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Strong Convergence of Mann's Iteration Process in Banach Spaces

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Abstract: Mann's iteration process for finding a fixed point of a nonexpansive mapping in a Banach space is considered. This process is known to converge weakly in some class of infinite-dimensional Banach spaces (e.g., uniformly convex Banach spaces with a Fréchet differentiable norm), but not strongly even in a Hilbert space. Strong convergence is therefore a nontrivial problem. In this paper we provide certain conditions either on the underlying space or on the mapping under investigation so as to guarantee the strong convergence of Mann's iteration process and its variants.

Keywords: nonexpansive mapping; mann iteration; strong convergence; duality map; banach space

MSC: 47H09; 47H10; 47J25

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$). We use $\text{Fix}(T)$ to denote the set of fixed points of T; i.e., $\text{Fix}(T) = \{x \in C : Tx = x\}$. It is known that $\text{Fix}(T)$ is nonempty if X is uniformly convex and C is bounded. Mann's iteration process [1], an averaged iterative scheme, is used to find a point in $\text{Fix}(T)$. This process generates a sequence $\{x_n\}$ via the recursive process:

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n = 0, 1, \cdots, \tag{1}$$

where the initial point $x_0 \in C$ is arbitrary and $(t_n) \subset [0, 1]$. It is known [2] that if X is uniformly convex with a Fréchet differentiable norm, if $\text{Fix}(T)$ is nonempty, and if $\sum_{n=0}^{\infty} t_n(1 - t_n) = \infty$, then the sequence (x_n) generated by (1) converges weakly to a point in $\text{Fix}(T)$. However, the counterexample in [3] shows that the algorithm (1) fails, in general, to converge strongly even in Hilbert space unless C is compact [4]. Therefore, efforts have been made to study sufficient conditions to guarantee strong convergence of Mann's algorithm (1) without assuming compactness of C. For instance, Gwinner [5] imposed the φ-accretiveness condition on $I - T$ to prove strong convergence of (1) in a uniformly convex Banach space. Some authors have made modifications of Mann's iteration process in order to get strong convergence (see, e.g., [6–8]).

In this paper we continue to study the strong convergence of Mann's iteration process. We improve Gwinner's strongly convergent result ([5], Theorem 1) by removing the condition $t_n \leq b$ with $b \in (0, 1)$. We also prove strong convergence of Mann's iteration process in a reflexive Banach space with Opial's and Kadec-Klee properties when $I - T$ is φ-accretive. A regularization method is introduced to approximate a fixed point of T. This method implicitly yields a sequence of approximate solutions and we shall prove (in Theorem 4) its strong convergence to a solution of a variational inequality. Combining this regularization method with Mann's method, we obtain a new iteration process (see (22)) when the regularizer $R = I - f$ with f a contraction. We will prove that this process converges strongly in a Banach space whenever the sequence of approximate solutions of the implicit regularization converges strongly.

We use the notation:

- "$x_n \rightharpoonup x$" stands for the weak convergence of (x_n) to x,
- "$x_n \to x$" stands for the strong convergence of (x_n) to x,
- $\omega_w(x_n) := \{x : \exists x_{n_k} \rightharpoonup x\}$ is the set of all weak accumulation points of the sequence (x_n).

2. Preliminaries

2.1. Uniform Convexity

Let X be a real Banach space with norm $\|\cdot\|$. Recall that the modulus of convexity of X is defined as (cf. [9])

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}, \quad \varepsilon \in [0,2].$$

We say that X is uniformly convex if

$$\delta_X(\varepsilon) > 0 \quad \text{for all } \varepsilon \in (0,2].$$

Examples of uniformly convex Banach spaces include Hilbert spaces H and l^p (and also L^p) spaces for $1 < p < \infty$. As a matter of fact, the moduli of convexity of these spaces are

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}$$

and

$$\delta_{l^p}(\varepsilon) = \delta_{L^p}(\varepsilon) \begin{cases} = 1 - \sqrt[p]{1 - \left(\frac{\varepsilon}{2}\right)^p} & \text{if } 2 \le p < \infty, \\ \ge 1 - \sqrt{1 - (p-1)\left(\frac{\varepsilon}{2}\right)^2} & \text{if } 1 < p < 2. \end{cases}$$

The following inequality characterization of uniform convexity is convenient in application.

