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Nontrivial Solutions for a System of Fractional q-Difference Equations Involving q-Integral Boundary Conditions

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Abstract: In this paper, we study the existence of nontrivial solutions for a system of fractional q-difference equations involving q-integral boundary conditions, and we use the topological degree to establish our main results by considering the first eigenvalue of some associated linear integral operators.

Keywords: fractional q-difference equations; q-integral boundary conditions; topological degree; nontrivial solutions

1. Introduction

The initial work of q-difference calculus can be dated back to Jackson [1,2] and for results on fractional q-difference calculus or quantum calculus we refer the reader to [3–28] and the references therein. For example, in [3,4] the author studied some basic properties of fractional q-fractional integral and differential operators, and used fixed point theorems in cones to investigate the existence of nontrivial solutions for the fractional q-difference equation

$$\left(D_q^\alpha y\right)(x) = -f(x, y(x)),\ 0 < x < 1, \tag{1}$$

with the boundary conditions

$$y(0) = y(1) = 0, \text{ as } \alpha \in (1,2], \tag{2}$$

or

$$y(0) = \left(D_q y\right)(0) = 0,\ \left(D_q y\right)(1) = \beta \geq 0, \text{ as } \alpha \in (2,3]. \tag{3}$$

In [5] the author considered the three-point boundary value problem of fractional q-difference equations

$$\begin{cases} D_q^\alpha + f(t, x(t), x(t)) + g(t, x(t)) = 0, 0 < t < 1, \\ x(0) = D_q x(0) = 0, D_q x(1) = \beta D_q x(\eta), \end{cases} \tag{4}$$

where $\beta\eta^{\alpha-2} \in (0,1)$, $q \in (0,1)$, $\alpha \in (2,3)$ and based on fixed point theorems on mixed monotone operators, some sufficient conditions are used to guarantee the existence and uniqueness of positive solutions for the above problem. In [6] the authors discussed the following nonhomogeneous boundary value problem with fractional q-derivatives

$$\begin{cases} \left(D_q^\alpha u\right)(t) + f(t, u(t)) = 0, t \in (0,1), \\ u(0) = \left(D_q u\right)(0) = 0, \gamma\left(D_q u\right)(1) + \beta\left(D_q^2 u\right)(1) = \lambda, \end{cases} \tag{5}$$

where $q \in (0,1), 2 < \alpha \le 3, \gamma \ge 0, \beta > 0$, and λ is a parameter. Using the generalized Banach contraction principle and Krasnoselskii's fixed point theorem, uniqueness, existence, and multiplicity of positive solutions for the above problem were obtained in terms of explicit intervals for the nonhomogeneous term.

Coupled systems of fractional q-difference equations were investigated in [23–28] (also see [29–35]). In [23] the authors studied the following system of fractional q-difference equations with four-point boundary conditions

$$\begin{cases} D_q^\alpha u(t) + f(t, v(t)) = 0,\ 0 < t < 1, \\ D_q^\beta v(t) + g(t, u(t)) = 0,\ 0 < t < 1, \\ u(0) = 0, u(1) = \gamma_1 u(\eta_1), \\ v(0) = 0, v(1) = \gamma_2 u(\eta_2), \end{cases} \tag{6}$$

where $0 < q < 1$, $1 < \beta \le \alpha \le 2$, $0 < \eta_1,\ \eta_2 < 1$, $0 < \gamma_1 \eta_1^{\alpha-1} < 1$, $0 < \gamma_2 \eta_2^{\beta-1} < 1$ and using the monotone iterative approach they constructed two convergent monotone iterative schemes and obtained two positive solutions for the above problem. In [24] the authors studied the coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions

$$\begin{cases} D_q^\alpha x(t) = f\left(t, x(t), I_r^\delta y(t)\right), t \in [0,T], 1 < \alpha \le 2, \\ D_p^\beta y(t) = g\left(t, y(t), I_z^\varepsilon x(t)\right), t \in [0,T], 1 < \beta \le 2, \\ x(0) = 0, \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, \lambda_2 I_h^\mu y(\theta) = I_k^v x(\tau), \end{cases} \tag{7}$$

where $0 < p, q, r, z, m, n, h, k < 1$, $\eta, \xi, \theta, \tau \in (0,T)$, $\delta, \varepsilon, \gamma, \kappa, \mu, v > 0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, I_ϕ^ψ is the fractional ϕ-integral of order ψ with $\phi \in \{r, z, m, n, h, k\}$ and $\psi \in \{\delta, \varepsilon, \gamma, \kappa, \mu, v\}$. Using the Banach contraction principle and the Leray-Schauder alternative, they obtained existence and uniqueness of solutions under some appropriate conditions on f, g.

