4. Main Result

Before we state the main result, we will prove a lemma about the bound of the leading approximation function and the residual function.

Lemma 2. *For every $A(0) = \varphi + \eta$, where $\varphi \in H^k(\mathbb{R})$ with integer $k > 4$ such that $E[\varphi]$ is small, then there exist positive real constants C_X and C_R that depend on $\|A(0)\|_{L^\infty(\mathbb{R})}$ such that*

$$\begin{aligned} \|X_t(t)\|_{L^\infty(\mathbb{R})} + \|X(t)\|_{L^\infty(\mathbb{R})} &\leq \epsilon\, C_X, \\ \|\mathrm{Res}(t)\|_{L^\infty(\mathbb{R})} &\leq \epsilon^4\, C_R\,, \end{aligned} \tag{47}$$

for all $t \in [0, +\infty)$. Furthermore, we also have $X(t) \in C_b^{k-1}(\mathbb{R})$ and $X_t(t) \in C_b^{k-3}(\mathbb{R})$ for all $t \in [0, +\infty)$.

Proof. From Theorem 2, we have $\phi \in C\left([0, +\infty), H^k(\mathbb{R})\right)$ for integer $k \geq 1$. Since $H^k(\mathbb{R})$ is embedded into $L^\infty(\mathbb{R})$ for $k \geq 1$, then using the decomposition (8) and the fact that $A_p(\tau) \in L^\infty(\mathbb{R})$, we get

$$\|A(\tau)\|_{L^\infty(\mathbb{R})} \leq C_A. \tag{48}$$

Since $L^\infty(\mathbb{R})$ is a Banach algebra with respect to pointwise multiplication, then we can estimate Equation (3),

$$\|X(t)\|_{L^\infty(\mathbb{R})} \leq \epsilon\, C_1.$$

Since $k > 4$, $\|\phi_{\xi\xi}\|_{H^{k-2}(\mathbb{R})} \leq C\|\phi\|_{H^k(\mathbb{R})}$ then from Equation (12), we get

$$\|\phi_\tau\|_{H^k(\mathbb{R})} \leq C\,.$$

Thus, using decomposition (8), we have an estimate for the first derivative of A with respect to τ and conclude that

$$\|X_t(t)\|_{L^\infty(\mathbb{R})} \leq \epsilon\, C_2\,,$$

hence proved the first inequality.

The residual terms consist of powers and higher derivatives of A up to second order (both time and space). Since $k > 4$, then using Sobolev embedding and Equation (12), we get the bound for second derivative of ϕ with respect to both space and time. Then, using (8), we proved the second inequality.

For the second part of the theorem, note that $A_p(\tau, \xi)$ is smooth and bounded function. Since $\phi(\tau) \in H^k(\mathbb{R})$ and $\phi_\tau(\tau) \in H^{k-2}(\mathbb{R})$ and $k > 4$, then by Sobolev embedding we have

$$\|\phi(\tau)\|_{C_b^m(\mathbb{R})} \leq C\|f\|_{H^k(\mathbb{R})}\,,$$

for $k > m + 1$, hence we proved the theorem. □

We define the error term by writing $\epsilon^2 y(t,x) = u(t,x) - X(t,x)$, with $X(t,x)$ the leading approximation term (3) and $u(t,x)$ the exact solution of our ϕ^4 equation. The evolution equation for the error term is given by

$$\begin{cases} y_{tt} + \epsilon^2 \alpha y_t - \beta y_{xx} + \gamma y - \lambda(\epsilon^4 y^3 + 3X^2 y + 3\epsilon^2 X y^2) + \epsilon^{-2}\mathrm{Res}(t,x) = 0, \\ y(0,\cdot) = f, \\ y_t(0,\cdot) = \epsilon g. \end{cases} \tag{49}$$

