mathematics

MDPI

Article

A Family of Theta-Function Identities Based upon Combinatorial Partition Identities Related to Jacobi's Triple-Product Identity

Hari Mohan Srivastava [1,2,3], Rekha Srivastava [1,*], Mahendra Pal Chaudhary [4] and Salah Uddin [5]

1. Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
2. Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3. Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
4. Department of Mathematics, Netaji Subhas University of Technology, Sector 3, Dwarka, New Delhi 110078, India; dr.m.p.chaudhary@gmail.com
5. Department of Mathematics, PDM University, Bahadurgarh 124507, Haryana State, India; vsludn@gmail.com

* Correspondence: rekhas@math.uvic.ca

Received: 11 May 2020; Accepted: 3 June 2020; Published: 5 June 2020

Abstract: The authors establish a set of six new theta-function identities involving multivariable R-functions which are based upon a number of q-product identities and Jacobi's celebrated triple-product identity. These theta-function identities depict the inter-relationships that exist among theta-function identities and combinatorial partition-theoretic identities. Here, in this paper, we consider and relate the multivariable R-functions to several interesting q-identities such as (for example) a number of q-product identities and Jacobi's celebrated triple-product identity. Various recent developments on the subject-matter of this article as well as some of its potential application areas are also briefly indicated. Finally, we choose to further emphasize upon some close connections with combinatorial partition-theoretic identities and present a presumably open problem.

Keywords: theta-function identities; multivariable R-functions; Jacobi's triple-product identity; Ramanujan's theta functions; q-product identities; Euler's pentagonal number theorem; Rogers-Ramanujan continued fraction; Rogers-Ramanujan identities; combinatorial partition-theoretic identities; Schur's, the Göllnitz-Gordon's and the Göllnitz's partition identities; Schur's second partition theorem

1. Introduction and Definitions

Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

In what follows, we shall make use of the following q-notations for the details of which we refer the reader to a recent monograph on q-calculus by Ernst [1] and also to the earlier works on the subject by Slater [2] (Chapter 3, Section 3.2.1), and by Srivastava et al. $\big($[3] (pp. 346 et seq.) and [4] (Chapter 6)$\big)$.

The q-shifted factorial $(a;q)_n$ is defined (for $|q|<1$) by

$$(a;q)_n := \begin{cases} 1 & (n=0) \\ \prod_{k=0}^{n-1}(1-aq^k) & (n\in\mathbb{N}), \end{cases} \tag{1}$$

where $a,q\in\mathbb{C}$, and it is assumed *tacitly* that $a\neq q^{-m}\quad(m\in\mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) = \prod_{k=1}^{\infty}(1-aq^{k-1}) \qquad (a,q\in\mathbb{C};\ |q|<1). \tag{2}$$

It should be noted that, when $a\neq 0$ and $|q|\geqq 1$, the infinite product in Equation (2) diverges. Thus, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1,a_2,\cdots,a_m;q)_n := (a_1;q)_n\,(a_2;q)_n\,\cdots\,(a_m;q)_n \tag{3}$$

and

$$(a_1,a_2,\cdots,a_m;q)_\infty := (a_1;q)_\infty\,(a_2;q)_\infty\,\cdots\,(a_m;q)_\infty. \tag{4}$$

Ramanujan (see [5,6]) defined the general theta function $\mathfrak{f}(a,b)$ as follows (see, for details, in [7] (p. 31, Equation (18.1)) and [8,9]):

$$\begin{aligned}\mathfrak{f}(a,b) &= 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}\,(a^n+b^n)\\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\,b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b,a) \qquad (|ab|<1).\end{aligned} \tag{5}$$

We find from this last Equation (5) that

$$\mathfrak{f}(a,b) = a^{\frac{n(n+1)}{2}}\,b^{\frac{n(n-1)}{2}}\,\mathfrak{f}\big(a(ab)^n, b(ab)^{-n}\big) = \mathfrak{f}(b,a) \qquad (n\in\mathbb{Z}). \tag{6}$$

In fact, Ramanujan (see [5,6]) also rediscovered Jacobi's famous triple-product identity, which, in Ramanujan's notation, is given by (see [7] (p. 35, Entry 19)):

$$\mathfrak{f}(a,b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty \tag{7}$$

or, equivalently, by (see [10])

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2}\,z^n &= \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+zq^{2n-1}\right)\left(1+\frac{1}{z}\,q^{2n-1}\right)\\ &= \left(q^2;q^2\right)_\infty\,\left(-zq;q^2\right)_\infty\,\left(-\frac{q}{z};q^2\right)_\infty \qquad (|q|<1;\ z\neq 0).\end{aligned}$$

Remark 1. *Equation* (6) *holds true as stated only if n is any integer. In case n is not an integer, this result (6) is only approximately true* (see, for details, [5] (Vol. 2, Chapter XVI, p. 193, Entry 18 (iv))). *Moreover, historically speaking, the q-series identity* (7) *or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).*

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worthy of note here (see, for details, (pp. 36–37, Entry 22) in [7]):

$$\begin{aligned}\varphi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \\ &= \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty\,(q^2;q^2)_\infty}{(q;q^2)_\infty\,(-q^2;q^2)_\infty};\end{aligned} \tag{8}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{9}$$

$$\begin{aligned} f(-q) &:= \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty}(-1)^n\, q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty}(-1)^n\, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \end{aligned} \tag{10}$$

Equation (10) is known as Euler's *Pentagonal Number Theorem*. Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{11}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [11,12]).