Motivated by the mentioned works above, in this paper we use topological degree theory to study nontrivial solutions for the following system of fractional q-difference equations with q-integral boundary conditions:

$$\begin{cases} D_q^\alpha x(t) + f_1(t, y(t)) = 0, t \in (0,1), \\ D_q^\alpha y(t) + f_2(t, x(t)) = 0, t \in (0,1), \\ x(0) = 0, D_q x(0) = 0, D_q^v x(1) = \displaystyle\int_0^1 h(t) D_q^v x(t) d_q t, \\ y(0) = 0, D_q y(0) = 0, D_q^v y(1) = \displaystyle\int_0^1 h(t) D_q^v y(t) d_q t, \end{cases} \tag{8}$$

where $\alpha \in (2,3), v \in (1,2)$, D_q^α is the α-order Riemann-Liouville's fractional q-derivative.

Now, we list our assumptions for $h, f_i (i = 1,2)$:

Hypothesis 1 (H1). *$h \ge 0$ and $1 - \int_0^1 h(t) t^{\alpha - v - 1} d_q t := A > 0$.*

Hypothesis 2 (H2). $f_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Hypothesis 3 (H3). *There exist* $b_i(t), c_i(t) \in C([0,1], \mathbb{R}^+)$ *with* $c_i(t) \not\equiv 0$ *and* $K_1(y), K_2(x) \in C[\mathbb{R}, \mathbb{R}^+]$ *such that*

$$f_1(t,y) \geq -b_1(t) - c_1(t)K_1(y),\ f_2(t,x) \geq -b_2(t) - c_2(t)K_2(x),\ \forall x, y \in \mathbb{R}, t \in [0,1], i = 1,2.$$

Hypothesis 4 (H4). $\lim_{|y| \to +\infty} \frac{K_1(y)}{|y|} = 0, \lim_{|x| \to +\infty} \frac{K_2(x)}{|x|} = 0.$

Hypothesis 5 (H5). $\liminf_{|y| \to +\infty} \frac{f_1(t,y)}{|y|} > \lambda_1, \liminf_{|x| \to +\infty} \frac{f_2(t,x)}{|x|} > \lambda_1$, *uniformly for* $t \in [0,1]$.

Hypothesis 6 (H6). $\limsup_{|y| \to 0} \frac{|f_1(t,y)|}{|y|} < \lambda_1, \limsup_{|x| \to 0} \frac{|f_2(t,x)|}{|x|} < \lambda_1$, *uniformly for* $t \in [0,1]$, *where* λ_1 *is the first eigenvalue of the following eigenvalue problem*

$$\begin{cases} D_q^\alpha x(t) + \lambda x(t) = 0, t \in (0,1), \\ x(0) = 0, D_q x(0) = 0, D_q^v x(1) = \int_0^1 h(t) D_q^v x(t) d_q t, \end{cases} \tag{9}$$

where λ *is a parameter, and* α, v, h *are as in* (8).

Finally, we state our main result in this paper:

Theorem 1. *Suppose that (H1)–(H6) hold. Then* (8) *has at least one nontrivial solution.*

Remark 1. *In (H1), the function* h *is a non-negative function on* $[0,1]$ *(it also can be a zero function). If* $h \not\equiv 0$ *on* $[0,1]$, *we demand that* $\int_0^1 h(t) t^{\alpha - v - 1} d_q t \in [0,1)$.

2. Preliminaries

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^0 = 1, \quad (a-b)^n = \prod_{k=0}^{n-1} \left(a - bq^k\right), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Please note that if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \backslash \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

Definition 1 (see [3], Definition 2.2). *Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $\left(I_q^0 f\right)(x) = f(x)$ and*

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0,1].$$

The fractional q-derivative of order $\alpha \geq 0$ is defined by $\left(D_q^0 f\right)(x) = f(x)$ and $\left(D_q^\alpha f\right)(x) = \left(D_q^m I_q^{m-\alpha} f\right)(x)$ for $\alpha > 0$, where m is the smallest integer greater or equal than α.

Lemma 1 (see [3], Lemma 2.3). *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:*
(i) $\left(I_q^\beta I_q^\alpha f\right)(x) = \left(I_q^{\alpha+\beta} f\right)(x)$;
(ii) $\left(D_q^\alpha I_q^\alpha f\right)(x) = f(x)$.