Since we expect ϵ is small, then we assume that $\gamma > \frac{\epsilon^2\alpha}{2}$. We prove that the error function $y(t)$ remains bounded over time. First, we can convert the differential Equation (49) into an integral equation [30]

$$y(t,x) = \Phi[y](t,x) = A[f](t,x) + B[\epsilon g](t,x) + M\left[\lambda(y^3 + 3X^2y + 3Xy^2) + \text{Res}\right], \tag{50}$$

where A, B and M are integral operators defined as

$$\begin{aligned} A[f](t,x) &= \frac{1}{2}e^{-\hat{\alpha}t}\left[f(x+t) + f(x-t) + \hat{\alpha}\int_{x-t}^{x+t} f(z)J_0(\epsilon w\zeta_0)\,dz + \int_{x-t}^{x+t} f(z)\frac{\partial J_0(\epsilon w\zeta_0)}{\partial t}\,dz\right], \\ B[\epsilon g](t,x) &= \frac{\epsilon}{2}e^{-\hat{\alpha}t}\int_{x-t}^{x+t} g(z)J_0(\epsilon w\zeta_0)\,dz, \\ M[h](t,x) &= -\frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s} e^{-\hat{\alpha}(t-s)}h(z,s)J_0(\epsilon w\zeta)\,dz\,ds, \end{aligned}$$

where $\hat{\alpha} = \epsilon^2\alpha/2$, J_0 is a zeroth order Bessel function, and

$$\begin{aligned} \epsilon w &= \sqrt{\gamma^2 - \hat{\alpha}^2}, \\ \zeta^2 &= (t-s)^2 - (x-z)^2, \\ \zeta_0^2 &= t^2 - (x-z)^2. \end{aligned}$$

Since the intervals of integration are $s \in [0,t]$ and $z \in [x-t+s, x+t-s]$, then $\zeta^2 \geq 0$. Thus we can set $\zeta \geq 0$.

Note that

$$|J_n(z)| \leq \frac{1}{\Gamma(1+n)}\left(\frac{|z|}{2}\right)^n e^{\text{Im}(z)}\,. \tag{51}$$

Since $w\zeta \in \mathbb{R}$, then we have,

$$\begin{aligned} |J_0(\epsilon w\zeta)| &\leq 1, \\ \left|\frac{\partial J_0}{\partial t}(\epsilon w\zeta)\right| = \left|-\frac{\epsilon w}{\zeta}(t-s)J_1(\epsilon w\zeta)\right| &\leq \frac{\epsilon^2 w^2}{2}|t-s|\,. \end{aligned} \tag{52}$$

Using these two estimates, now we can estimate the integral operators

$$\begin{aligned} |A[f](t,x)| &\leq e^{-\hat{\alpha}t}\|f\|_{L^\infty(\mathbb{R})}\left(1 + \hat{\alpha}t + \frac{\epsilon^2w^2t^2}{2}\right), \\ |B[\epsilon g](t,x)| &\leq \epsilon e^{-\hat{\alpha}t}t\|g\|_{L^\infty(\mathbb{R})}, \\ |M[h](t,x)| &\leq \int_0^t e^{-\hat{\alpha}(t-s)}(t-s)\|h(s)\|_{L^\infty(\mathbb{R})}\,ds. \end{aligned} \tag{53}$$

We assume that $\|f\|_{L^\infty(\mathbb{R})}, \|g\|_{L^\infty(\mathbb{R})} \leq C_0$. The Banach algebra property of $L^\infty(\mathbb{R})$ enables us to bound the nonlinear term for each $D > 0$ and for all $\|y\|_{L^\infty(\mathbb{R})} \leq D$ and then we have

$$\begin{aligned} \|y(t)\|_{L^\infty(\mathbb{R})} \leq\; & \left[\left(1 + \hat{\alpha}t + \epsilon t + \frac{\epsilon^2w^2t^2}{2}\right)C_0 + \frac{\epsilon^2t^2}{2}C_R + \frac{|\lambda|\epsilon^2t^2}{2}\left(\epsilon^4D^3 + \epsilon^3C_XD^2\right)\right] \\ & + |\lambda|\epsilon^2C_X^2\int_0^t (t-s)\|y(s)\|_{L^\infty(\mathbb{R})}\,ds\,, \end{aligned} \tag{54}$$

where we already used the fact that $e^{-\hat{\alpha}t} \leq 1$ for $t \geq 0$. If $\epsilon > 0$ is sufficiently small, that is, $\epsilon \in (0, \epsilon_0)$ with ϵ_0 a positive real constant, then for each $D > 0$ and for every $\|y(t)\|_{L^\infty(\mathbb{R})} \leq D$, we can find a positive real constant M independent of ϵ such that

$$\frac{1}{2}\left(|\lambda|\epsilon^4 D^3 + |\lambda|\epsilon^3 C_X D^2\right) < M\,.$$