Theorem 1. *(Euler's Pentagonal Number Theorem) The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$\begin{aligned} R(q) &:= q^{\frac{1}{5}}\,\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\,\frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}}\,\frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+} \qquad (|q|<1). \end{aligned} \tag{12}$$

Here, $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger-Ramanujan identities, are defined as follows:

$$\begin{aligned} G(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)} \\ &= \frac{1}{(q;q^5)_\infty\,(q4;q^5)_\infty} = \frac{(q^2;q^5)_\infty\,(q^3;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty} \end{aligned} \tag{13}$$

and

$$\begin{aligned} H(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty} \\ &= \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty}, \end{aligned} \tag{14}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by Equations (5) and (10), respectively.

For a detailed historical account of (and for various related developments stemming from) the Rogers-Ramanujan continued fraction (12) as well as the Rogers-Ramanujan identities (13) and (14), the interested reader may refer to the monumental work [7] (p. 77 et seq.) (see also [4,8]).

The following continued-fraction results may be recalled now (see, for example, (p. 5, Equation (2.8)) in [13]).

Theorem 2. *Suppose that $|q| < 1$. Then,*

$$\begin{aligned} A(q) &:= (q^2;q^2)_\infty(-q;q)_\infty \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\;\frac{q}{1+}\;\frac{q(1-q)}{1-}\;\frac{q^3}{1+}\;\frac{q^2(1-q^2)}{1-}\;\frac{q^5}{1+}\;\frac{q^3(1-q^3)}{1-\cdots} \\ &= \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\cdots}}}}}}}, \end{aligned} \tag{15}$$

$$\begin{aligned} B(q) := \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} &= \frac{1}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\frac{q^5}{1+}\;\frac{q^6}{1+}\cdots \\ &= \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\cdots}}}}}}} \end{aligned} \tag{16}$$

and

$$\begin{aligned} C(q) &:= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1+\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\frac{q^5}{1+}\;\frac{q^6}{1+}\cdots \\ &= 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\cdots}}}}}}. \end{aligned} \tag{17}$$

By introducing the general family $R(s,t,l,u,v,w)$, Andrews et al. [14] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l,u,v,w;n), \tag{18}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\,(q^{uv};q^{uv})_j}. \tag{19}$$

We also recall the following interesting special cases of (18) (see, for details, (p. 106, Theorem 3) in [14]; see also [8]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty, \tag{20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty \tag{21}$$

and

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{22}$$

For the sake of brevity in our presentation of the main results, we now introduce the following notations:

$$R_\alpha = R(2,1,1,1,2,2),$$
$$R_\beta = R(2,2,1,1,2,2)$$

and

$$R_m = R(m,m,1,1,1,2) \qquad (m \in \mathbb{N}).$$

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [8,15–24]). In particular, Chaudhary et al. generalized several known results on character formulas (see [22]), Roger-Ramanujan type identities (see [19]), Eisenstein series, the Ramanujan-Göllnitz-Gordon continued fraction (see [20]), the 3-dissection property (see [18]), Ramanujan's modular equations of degrees 3, 7, and 9 (see [16,17]), and so on, by using combinatorial partition-theoretic identities. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn et al. [25] is also worth mentioning in this connection.

Here, in this paper, our main objective is to establish a set of six new theta-function identities which depict the inter-relationships that exist between the multivariable R-functions, q-product identities, and partition-theoretic identities.

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [26] (pp. 1749–1750 and 1752–1754)):

I. If

$$P = \frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)} \qquad \text{and} \qquad Q = \frac{\psi(q^3)}{q^{\frac{3}{2}}\psi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 + 3\left(\frac{Q}{P} + \frac{P}{Q}\right). \tag{23}$$

II. If

$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\psi(q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^5)}{q^{\frac{5}{4}}\psi(q^{15})},$$

then

$$(PQ)^2 + \left(\frac{3}{PQ}\right)^2 = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 - 5\left(\frac{Q}{P} - \frac{P}{Q}\right) + 5\left(\frac{P}{Q}\right)^2 + 5\left(\frac{Q}{P}\right)^2. \tag{24}$$