Lemma 2 (see [3], Theorem 2.4). *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^k f\right)(0).$$

Lemma 3. *Suppose that (H1) holds, and $\alpha \in (2,3), v \in (1,2)$. If $g \in C[0,1]$, then the following boundary value problem*

$$\begin{cases} D_q^\alpha x(t) + g(t) = 0, t \in (0,1), \\ x(0) = 0, D_q x(0) = 0, D_q^v x(1) = \int_0^1 h(t) D_q^v x(t) d_q t, \end{cases} \tag{10}$$

has a unique solution

$$x(t) = \int_0^1 G(t, qs) g(s) d_q s,$$

where

$$G(t,qs) = G_0(t,qs) + \frac{t^{\alpha-1}}{A} \int_0^1 h(t) G_1(t,qs) d_q t,$$

$$G_0(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} t^{\alpha-1}(1-qs)^{(\alpha-v-1)} - (t-qs)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-qs)^{(\alpha-v-1)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_1(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} t^{\alpha-v-1}(1-qs)^{(\alpha-v-1)} - (t-qs)^{(\alpha-v-1)}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-v-1}(1-qs)^{(\alpha-v-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. Using Definition 2 and Lemmas 2 and 3 we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_q^\alpha g(t), c_i \in \mathbb{R}, i = 1,2,3.$$

Note from $x(0) = D_q x(0) = 0$ we have $c_2 = c_3 = 0$. Hence,

$$x(t) = c_1 t^{\alpha-1} - I_q^\alpha g(t),$$

and

$$D_q^v x(t) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-v)} c_1 t^{\alpha-v-1} - \frac{1}{\Gamma_q(\alpha-v)} \int_0^t (t-qs)^{(\alpha-v-1)} g(s) d_q s.$$

Consequently, we have

$$\begin{aligned} D_q^v x(1) &= \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-v)}c_1 - \frac{1}{\Gamma_q(\alpha-v)}\int_0^1 (1-qs)^{(\alpha-v-1)}g(s)d_qs \\ &= \int_0^1 h(t)D_q^v x(t)d_qt \\ &= \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-v)}c_1\int_0^1 h(t)t^{\alpha-v-1}d_qt - \frac{1}{\Gamma_q(\alpha-v)}\int_0^1 h(t)\int_0^t (t-qs)^{(\alpha-v-1)}g(s)d_qsd_qt. \end{aligned}$$

This, together with (H1), implies that

$$c_1 = \frac{1}{A\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-v-1)}g(s)d_qs - \frac{1}{A\Gamma_q(\alpha)}\int_0^1 h(t)\int_0^t (t-qs)^{(\alpha-v-1)}g(s)d_qsd_qt.$$

Thus, we have

$$\begin{aligned} x(t) &= \frac{1}{A\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-v-1)}g(s)d_qs - \frac{t^{\alpha-1}}{A\Gamma_q(\alpha)}\int_0^1 h(t)\int_0^t (t-qs)^{(\alpha-v-1)}g(s)d_qsd_qt \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}g(s)d_qs \\ &= \frac{1}{A\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-v-1)}g(s)d_qs - \frac{t^{\alpha-1}}{A\Gamma_q(\alpha)}\int_0^1 h(t)\int_0^t (t-qs)^{(\alpha-v-1)}g(s)d_qsd_qt \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}g(s)d_qs + \frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-v-1)}g(s)d_qs \\ &\quad - \frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-v-1)}g(s)d_qs \\ &= \int_0^1 G_0(t,qs)g(s)d_qs + \frac{t^{\alpha-1}}{A\Gamma_q(\alpha)}\int_0^1 h(t)t^{\alpha-v-1}d_qt\int_0^1 (1-qs)^{(\alpha-v-1)}g(s)d_qs \\ &\quad - \frac{t^{\alpha-1}}{A\Gamma_q(\alpha)}\int_0^1 h(t)\int_0^t (t-qs)^{(\alpha-v-1)}g(s)d_qsd_qt \\ &= \int_0^1 G_0(t,qs)g(s)d_qs + \frac{t^{\alpha-1}}{A\Gamma_q(\alpha)}\int_0^1 \left[\int_0^1 h(t)t^{\alpha-v-1}(1-qs)^{(\alpha-v-1)}d_qt - \int_s^1 h(t)(t-qs)^{(\alpha-v-1)}d_qt\right]g(s)d_qs \\ &= \int_0^1 G_0(t,qs)g(s)d_qs + \frac{t^{\alpha-1}}{A}\int_0^1\int_0^1 h(t)G_1(t,qs)d_qtg(s)d_qs \\ &= \int_0^1 G(t,qs)g(s)d_qs. \end{aligned}$$

This completes the proof. □

Lemma 4 (see ([7], Lemma 2.2), ([4], Lemma 3.0.7), ([28], Lemma 2.7)). *The functions $G_i(i=0,1)$ has the following properties*

(i) $G_i(t,qs) \geq 0$ for $t,s \in [0,1]$,
(ii) $t^{\alpha-1}G_0(1,qs) \leq G_0(t,qs) \leq G_0(1,qs)$ for $t,s \in [0,1]$.