Proposition 1 ([10])**.** *Let X be a uniformly convex Banach space. Then for each fixed real number $r > 0$, there exists a strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$, $h(0) = 0$, satisfying the property:*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)h(\|x-y\|)$$

for all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$, and $0 \le t \le 1$.

Recall that a Banach space X is said to satisfy the Kadec-Klee property (also known as property (H)) if the (sequential) weak and strong topologies on the unit sphere coincide; equivalently, given any sequence $\{u_n\}$ and a point u in X, the following implication holds:

$$(u_n \to u \text{ weakly and } \|u_n\| \to \|u\|) \quad \Longrightarrow \quad u_n \to u \text{ strongly}.$$

Every uniformly convex Banach space satisfies the Kadec-Klee property.

2.2. Duality Maps

Let X be a real Banach space with norm $\|\cdot\|$. The notion of general duality maps on X was introduced by Browder [11–14]. By a gauge we mean a function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the properties:

(i) $\mu(0) = 0$,
(ii) μ is continuous and strictly increasing, and
(iii) $\lim_{t\to\infty} \mu(t) = \infty$.

Associated with a gauge μ is the duality map $J_\mu : X \to X^*$ ([12]) defined by

$$J_\mu(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|,\ \mu(\|x\|) = \|x^*\|\}. \tag{2}$$

A special case is given by choosing the gauge $\mu(t) = t^{p-1}$ for $t \geq 0$, where $1 < p < \infty$. In this case the corresponding duality map, which is denoted by J_p and referred to as the generalized duality map of order p of X, is given by

$$J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\}. \tag{3}$$

In particular, we denote J for J_2 and call it the normalized duality map. In more detail, J is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{4}$$

For a Hilbert space H, its normalized duality map is identified with the identity map I (with the dual space H^* being identified with H through the Riesz canonical embedding).

Recall that a Banach space X is said to have a (sequentially) weakly continuous duality map if, for some gauge μ, the duality map $J_\mu : X \to X^*$ is (sequentially) continuous when X is endowed with the weak topology and the dual space X^* endowed with the weak-star topology. It is known that, for $1 < p < \infty$ $(p \neq 2)$, the sequence space ℓ_p has a sequentially weakly continuous duality map J_p; while the function space L^p does not [15]. However, the normalized duality map J of ℓ_p $(p \neq 2)$ is not weakly continuous [16], Proposition 3.2.

Proposition 2. *[17] Assume a Banach space X has a weakly continuous duality map J_μ for some gauge μ. If (u_n) is a sequence in X weakly convergent to a point u, then we have*

$$\limsup_{n\to\infty} \Psi(\|u_n - v\|) = \limsup_{n\to\infty} \Psi(\|u_n - u\|) + \Psi(\|u - v\|), \quad v \in X, \tag{5}$$

where Ψ is defined by $\Psi(t) = \int_0^t \mu(s)ds$, $t \geq 0$. In particular, X satisfies the Opial property ([15]):

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\| \quad \forall u \neq v \in X \tag{6}$$

whenever (u_n) is a sequence in X weakly convergent to u.

2.3. Demiclosedness Principle for Nonexpansive Mappings

Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping. Recall that $I - T$ is said to be demiclosed if the graph of $I - T$, $G(I - T) := \{(x, y) : x \in C,\ y = (I - T)x\}$ is closed in the product space $X_w \times X$, where X_w is endowed with the weak topology. Equivalently, $I - T$ is demiclosed if and only if the implication below holds:

$$((x_n) \subset C,\ x_n \rightharpoonup x,\ x_n - Tx_n \to y) \quad \Longrightarrow \quad x - Tx = y.$$

This is called the demiclosedness principle for nonexpansive mappings, which holds in the following Banach spaces:

- Uniformly convex Banach spaces [12];
- Banach spaces satisfying Opial's property [15], in particular, Banach spaces with a weakly continuous duality map J_μ for some gauge μ.