III. If
$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\psi(q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^7)}{q^{\frac{7}{4}}\psi(q^{21})},$$
then
$$(PQ)^3\left[\left(\frac{P}{Q}\right)^8 - 1\right] + 14P^5Q\left[\left(\frac{P}{Q}\right)^4 - 1\right] = P^6Q^2(7-P^4) + \frac{7P^6}{Q^2}(P^4-3)$$
$$-\left\{27\left(\frac{P}{Q}\right)^4 - 7P^4\left[3+3\left(\frac{P}{Q}\right)^4 - P^4\right]\right\}. \tag{25}$$

IV. If
$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\ \psi(q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q^{\frac{1}{2}};\psi(q^6)},$$
then
$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 = 0. \tag{26}$$

V. If
$$P = \frac{\psi(-q)}{q^{\frac{1}{4}}\ \psi(-q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q^{\frac{1}{2}}\ \psi(q^6)},$$
then
$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} + P^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{27}$$

VI. If
$$P = \frac{\psi(-q)}{q^{\frac{1}{4}}\ \psi(-q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q)}{q^{\frac{1}{4}}\ \psi(q^3)},$$
then
$$\left[\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right]\cdot\left[\left(\frac{3}{PQ}\right)^2 - (PQ)^2\right] + \left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 - 10 = 0. \tag{28}$$

2. A Set of Main Results

In this section, we state and prove a set of six new theta-function identities which depict inter-relationships among q-product identities and the multivariate R-functions.

Theorem 3. *Each of the following relationships holds true*:
$$\frac{R_1R_3}{R_5R_{15}} = \left(\frac{R_3R_5}{R_1R_{15}}\right)^2 - \left(\frac{q^2R_1R_{15}}{R_3R_5}\right)^2$$
$$+\left(\frac{3qR_3R_5}{R_1R_{15}}\right\} + \left(\frac{3q^3R_1R_{15}}{R_3R_5}\right) - \left(\frac{5q^4R_5R_{15}}{R_1R_3}\right) \tag{29}$$

and
$$\left(\frac{R_1R_5}{R_3R_{15}}\right)^2 = \left(\frac{R_3R_5}{R_1R_{15}}\right)^3 - \left(\frac{q^2R_1R_{15}}{R_3R_5}\right)^3 - \left(\frac{5q^2R_3R_5}{R_1R_{15}}\right) + \left(\frac{5q^4R_1R_{15}}{R_3R_5}\right)$$
$$+5q^5\left(\frac{R_1R_{15}}{R_3R_5}\right)^2 + 5q\left(\frac{R_3R_5}{R_1R_{15}}\right)^2 - \left(\frac{3q^3R_3R_{15}}{R_1R_5}\right)^2. \tag{30}$$

Equations (29) *and* (30) *give inter-relationships between* R_1, R_3, R_5 *and* R_{15}.

$$\left(\frac{R_1R_7}{q^2R_3R_{21}}\right)^3\cdot\left(\frac{q^{12}[R_1R_{21}]^8}{[R_3R_7]^8}-1\right)=\frac{1}{q^5}\left(\frac{[R_1]^3R_7}{[R_3]^3R_{21}}\right)^2\left(7-\frac{[R_1]^4}{q[R_3]^4}\right)$$
$$+\left(\frac{q[R_1]^3R_{21}}{[R_3]^3R_7}\right)^2\left(\frac{[R_1]^4}{q[R_3]^4}-3\right)-\frac{14}{q^3}\left(\frac{[R_1]^5R_7}{[R_3]^5R_{21}}\right)\cdot\left(\frac{q^6[R_1R_{21}]^4}{[R_3R_7]^4}-1\right)$$
$$-27q^6\left(\frac{R_1R_{21}}{R_3R_7}\right)^4+\frac{21}{q}\left(\frac{R_1}{R_3}\right)^4+21q^5\left(\frac{[R_1]^2R_{21}}{[R_3]^2R_7}\right)^4-\frac{7}{q^2}\left(\frac{R_1}{R_3}\right)^8. \tag{31}$$

Equation (31) *gives inter-relationships between* R_1, R_3, R_7, *and* R_{21}.

$$\left(\frac{R_1}{R_3}\right)^2=\left(\frac{q^{\frac{1}{2}}R_1R_6}{R_2R_3}\right)^2+\left(\frac{(3q)^{\frac{1}{2}}R_3}{R_1}\right)^2+\left(\frac{R_2R_3}{R_1R_6}\right)^2. \tag{32}$$

Equation (32) *gives inter-relationships between* R_1, R_2, R_3, *and* R_6.

$$\left(\frac{R_\alpha R_2(q^6;q^6)_\infty}{R_6(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}\right)^2=\left(\frac{(3q)^{\frac{1}{2}}\,R_\alpha(q^6;q^6)_\infty}{(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}\right)^2+\left(\frac{(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}{R_\alpha(q^6;q^6)_\infty}\right)^2$$
$$+\left(\frac{q^{\frac{1}{2}}\,R_6(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}{R_\alpha R_2(q^6;q^6)_\infty}\right)^2. \tag{33}$$