Lemma 5. *The function G satisfies*

$$t^{\alpha-1}\varphi_1(qs) \leq G(t,qs) \leq \varphi_1(qs), \text{ for } t,s \in [0,1], \tag{11}$$

and

$$G(t,qs) \leq t^{\alpha-1}\varphi_2(qs), \text{ for } t,s \in [0,1], \tag{12}$$

where

$$\varphi_1(s) = G_0(1,s) + \frac{1}{A}\int_0^1 h(t)G_1(t,s)d_qt,\ s \in [0,1],$$

and

$$\varphi_2(s) = \frac{1}{\Gamma_q(\alpha)}(1-s)^{(\alpha-v-1)} + \frac{1}{A}\int_0^1 h(t)G_1(t,s)d_qt,\ s \in [0,1].$$

This is the direct result from Lemma 4, so we omit the proof.

Let $E := C[0,1], \|x\| := \max_{t\in[0,1]} |x(t)|$ *and* $P := \{x \in E : x(t) \geq 0, \forall t \in [0,1]\}$*. Then* $(E, \|\cdot\|)$ *is a real Banach space and P is a cone on E. Moreover,* $E^2 = E \times E$ *is a Banach space with the norm* $\|(u,v)\| = \|u\| + \|v\|$*, and* $P^2 = P \times P$ *is a cone on* E^2*. From Lemma 3 we can define operators* $T_i(i = 1,2) : E \to E$*, and* $T : E^2 \to E^2$ *as follows:*

$$(T_1y)(t) := \int_0^1 G(t,qs)f_1(s,y(s))d_qs,$$

$$(T_2x)(t) := \int_0^1 G(t,qs)f_2(s,x(s))d_qs,$$

and

$$T(x,y)(t) = ((T_1y),(T_2x))(t), t \in [0,1], x,y \in E,$$

where G is determined in Lemma 3. Please note that $T_i(i = 1,2)$ *and T are completely continuous operators, and* (x,y) *solves* (8) *if and only if* (x,y) *is a fixed point of the operator T.*

In addition, from Lemma 3 we can obtain that (9) *is equivalent to*

$$x(t) = \lambda \int_0^1 G(t,qs)x(s)d_qs, t \in [0,1].$$

For our purposes, we need to define the operator L by

$$(Lx)(t) = \int_0^1 G(t,qs)x(s)d_qs, t \in [0,1], x \in E.$$

It is not difficult to prove that $L : E \to E$ *is a linear completely continuous and* $T(P) \subset P$*. From Lemmas 2 and 3 in [7] we obtain that the spectral radius, denoted by* $r(L)$*, is not equal to* 0*, and L has a positive eigenfunction* φ^* *corresponding to its first eigenvalue* $\lambda_1 = (r(L))^{-1}$*, i.e.,* $\varphi^* = \lambda_1 L\varphi^*$*.*

Lemma 6. *Let* $P_0 = \{x \in P : x(t) \geq t^{\alpha-1}\|x\|, \forall t \in [0,1]\}$*. Then* $L(P) \subset P_0$*.*

Proof. If $x \in P$, and from (11) we have

$$t^{\alpha-1}\int_0^1 \varphi_1(qs)x(s)d_qs \leq \int_0^1 G(t,qs)x(s)d_qs \leq \int_0^1 \varphi_1(qs)x(s)d_qs, \text{ for } t \in [0,1].$$

Therefore, we have

$$(Lx)(t) \geq t^{\alpha-1}\int_0^1 \varphi_1(qs)x(s)d_qs \geq t^{\alpha-1}\|Lx\|, \text{ for } t \in [0,1].$$

This completes the proof. □

Remark 2. *From Lemma 6 we have* $\varphi^* \in P_0$*.*

We recall the following topological degree theorems, which will play important roles in proving our main results.

Lemma 7 (see ([36], Theorem A.3.3))**.** *[Let Ω be a bounded open set in a Banach space E, and $T : \Omega \to E$ a continuous compact operator. If there exists $x_0 \in E \backslash \{0\}$ such that*

$$x - Tx \neq \mu x_0,\ \forall x \in \partial\Omega,\ \mu \geq 0,$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 8 (see ([36], Lemma 2.5.1))**.** *Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$, and $T : \Omega \to E$ a continuous compact operator. If*

$$Tx \neq \mu x,\ \forall x \in \partial\Omega,\ \mu \geq 1,$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

3. Proof of Theorem 1

In this section, we present the detailed proof of Theorem 1. From (H6) there exist $\varepsilon_0 \in (0, \lambda_1)$ and $r_1 > 0$ such that

$$|f_1(t, y)| \leq (\lambda_1 - \varepsilon_0)|y|,\ |f_2(t, x)| \leq (\lambda_1 - \varepsilon_0)|x|, \forall t \in [0, 1], x, y \in \mathbb{R} \text{ with } |x|, |y| \leq r_1. \tag{13}$$