Thus, as long as we have $\|y(t)\|_{L^\infty(\mathbb{R})}$ staying in the ball of radius D, we have

$$\|y(t)\|_{L^\infty(\mathbb{R})} \leq a(t) + \int_0^t b(s)\|y(s)\|_{L^\infty(\mathbb{R})}\, ds\,,$$

where

$$a(t) = \left[\left(1 + \hat{\alpha}t + \epsilon t + \frac{\epsilon^2 w^2 t^2}{2}\right) C_0 + \frac{\epsilon^2 t^2}{2} C_R + \epsilon^2 M t^2\right], \qquad b(s) = |\lambda|\epsilon^2 C_X^2 (t - s)\,. \tag{55}$$

The function $a(t)$ is continuous and non-decreasing and $b(t)$ is positive for $t \in [0, T_0/\epsilon]$. Then applying Gronwall's inequality we get

$$\|y(t)\|_{L^\infty(\mathbb{R})} \leq a(t) e^{|\lambda| C_X^2 \epsilon^2 t^2/2}\,. \tag{56}$$

Therefore,

$$\|y(t)\|_{L^\infty(\mathbb{R})} \leq \left[\left(1 + \frac{\epsilon\alpha T_0}{2} + T_0 + \frac{w^2 T_0^2}{2}\right) C_0 + \frac{T_0^2}{2} C_R + M T_0^2\right] e^{|\lambda| C_X^2 T_0^2/2}\,. \tag{57}$$

Let $C_y = \left(1 + T_0 + \frac{w^2 T_0^2}{2}\right) C_0 + C M T_0^2$ and $D = C_y e^{|\lambda| C_X^2 T_0^2/2}$ and make ϵ_0 smaller such that $\epsilon < \frac{2M}{\alpha} T_0$. Then we have

$$\|y(t)\|_{L^\infty(\mathbb{R})} \leq D\,, \tag{58}$$

for $t \in [0, T_0/\epsilon]$. Hence, we proved the following theorem

Theorem 3. *Let $A(\tau, \xi)$ be the solution of Equation (1) such that $A \in C^2\left([0, T_1], C_b^k(\mathbb{R})\right)$ for integer $k \geq 0$ and X be the leading approximation function (3). Let $u(t, x)$ be a solution of Equation (2). Then for each $T_0 < T_1$ and each $C_0 > 0$, there exist ϵ_0 and $D > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ with*

$$\|u(0, \cdot) - X(0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \epsilon^2 C_0, \qquad \|u_t(0, \cdot) - X_t(0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \epsilon^3 C_0\,,$$

the following inequality

$$\|u(t, \cdot) - X(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \epsilon^2 D$$

holds for $t \in [0, T_0/\epsilon]$.

5. Conclusions

We have considered the Lugiato-Lefever equation and how it can be derived from a damped driven ϕ^4 equation. In particular, we provided a justification of the former equation as a slowly modulated approximation of the latter. Compared to previous works, the main obstacle in here is the driving term that is not square integrable, which yields non-integrable solutions. We introduce an idea to handle the non-integrability by decomposing the solutions into integrable and non-integrable parts. This is the main novelty of the present paper.

Even though the parameters in (1) were considered to be real, our method can also be applied when they are complex (including complex-valued dispersion coefficient), known as the complex Ginzburg-Landau equation with external drive (see, e.g., References [31,32]) . Such equations generically describe the dynamics of oscillating, spatially extended systems near the threshold of long-wavelength supercritical oscillatory instability, that is, Hopf bifurcation [33,34]. We can also extend the problem and apply the method to counter propagating waves of (2) from which one will obtain coupled Lugiato-Lefever equations [32]. In that case, we have an analogue of the coupled Lugiato-Lefever equations that describe, for example, two orthogonal polarization modes in optical resonators [35,36].

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