Equation (33) *gives inter-relationships between* R_2, R_6, *and* R_α. *Furthermore, it is asserted that*

$$\left(\frac{R_3(q^2;q^2)_\infty(-q^3;q^6)_\infty}{R_1R_\alpha(q^6;q^6)_\infty}\right)^4+\left(\frac{R_1R_\alpha(q^6;q^6)_\infty}{R_3(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}\right)^4$$
$$+\left[\left(\frac{R_3(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}{R_1R_\alpha(q^6;q^6)_\infty}\right)^2-\left(\frac{R_1R_\alpha(q^6;q^6)_\infty}{R_3(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}\right)^2\right]$$
$$\cdot\left[\left(\frac{3q^{\frac{1}{2}}\,R_\alpha R_3(q^6;q^6)_\infty}{R_1(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}\right)^2-\left(\frac{R_1(q^2;q^2)_\infty(-q^3;q^6)_\infty}{q^{\frac{1}{2}}\,R_\alpha R_3(q^6;q^6)_\infty}\right)^2\right]-10=0. \tag{34}$$

Equation (34) *gives inter-relationships between* R_1, R_3 *and* R_α.
It is assumed that each member of the assertions (29) *to* (34) *exists.*

Proof. First of all, in order to prove the assertion (29) of Theorem 3, we apply the identity (9) (with q replaced by q^3, q^5 q^{15}) under the given precondition of result (23). Thus, by using (20) and (21), and, after some simplifications, we get the values for P and Q as follows:

$$P=\frac{\psi(q)}{q^{\frac{1}{2}}\,\psi(q^5)}=\frac{R_1}{q^{\frac{1}{2}}\,R_5} \tag{35}$$

and

$$Q=\frac{\psi(q^3)}{q^{\frac{3}{2}}\,\psi(q^{15})}=\frac{R_3}{q^{\frac{3}{2}}\,R_{15}}. \tag{36}$$

Now, upon substituting from these last results (35) and (36) into (23), if we rearrange the terms and use some algebraic manipulations, we are led to the first assertion (29) of Theorem 3.

Secondly, we prove the second relationship (30) of Theorem 3. Indeed, if we first apply the identity (9) (with q replaced by q^3, q^5 and q^{15}) under the given precondition of the assertion (24), and then make use of (20) and (21), after some simplifications, the following values for P and Q would follow:

$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\ \psi(q^3)} = \frac{R_1}{q^{\frac{1}{4}}\ R_3} \tag{37}$$

and

$$Q = \frac{\psi(q^5)}{q^{\frac{5}{4}}\ \psi(q^{15})} = \frac{R_5}{q^{\frac{5}{4}}\ R_{15}}. \tag{38}$$

Now, upon substituting from these last results (37) and (38) into (24), if we rearrange the terms and use some algebraic manipulations, we obtain the second assertion (30) of Theorem 3.

Thirdly, we prove the third relationship (31) of Theorem 3. For this purpose, we first apply the identity (9) (with q replaced by q^3, q^7 and q^{21}) under the given precondition of (25), and then use (20) and (21). We thus find for the values of P and Q that

$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\ \psi(q^3)} = \frac{R_1}{q^{\frac{1}{4}}\ R_3} \tag{39}$$

and

$$Q = \frac{\psi(q^7)}{q^{\frac{7}{4}}\ \psi(q^{21})} = \frac{R_7}{q^{\frac{7}{4}}\ R_{21}}, \tag{40}$$

which, in view of (25) and after some rearrangements of the terms and the resulting algebraic manipulations, yields the third assertion (31) of Theorem 3.

Fourthly, we prove the identity (32) by applying the identity (9) (with the parameter q replaced by q^2, q^3 and q^6) under the given precondition of (26), we further use the assertions (20) and (21). Then, upon simplifications, we get the values for P and Q as follows:

$$P = \frac{\psi(q)}{q^{\frac{1}{4}}\ \psi(q^3)} = \frac{R_1}{q^{\frac{1}{2}}\ R_3} \tag{41}$$

and

$$Q = \frac{\psi(q^2)}{q^{\frac{1}{2}}\ \psi(q^6)} = \frac{R_2}{q^{\frac{1}{2}}\ R_6}. \tag{42}$$

Now, after using (41) and (42) in (26), if we rearrange the terms and and apply some algebraic manipulations, we get required result (32) asserted by Theorem 3.

We next prove the fifth identity (33). We apply the identity (9) (with the parameter q replaced by $-q$, $-q^3$, q^2 and q^6) under the given precondition of (27). We then further use the results (20) and (21). After simplification, we find the values for P and Q as follows:

$$P = \frac{\psi(-q)}{q^{\frac{1}{4}}\ \psi(-q^3)} = \frac{(q^2;q^2)_\infty\ (-q^3;q^6)_\infty}{q^{\frac{1}{4}}\ R_\alpha(q^6;q^6)_\infty} \tag{43}$$

and

$$Q = \frac{\psi(q^2)}{q^{\frac{1}{2}}\ \psi(q^6)} = \frac{R_2}{q^{\frac{1}{2}}\ R_6}. \tag{44}$$

Now, after using (43) and (44) in (27), we rearrange the terms and apply some algebraic manipulations. We are thus led to the required result (33).