This implies that

$$|(T_1 y)(t)| \leq \int_0^1 G(t, qs)|f_1(s, y(s))|d_q s \leq (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs)|y(s)|d_q s,$$

and

$$|(T_2 x)(t)| \leq \int_0^1 G(t, qs)|f_2(s, x(s))|d_q s \leq (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs)|x(s)|d_q s.$$

Now we prove that

$$(x, y) \neq \mu T(x, y) \text{ for all } x, y \in \partial B_{r_1} \text{ and } \mu \in [0, 1]. \tag{14}$$

We argue by contradiction. Suppose there exist $x, y \in \partial B_{r_1}$ and $\mu \in [0, 1]$ such that

$$(x, y) = \mu T(x, y).$$

Therefore,

$$x = \mu T_1 y, \text{ and } y = \mu T_2 x.$$

This implies that

$$|x(t)| = \mu|(T_1 y)(t)| \leq (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs)|y(s)|d_q s, t \in [0, 1],$$

and

$$|y(t)| = \mu|(T_2 x)(t)| \leq (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs)|x(s)|d_q s, t \in [0, 1].$$

Consequently, we have

$$|x(t)| + |y(t)| \leq (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs)(|x(s)| + |y(s)|)d_q s, t \in [0, 1].$$

Let $z(t) = |x(t)| + |y(t)|$.

Then $z \in P$ and

$$z(t) \le (\lambda_1 - \varepsilon_0) \int_0^1 G(t, qs) z(s) d_q s = (\lambda_1 - \varepsilon_0)(Lz)(t), t \in [0,1].$$

The nth iteration of this inequality shows that

$z(t) \le (\lambda_1 - \varepsilon_0)^n (L^n z)(t)(n = 1, 2, \ldots)$, and then $\|z\| \le (\lambda_1 - \varepsilon_0)^n \|L^n\| \cdot \|z\|$, i.e., $1 \le (\lambda_1 - \varepsilon_0)^n \|L^n\|$.

This yields

$$1 \le (\lambda_1 - \varepsilon_0) \lim_{n\to\infty} \sqrt[n]{\|L^n\|} = (\lambda_1 - \varepsilon_0) r(L) = \frac{\lambda_1 - \varepsilon_0}{\lambda_1} < 1,$$

which is a contradiction. Hence, (14) holds. It follows from Lemma 8 that

$$\deg(I - T, B_{r_1}, 0) = 1. \tag{15}$$

On the other hand, from (H5) there exist $\varepsilon_1 > 0$ and $r_2 > 0$ such that

$$f_1(t, y) \ge (\lambda_1 + \varepsilon_1)|y|,\ f_2(t, x) \ge (\lambda_1 + \varepsilon_1)|x|,\ \forall t \in [0,1],\ |x|, |y| > r_2.$$

Let $M_1 = \max_{t\in[0,1],|y|\le r_2}[|f_1(t, y)| + (\lambda_1 + \varepsilon_1)|y|], M_2 = \max_{t\in[0,1],|x|\le r_2}[|f_2(t, x)| + (\lambda_1 + \varepsilon_1)|x|]$. Then

$$f_1(t, y) \ge (\lambda_1 + \varepsilon_1)|y| - M_1,\ f_2(t, x) \ge (\lambda_1 + \varepsilon_1)|x| - M_2,\ \forall t \in [0,1],\ x, y \in R. \tag{16}$$

For any given $\varepsilon, \widetilde{\varepsilon}$ with $\varepsilon_1 - \|c_1\|\varepsilon > 0, \varepsilon_1 - \|c_2\|\widetilde{\varepsilon} > 0$, and from (H4) there exists $r_3 > r_2$ such that

$$K_1(y) \le \varepsilon|y|,\ K_2(x) \le \widetilde{\varepsilon}|x|,\ \forall |x|, |y| > r_3.$$

Let $K_1^* = \max_{|y|\le r_3} K_1(y)$, and $K_2^* = \max_{|x|\le r_3} K_2(x)$. Then we have

$$K_1(y) \le \varepsilon|y| + K_1^*,\ K_2(x) \le \widetilde{\varepsilon}|x| + K_2^*,\ \forall x, y \in \mathbb{R}. \tag{17}$$