Finally, we proceed to prove the last identity (34) asserted by Theorem 3. We make use of the identity (9) (with the parameter q replaced by $-q$, $-q^3$ and q^3) under the given precondition of (28). Then, by applying the identities (20) and (21), we obtain the values for P and Q as follows:

$$P = \frac{\psi(-q)}{q^{\frac{1}{4}}\,\psi(-q^3)} = \frac{(q^2;q^2)_\infty\,(-q^3;q^6)_\infty}{q^{\frac{1}{4}}\,R_\alpha(q^6;q^6)_\infty} \tag{45}$$

and

$$Q = \frac{\psi(q)}{q^{\frac{1}{4}}\,\psi(q^3)} = \frac{R_1}{q^{\frac{1}{4}}\,R_3}. \tag{46}$$

Thus, upon using (45) and (46) in (28), we rearrange the terms and apply some algebraic simplifications. This leads us to the required result (34), thereby completing the proof of Theorem 3. □

3. Applications Based upon Ramanujan's Continued-Fraction Identities

In this section, we first suggest some possible applications of our findings in Theorem 3 within the context of continued fraction identities. We begin by recalling that Naika et al. [27] studied the following continued fraction:

$$U(q) := \frac{q(1-q)}{(1-q^3)+}\,\frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+\cdots+}\,\frac{q^3(1-q^{6n-4})(1-q^{6n-2})}{(1-q^3)(1+q^{6n})+\cdots}, \tag{47}$$

which is a special case of a fascinating continued fraction recorded by Ramanujan in his second notebook [5,28,29]. On the other hand, Chaudhary et al. (see p. 861, Equations (3.1) to (3.5)) developed the following identities for the continued fraction $U(q)$ in (47) by using such R-functions as (for example) $R(1,1,1,1,1,2)$, $R(2,2,1,1,2,2)$, $R(2,1,1,1,2,2)$, $R(3,3,1,1,1,2)$ and $R(6,6,1,1,1,2)$:

$$\frac{1}{U(q)} + U(q) = \frac{R(1,1,1,1,1,2)R(2,2,1,1,2,2)}{\{R(2,1,1,1,2,2)\}^2}\cdot\{(q^3;q^6)_\infty(q^6;q^{12})_\infty\}^3, \tag{48}$$

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} = \frac{R(2,1,1,1,2,2)}{R(2,2,1,1,2,2)}\left\{\frac{R(1,1,1,1,1,2)R(2,2,1,1,1,2)}{q\,R(3,3,1,1,1,2)R(6,6,1,1,1,2)}\right\}^{\frac{1}{2}}, \tag{49}$$

$$\begin{aligned}\frac{1}{\sqrt{U(q)}} - \sqrt{U(q)} &= f(-q,q^3)\\ &\quad\cdot\left\{\frac{R(1,1,1,1,1,2)\{R(2,2,1,1,2,2)\}^2}{q\,R(6,6,1,1,1,2)R(3,3,1,1,1,2)R(2,2,1,1,1,2)}\right\}^{\frac{1}{2}},\end{aligned} \tag{50}$$

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} + 2 = \frac{R(2,1,1,1,2,2)\{R(1,1,1,1,1,2)\}^2}{q\,R(6,6,1,1,1,2)R(3,3,1,1,1,2)R(2,2,1,1,2,2)} \tag{51}$$

and

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} - 2 = \frac{R(2,2,1,1,1,2)\{R(3,3,1,1,1,2)\}^3}{q\,R(1,1,1,1,1,2)\{R(6,6,1,1,1,2)\}^3}. \tag{52}$$

By using the above formulas (48) to (52), we can express our results (29) to (34) in Theorem 3 in terms of Ramanujan's continued fraction $U(q)$ given here by (47).

Remark 2. *Even though the results of Theorem 3 are apparently considerably involved, each of the asserted theta-function identities does have the potential for other applications in analytic number theory and partition theory (see, for example, [30,31]) as well as in real and complex analysis, especially in connection with a significant number of wide-spread problems dealing with various basic (or q-) series and basic (or q-) operators* (see, for example, [32,33]).

Each of the theta-function identities (29) to (34), which are asserted by Theorem 3, obviously depict the inter-relationships that exist between q-product identities and the multivariate R-functions. Some corollaries and consequences of Theorem 3 may be worth pursuing for further research in the direction of the developments which we have presented in this article.

4. Connections with Combinatorial Partition-Theoretic Identities

Various extensions and generalizations of partition-theoretic identities and other q-identities, which we have investigated in this paper, as well as their connections with combinatorial partition-theoretic identities, can be found in several recent works (see, for example, [31,34,35]). The demonstrations in some of these recent developments are also based upon their combinatorial interpretations and generating functions (see also [25]).

As far as the connections with many different partition-theoretic identities are concerned, the existing literature is full of interesting findings and observations on the subject. In fact, in the year 2015, valuable progress in this direction was made by Andrews et al. [14], who established a number of interesting results including those for the q-series, q-products, and q-hypergeometric functions, which are associated closely with Schur's partitions, the Göllnitz-Gordon's partitions, and the Göllnitz's partitions in terms of multivariate R-functions. With a view to making our presentation to be self-sufficient, we choose to recall here some relevant parts of the developments in the remarkable investigation by Andrews et al. (see, for details, [14]).

We consider an integer partition of λ with parts $\lambda_1 \geqq \cdots \geqq \lambda_\ell$ and denote, as usual, its size by

$$|\lambda| := \lambda_1 + \cdots + \lambda_\ell$$

and its length (that is, the number of parts) by $\ell(\lambda)$ (see, for details, [36]).

Let us now assume that S denotes the set of Schur's partitions of λ such that

$$\lambda_j - \lambda_{j+1} > 3 \qquad (1 \leqq j \leqq \ell - 1),$$

with a strict inequality. We recall Schur's partitions as follows:

$$f_{\mathrm{S}}(x;q) := \sum_{\lambda \in \mathrm{S}} x^{\ell(\lambda)}\, q^{|\lambda|}, \tag{53}$$

which is of special interest here due to the following strikingly important infinite-product identity known as *Schur's Second Partition Theorem* (see [37]):

$$f_{\mathrm{S}}(1;q) = (-q;q^3)_\infty\, (-q^2;q^3)_\infty \tag{54}$$

In fact, Equation (54) yields a double-series representation for the two-parameter generating function for Schur partitions, which is given below:

$$f_{\mathrm{S}}(x;q) = \sum_{m,n \geqq 0} \frac{(-1)^n\, x^{m+2n}\, q^{(m+3n)^2 + \frac{m(m-1)}{2}}}{(q;q)_m (q^6;q^6)_n} \tag{55}$$

or, alternatively, as follows (see [14] (p. 103)):

$$f_{\mathrm{S}}(x;q) = (x;q^3)_\infty \sum_{n \geqq 0} \frac{x^n\, (-q,-q^2;q^3)_n}{(q^3;q^3)_n}.$$

We next suppose that GG denotes the set of the Göllnitz-Gordon partitions which satisfy the following inequality:

$$\lambda_j - \lambda_{j+1} \geqq 2 \qquad (1 \leqq j \leqq \ell - 1) \tag{56}$$

with strict inequality if either part is even. A direct combinatorial argument would now show that

$$f_{\mathrm{GG}}(x;q) := \sum_{\lambda \in \mathrm{GG}} x^{\ell(\lambda)}\, q^{|\lambda|} = \sum_{n \geqq 0} \frac{x^n\, q^{n^2} (-q;q^2)_n}{(q^2;q^2)_n}. \tag{57}$$

Hence, clearly, we have a new double-series representation of the generating function for the Göllnitz-Gordon partitions, which is given below:

$$f_{\mathrm{GG}}(x;q) = \sum_{k,m \geqq 0} \frac{(-1)^k\, x^{m++2k}\, q^{m^2+4mk+6k^2}}{(q;q)_m\, (q^4;q^4)_k}. \tag{58}$$

We also let G denote the set of the Göllnitz partitions which satisfy the following inequality:

$$\lambda_j - \lambda_{j+1} \geqq 2 \qquad (1 \leqq j \leqq \ell - 1)$$

with strict inequality if either part is odd. Then, the corresponding generating function for the Göllnitz partitions is given by

$$f_{\mathrm{G}}(x;q) := \sum_{\lambda \in \mathrm{G}} x^{\ell(\lambda)}\, q^{|\lambda|}. \tag{59}$$

We thus find the following double-series representation of the generating function for the Göllnitz partitions:

$$f_{\mathrm{G}}(x;q) = \sum_{k,m \geqq 0} \frac{(-1)^k\, x^{m++2k}\, q^{m^2+4mk+6k^2-2k}}{(q;q)_m\, (q^4;q^4)_k}. \tag{60}$$

Remark 3. *As pointed out by Andrews et al.* [14] (p. 105, Equations (1.8) and (1.9)), *alternative double-series representations for the double series in Equations (58) and (60) were given in an earlier publication by Alladi and Berkovich [38].*

In order to illustrate the connections of the above-mentioned partition-theoretic identities with the multivariable R-functions given by Equations (18) and (19), we note that the Schur's, the Göllnitz-Gordon and the Göllnitz partition identities can be expressed as follows:

$$R(3,t,0,2,3,4) = f_{\mathrm{S}}(q^{t-1};q), \tag{61}$$

$$R(2,t,0,2,2,2) = f_{\mathrm{GG}}(q^{t-1};q) \tag{62}$$

and

$$R(2,t,1,2,2,2) = f_{\mathrm{G}}(q^{t-1};q). \tag{63}$$

5. An Open Problem

Based upon the work presented in this paper, we find it to be worthwhile to motivate the interested reader to consider the following related open problem.