Please note that $\varepsilon, \widetilde{\varepsilon}$ can be chosen arbitrarily small, so we can let $R_1 > \max\{r_1, N_1, N_2, N_3, N_4\}$, where r_1 is defined by (14), and

$$\begin{aligned}
N_1 &= \frac{(2\|b_1\| + 2\|c_1\|K_1^* + M_1) \int_0^1 \varphi_1(qs) d_q s}{\frac{1}{2} - \varepsilon\|c_1\| \int_0^1 \varphi_1(qs) d_q s}, \\
N_2 &= \frac{(2\|b_2\| + 2\|c_2\|K_2^* + M_2) \int_0^1 \varphi_1(qs) d_q s}{\frac{1}{2} - \widetilde{\varepsilon}\|c_2\| \int_0^1 \varphi_1(qs) d_q s}, \\
N_3 &= \frac{N_5(2\|b_1\| + 2\|b_2\| + 2\|c_1\|K_1^* + 2\|c_2\|K_2^* + M_1 + M_2)}{(\varepsilon_1 - \|c_1\|\varepsilon) - N_5(\|c_1\|\varepsilon + \|c_2\|\widetilde{\varepsilon})}, \\
N_4 &= \frac{N_6(2\|b_1\| + 2\|b_2\| + 2\|c_1\|K_1^* + 2\|c_2\|K_2^* + M_1 + M_2)}{(\varepsilon_1 - \|c_2\|\widetilde{\varepsilon}) - N_6(\|c_1\|\varepsilon + \|c_2\|\widetilde{\varepsilon})}, \\
N_5 &= (\varepsilon_1 - \|c_1\|\varepsilon) \int_0^1 \varphi_1(qs) d_q s + (\lambda_1 + \varepsilon_1 - \|c_1\|\varepsilon) \int_0^1 \varphi_2(qs) d_q s, \\
N_6 &= (\varepsilon_1 - \|c_2\|\widetilde{\varepsilon}) \int_0^1 \varphi_1(qs) d_q s + (\lambda_1 + \varepsilon_1 - \|c_2\|\widetilde{\varepsilon}) \int_0^1 \varphi_2(qs) d_q s.
\end{aligned}$$

Now we prove that

$$(x, y) - T(x, y) \ne \mu(\varphi^*, \varphi^*), \forall x, y \in \partial B_{R_1}, \mu \ge 0, \tag{18}$$

where φ^* is the positive eigenfunction of L corresponding to the eigenvalue λ_1. We argue by contradiction. Suppose there exist $x, y \in \partial B_{R_1}$ and $\mu \geq 0$ such that

$$(x,y) - T(x,y) = \mu(\varphi^*, \varphi^*),$$

and thus

$$x = T_1 y + \mu\varphi^*, \ y = T_2 x + \mu\varphi^*. \tag{19}$$

Let

$$\widetilde{x}(t) = \int_0^1 G(t,qs)[2b_2(s) + c_2(s)K_2(x(s)) + M_2 + \|c_2\|K_2^*]d_qs, \tag{20}$$

and

$$\widetilde{y}(t) = \int_0^1 G(t,qs)[2b_1(s) + c_1(s)K_1(y(s)) + M_1 + \|c_1\|K_1^*]d_qs. \tag{21}$$

Now we estimate the norms $\|\widetilde{x}\|$ and $\|\widetilde{y}\|$. Please note that $\|x\| = \|y\| = R_1$, and from (17) we have

$$\begin{aligned}\|\widetilde{y}\| &\leq \int_0^1 \varphi_1(qs)[2\|b_1\| + \|c_1\|(\varepsilon\|y\| + K_1^*) + M_1 + \|c_1\|K_1^*]d_qs \\ &= \int_0^1 \varphi_1(qs)d_qs \times (2\|b_1\| + 2\|c_1\|K_1^* + M_1 + \|c_1\|\varepsilon R_1) \\ &< \frac{1}{2}R_1,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\|\widetilde{x}\| &\leq \int_0^1 \varphi_1(qs)[2\|b_2\| + \|c_2\|(\widetilde{\varepsilon}\|x\| + K_2^*) + M_2 + \|c_2\|K_2^*]d_qs \\ &= \int_0^1 \varphi_1(qs)d_qs \times (2\|b_2\| + 2\|c_2\|K_2^* + M_2 + \|c_2\|\widetilde{\varepsilon}R_1) \\ &< \frac{1}{2}R_1.\end{aligned} \tag{23}$$

Furthermore, from (H3) and Lemma 6 we have $\widetilde{x}, \widetilde{y} \in P_0$. Consequently, we obtain

$$\begin{aligned}x(t) + \widetilde{y}(t) &= (T_1 y)(t) + \mu\varphi^*(t) + \widetilde{y}(t) \\ &= \int_0^1 G(t,qs)[f_1(s,y(s)) + 2b_1(s) + c_1(s)K_1(y(s)) + M_1 + \|c_1\|K_1^*]d_qs + \mu\varphi^*(t),\end{aligned} \tag{24}$$

and

$$\begin{aligned}y(t) + \widetilde{x}(t) &= (T_2 x)(t) + \mu\varphi^*(t) + \widetilde{x}(t) \\ &= \int_0^1 G(t,qs)[f_2(s,x(s)) + 2b_2(s) + c_2(s)K_2(x(s)) + M_2 + \|c_2\|K_2^*]d_qs + \mu\varphi^*(t).\end{aligned} \tag{25}$$