Open Problem. Find inter-relationships between R_β and R_α, R_m $(m \in \mathbb{N})$, q-product identities and continued-fraction identities.

6. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established a family of six presumably new theta-function identities which depict the inter-relationships that exist among q-product identities and combinatorial partition-theoretic identities. We have also considered several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identities. In addition, with a view to further motivating research involving theta-function identities and combinatorial partition-theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article.

The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further research and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Adiga et al. (see [28,39]), Cao et al. [40], Chaudhary et al. (see [13,21,22]), Hahn et al. [25], and Srivastava et al. (see [23,29,33,41–45]), and by Yee [35] and Yi [31], are worth mentioning here.

Author Contributions: Conceptualization, M.P.C., H.M.S.; Formal analysis, H.M.S.; Funding acquisition, R.S.; Investigation, R.S., M.P.C. and S.U.; Methodology, H.M.S., M.P.C. and S.U.; Supervision, H.M.S. and R.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: All four authors declare that they have no conflict of interest.

References

1. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Birkhäuser/Springer: Basel, Switzerland, 2012.
2. Slater, L.J. *Generalized Hypergeometric Functions*; Cambridge University Press: Cambridge, UK, 1966.
3. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press: Sydney, Australia; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
4. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands, 2012.
5. Ramanujan, S. *Notebooks*; Tata Institute of Fundamental Research: Bombay, India, 1957; Volumes 1 and 2.
6. Ramanujan, S. *The Lost Notebook and Other Unpublished Papers*; Narosa Publishing House: New Delhi, India, 1988.
7. Berndt, B.C. *Ramanujan's Notebooks*; Part III; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1991.
8. Srivastava, H.M.; Chaudhary, M.P. Some relationships between q-product identities, combinatorial partition identities and continued-fraction identities. *Adv. Stud. Contemp. Math.* **2015**, *25*, 265–272.
9. Baruah, N.D.; Bora, J. Modular relations for the nonic analogues of the Rogers-Ramanujan functions with applications to partitions. *J. Number Theory* **2008**, *128*, 175–206. [CrossRef]
10. Jacobi, C.G.J. *Fundamenta Nova Theoriae Functionum Ellipticarum*; Regiomonti, Sumtibus Fratrum Bornträger: Königsberg, Germany, 1829; Reprinted in *Gesammelte Mathematische Werke* **1829**, *1*, 497–538; American Mathematical Society: Providence, RI, USA, 1969; pp. 97–239.
11. Hardy, G.H.; Wright, E.M. *An Introduction to the Theory of Numbers*, 6th ed.; Oxford University Press: London, UK; New York, NY, USA, 2008.
12. Apostol, T.M. *Introduction to Analytic Number Theory*; Undergraduate Texts in Mathematics; Springer: Berlin/Heidelberg, Germany, 1976.
13. Chaudhary, M.P. Generalization of Ramanujan's identities in terms of q-products and continued fractions. *Glob. J. Sci. Front. Res. Math. Decis. Sci.* **2012**, *12*, 53–60.
14. Andrews, G.E.; Bringman, K.; Mahlburg, K.E. Double series representations for Schur's partition function and related identities. *J. Combin. Theory Ser. A* **2015**, *132*, 102–119. [CrossRef]
15. Chaudhary, M.P. Some relationships between q-product identities, combinatorial partition identities and continued-fractions identities. III. *Pac. J. Appl. Math.* **2015**, *7*, 87–95.