Using (H3), Lemma 6 and Remark 2 we have

$$x + \widetilde{y} \in P_0, \ y + \widetilde{x} \in P_0.$$

Please note that $\|x\| = \|y\| = R_1$, $y + \widetilde{y} + \widetilde{x} \in P_0$, and $x + \widetilde{y} + \widetilde{x} \in P_0$. Therefore, we get

$$x(t) + \widetilde{y}(t) + \widetilde{x}(t) \geq t^{\alpha-1}\|x + \widetilde{y} + \widetilde{x}\| \geq t^{\alpha-1}(\|x\| - \|\widetilde{y} + \widetilde{x}\|) \geq t^{\alpha-1}[\|x\| - (\|\widetilde{y}\| + \|\widetilde{x}\|)],$$

and

$$y(t) + \widetilde{y}(t) + \widetilde{x}(t) \geq t^{\alpha-1}\|y + \widetilde{y} + \widetilde{x}\| \geq t^{\alpha-1}(\|y\| - \|\widetilde{y} + \widetilde{x}\|) \geq t^{\alpha-1}[\|y\| - (\|\widetilde{y}\| + \|\widetilde{x}\|)].$$

Consequently, note that the range value of R_1, from (22) and (23) we find

$$
\begin{aligned}
&(\varepsilon_1 - \|c_1\|\varepsilon)\int_0^1 G(t,qs)(y(s)+\widetilde{y}(s)+\widetilde{x}(s))d_qs - (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs)(\widetilde{y}(s)+\widetilde{x}(s))d_qs \\
&\geq (\varepsilon_1 - \|c_1\|\varepsilon)\int_0^1 G(t,qs)s^{\alpha-1}(R_1-(\|\widetilde{y}\|+\|\widetilde{x}\|))d_qs - (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs) \\
&\quad \int_0^1 G(s,q\tau)[2b_1(\tau)+2b_2(\tau)+c_1(\tau)K_1(y(\tau))+c_2(\tau)K_2(x(\tau))+M_1+M_2+\|c_1\|K_1^*+\|c_2\|K_2^*]d_q\tau d_qs \\
&\geq (\varepsilon_1 - \|c_1\|\varepsilon)\int_0^1 G(t,qs)s^{\alpha-1}(R_1-(\|\widetilde{y}\|+\|\widetilde{x}\|))d_qs - (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs) \\
&\quad \int_0^1 s^{\alpha-1}\varphi_2(q\tau)[2\|b_1\|+2\|b_2\|+\|c_1\|(\varepsilon\|y\|+K_1^*)+\|c_2\|(\widetilde{\varepsilon}\|x\|+K_2^*)+M_1+M_2+\|c_1\|K_1^*+\|c_2\|K_2^*]d_q\tau d_qs \\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\varepsilon_1 - \|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)(x(s)+\widetilde{x}(s)+\widetilde{y}(s))d_qs - (\lambda_1+\varepsilon_1-\|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)(\widetilde{x}(s)+\widetilde{y}(s))d_qs \\
&\geq (\varepsilon_1 - \|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)s^{\alpha-1}(R_1-(\|\widetilde{y}\|+\|\widetilde{x}\|))d_qs - (\lambda_1+\varepsilon_1-\|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs) \\
&\quad \int_0^1 s^{\alpha-1}\varphi_2(q\tau)[2\|b_1\|+2\|b_2\|+\|c_1\|(\varepsilon\|y\|+K_1^*)+\|c_2\|(\widetilde{\varepsilon}\|x\|+K_2^*)+M_1+M_2+\|c_1\|K_1^*+\|c_2\|K_2^*]d_q\tau d_qs \\
&\geq 0.
\end{aligned}
$$

Consequently, from (16) and (17) we have

$$
\begin{aligned}
&\int_0^1 G(t,qs)[f_1(s,y(s))+2b_1(s)+c_1(s)K_1(y(s))+M_1+\|c_1\|K_1^*]d_qs \\
&\geq \int_0^1 G(t,qs)[(\lambda_1+\varepsilon_1)|y(s)|-M_1-b_1(s)-c_1(s)(\varepsilon|y(s)|+K_1^*)+b_1(s)+M_1+\|c_1\|K_1^*]d_qs \\
&\geq \int_0^1 G(t,qs)[(\lambda_1+\varepsilon_1)|y(s)|-\|c_1\|(\varepsilon|y(s)|+K_1^*)+\|c_1\|K_1^*]d_qs \\
&= (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs)|y(s)|d_qs \\
&\geq (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs)y(s)d_qs \\
&= (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs)(y(s)+\widetilde{y}(s)+\widetilde{x}(s))d_qs - (\lambda_1+\varepsilon_1-\|c_1\|\varepsilon)\int_0^1 G(t,qs)(\widetilde{y}(s)+\widetilde{x}(s))d_qs \\
&\geq \lambda_1 L(y+\widetilde{y}+\widetilde{x})(t) \\
&\geq \lambda_1 L(y+\widetilde{x})(t),
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
&\int_0^1 G(t,qs)[f_2(s,x(s))+2b_2(s)+c_2(s)K_2(x(s))+M_2+\|c_2\|K_2^*]d_qs \\
&\geq \int_0^1 G(t,qs)[(\lambda_1+\varepsilon_1)|x(s)|-M_2-b_2(s)-c_2(s)(\widetilde{\varepsilon}|x(s)|+K_2^*)+b_2(s)+M_2+\|c_2\|K_2^*]d_qs \\
&\geq \int_0^1 G(t,qs)[(\lambda_1+\varepsilon_1)|x(s)|-\|c_2\|(\widetilde{\varepsilon}|x(s)|+K_2^*)+\|c_2\|K_2^*]d_qs \\
&\geq (\lambda_1+\varepsilon_1-\|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)x(s)d_qs \\
&= (\lambda_1+\varepsilon_1-\|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)(x(s)+\widetilde{x}(s)+\widetilde{y}(s))d_qs - (\lambda_1+\varepsilon_1-\|c_2\|\widetilde{\varepsilon})\int_0^1 G(t,qs)(\widetilde{x}(s)+\widetilde{y}(s))d_qs \\
&\geq \lambda_1 L(x+\widetilde{x}+\widetilde{y})(t) \\
&\geq \lambda_1 L(x+\widetilde{y})(t).
\end{aligned} \tag{27}
$$

Now, using (26)–(27) we have

$$T_1 y + \widetilde{y} \geq \lambda_1 L(y + \widetilde{x}),\ T_2 x + \widetilde{x} \geq \lambda_1 L(x + \widetilde{y}).$$

Thus, from (19) we have

$$x + y + \widetilde{x} + \widetilde{y} = T_1 y + T_2 x + \widetilde{x} + \widetilde{y} + 2\mu\varphi^* \geq \lambda_1 L(x + y + \widetilde{x} + \widetilde{y}) + 2\mu\varphi^* \geq 2\mu\varphi^*.$$

Define $\mu^* = \sup S_\mu := \sup\{\mu > 0 : x + y + \widetilde{x} + \widetilde{y} \geq 2\mu\varphi^*\}$. Then $S_\mu(\neq \emptyset)$ is a limited set, $\mu^* \geq \mu$ and $x + y + \widetilde{x} + \widetilde{y} \geq 2\mu^*\varphi^*$. From $\varphi^* = \lambda_1 L\varphi^*$, we obtain

$$\lambda_1 L(x + y + \widetilde{x} + \widetilde{y}) \geq \lambda_1 L(2\mu^*\varphi^*) = 2\mu^*\lambda_1 L\varphi^* = 2\mu^*\varphi^*.$$

Hence

$$x + y + \widetilde{x} + \widetilde{y} \geq \lambda_1 L(x + y + \widetilde{x} + \widetilde{y}) + 2\mu\varphi^* \geq 2(\mu + \mu^*)\varphi^*,$$

which contradicts the definition of μ^*. Therefore, (18) holds, and from Lemma 7 we obtain

$$\deg(I - T, B_{R_1}, 0) = 0. \quad (28)$$

Now (15) and (28) together imply that

$$\deg(I - T, B_{R_1} \backslash \overline{B}_{r_1}, 0) = \deg(I - T, B_{R_1}, 0) - \deg(I - T, B_{r_1}, 0) = -1.$$

Therefore the operator T has at least one fixed point in $B_{R_1} \backslash \overline{B}_{r_1}$. Equivalently, (8) has at least one nontrivial solution. This completes the proof.

4. Conclusions

In this paper, we use topological degree to study nontrivial solutions for the system of fractional q-difference Equation (8) with q-integral boundary conditions. There are only a few papers in the literature which consider systems of fractional q-difference equations with q-integral boundary conditions where the nonlinear terms may be unbounded from below. Our main theorem is obtained under some conditions concerning the first eigenvalues corresponding to the relevant linear operators. As a result, our main result generalizes and improves the corresponding ones in the works cited in this paper.

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