16. Chaudhary, M.P.; Chaudhary, S. Note on Ramanujan's modular equations of degrees three and nine. *Pac. J. Appl. Math.* **2017**, *8*, 143–148.
17. Chaudhary, M.P.; Chaudhary, S.; Choi, J. Certain identities associated with 3-dissection property, continued fractions and combinatorial partition. *Appl. Math. Sci.* **2016**, *10*, 37–44. [CrossRef]
18. Chaudhary, M.P.; Chaudhary, S.; Choi, J. Note on Ramanujan's modular equation of degree seven. *Int. J. Math. Anal.* **2016**, *10*, 661–667. [CrossRef]
19. Chaudhary, M.P.; Choi, J. Note on modular relations for Roger-Ramanujan type identities and representations for Jacobi identities. *East Asian Math. J.* **2015**, *31*, 659–665. [CrossRef]
20. Chaudhary, M.P.; Choi, J. Certain identities associated with Eisenstein series, Ramanujan-Göllnitz-Gordon continued fraction and combinatorial partition identities. *Int. J. Math. Anal.* **2016**, *10*, 237–244. [CrossRef]
21. Chaudhary, M.P.; Choi, J. Certain identities associated with character formulas, continued fraction and combinatorial partition identities. *East Asian Math. J.* **2016**, *32*, 609–619. [CrossRef]
22. Chaudhary, M.P.; Uddin, S.; Choi, J. Certain relationships between q-product identities, combinatorial partition identities and continued-fraction identities. *Far East J. Math. Sci.* **2017**, *101*, 973–982. [CrossRef]
23. Srivastava, H.M.; Chaudhary, M.P.; Chaudhary, S. Some theta-function identities related to Jacobi's triple-product identity. *Eur. J. Pure Appl. Math.* **2018**, *11*, 1–9. [CrossRef]
24. Srivastava, H.M.; Chaudhary, M.P.; Chaudhary, S. A family of theta-function identities related to Jacobi's triple-product identity. *Russ. J. Math. Phys.* **2020**, *27*, 139–144. [CrossRef]
25. Hahn, H.-Y.; Huh, J.-S.; Lim, E.-S.; Sohn, J.-B. From partition identities to a combinatorial approach to explicit Satake inversion. *Ann. Combin.* **2018**, *22*, 543–562. [CrossRef]
26. Baruah, N.D.; Saikia, N. Two parameters for Ramanujan's theta-functions and their explicit values. *Rocky Mt. J. Math.* **2007**, *37*, 1747–1790. [CrossRef]
27. Naika, M.S.M.; Dharmendra, B.N.; Shivashankar, K. A continued fraction of order twelve. *Cent. Eur. J. Math.* **2008**, *6*, 393–404. [CrossRef]
28. Adiga, C.; Bulkhali, N.A.S.; Simsek, Y.; Srivastava, H.M. A continued fraction of Ramanujan and some Ramanujan-Weber class invariants. *Filomat* **2017**, *31*, 3975–3997. [CrossRef]
29. Srivastava, H.M.; Saikia, N. Some congruences for overpartitions with restriction. *Math. Notes* **2020**, *107*, 488–498. [CrossRef]
30. Liu, Z.-G. A three-term theta function identity and its applications. *Adv. Math.* **2005**, *195*, 1–23. [CrossRef]
31. Yi, J.-H. Theta-function identities and the explicit formulas for theta-function and their applications. *J. Math. Anal. Appl.* **2004**, *292*, 381–400. [CrossRef]
32. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
33. Srivastava, H.M.; Singh, S.N.; Singh, S.P. Some families of q-series identities and associated continued fractions. *Theory Appl. Math. Comput. Sci.* **2015**, *5*, 203–212.
34. Munagi, A.O. Combinatorial identities for restricted set partitions. *Discret. Math.* **2016**, *339*, 1306–1314. [CrossRef]
35. Yee, A.-J. Combinatorial proofs of generating function identities for F-partitions. *J. Combin. Theory Ser. A* **2003**, *102*, 217–228. [CrossRef]
36. Andrews, G.E. *The Theory of Partitions*; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1998.
37. Schur, I. *Zur Additiven Zahlentheorie*; Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl.: Berlin, Germany, 1926.
38. Alladi, K.; Berkovich, A. Göllnitz–Gordon partitions with weights and parity conditions. In *Zeta Functions, Topology and Quantum Physics*; Springer Series on Developments in Mathematics; Springer: New York, NY, USA, 2005; Volume 14, pp. 1–17.
39. Adiga, C.; Bulkhali, N.A.S.; Ranganatha, D.; Srivastava, H.M. Some new modular relations for the Rogers–Ramanujan type functions of order eleven with applications to partitions. *J. Number Theory* **2016**, *158*, 281–297. [CrossRef]
40. Cao, J.; Srivastava, H.M.; Luo, Z.-G. Some iterated fractional q-integrals and their applications. *Fract. Calc. Appl. Anal.* **2018**, *21*, 672–695. [CrossRef]
41. Srivastava, H.M. Some formulas of Srinivasa Ramanujan involving products of hypergeometric functions. *Indian J. Math.* (*Ramanujan Centen. Vol.*) **1987**, *29*, 91–100.

42. Srivastava, H.M. A note on a generalization of a q-series transformation of Ramanujan. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1987**, *63*, 143–145. [CrossRef]
43. Srivastava, H.M. Srinivasa Ramanujan and generalized basic hypergeometric functions. *Serdica* (*Academician Ljubomir G. Iliev Dedication Vol.*) **1993**, *19*, 191–197.
44. Srivastava, H.M.; Zhang, C.-H. A certain class of identities of the Rogers–Ramanujan type. *Pan Am. Math. J.* **2009**, *19*, 89–102.
45. Srivastava, H.M.; Arjika, S.; Kelil, A.S. Some homogeneous q-difference operators and the associated generalized Hahn polynomials. *Appl. Set-Valued Anal. Optim.* **2019**, *1*, 187–201.