2.4. Accretive Operators

Let C be a nonempty closed convex subset of a Banach space X. Recall that an operator $S : C \to X$ is said to be accretive (cf. [18]) if, for each $x, y \in C$,

$$\langle Sx - Sy, J(x-y) \rangle \geq 0.$$

Here $J : X \to X^*$ is the normalized duality map (or a selection in the case of multivalued).

Definition 1 ([5,19]). *Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function with $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then we say that an operator $S : C \to X$ is φ-accretive (or uniformly φ-accretive, respectively) if*

$$\langle Sx - Sy, J(x-y) \rangle \geq [\varphi(\|x\|) - \varphi(\|y\|)] \cdot (\|x\| - \|y\|) \tag{7}$$

for all $x, y \in C$, or respectively,

$$\langle Sx - Sy, J(x-y) \rangle \geq \varphi(\|x-y\|) \cdot \|x-y\| \tag{8}$$

for all $x, y \in C$, where J is the normalized duality map of X.

Notice that if S is φ-accretive, then the set of zeros of S, $S^{-1}(0) = \{x \in C : Sx = 0\}$, consists of at most one point. In fact, if $S\hat{x} = S\tilde{x} = 0$, we must have $[\varphi(\|\hat{x}\|) - \varphi(\|\tilde{x}\|)] \cdot (\|\hat{x}\| - \|\tilde{x}\|) = 0$. This implies $\hat{x} = \tilde{x}$.

2.5. A Useful Lemma

The lemma below is helpful in proving strong convergence of a sequence (x_n) to a point x in a Banach space by proving convergence to zero of the real sequence $(\|x_n - x\|)$.

Lemma 1 ([20,21] Lemma 2.2). *Assume (s_n) is a sequence of nonnegative real numbers satisfying the condition:*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\beta_n + \delta_n, \quad n \geq 0, \tag{9}$$

where (λ_n) and (δ_n) are sequences in (0,1) and (β_n) is a sequence in $\mathbb{R}$. Assume

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *(or $\sum_{n=1}^{\infty} \lambda_n|\beta_n| < \infty$),*
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Strong Convergence Analysis of Mann's Iteration Process

3.1. Strong Convergence of Mann's Algorithm

Let X be a real Banach space, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a nonexpansive mapping with a fixed point. We use $\mathrm{Fix}(T)$ to note the set of fixed points of T. It is known that if X is strictly convex, then $\mathrm{Fix}(T)$ is convex. For finding a fixed point of T, Mann's iterative algorithm [1] (see also [22]) is often used. This algorithm generates a sequence (x_n) as follows:

$$x_{n+1} = (1-t_n)x_n + t_nTx_n, \quad n = 0, 1, 2, \cdots, \tag{10}$$

where the initial point $x_0 \in C$ is arbitrary, and $t_n \in [0, 1]$ for all $n \geq 0$.

Lemma 2. *[23] In a Banach space X, let $T: C \to C$ be a nonexpansive mapping and let (x_n) be a sequence generated by Mann's method (10). Then*

(i) for each $z \in \mathrm{Fix}(T)$, the sequence $\{\|x_n - z\|\}$ is nonincreasing, and
(ii) the sequence $\{\|x_n - Tx_n\|\}$ is nonincreasing.

Consequently, both $\lim_{n\to\infty} \|x_n - z\|$ and $\lim_{n\to\infty} \|x_n - Tx_n\|$ exist.

We begin with a strong convergence result on the Mann algorithm (10), which improves ([5], Theorem 1) by removing the restriction of $t_n \le b$ with $b \in (0,1)$.

Theorem 1. *Let X be a real uniformly convex Banach space, let C be a nonempty closed convex subset of X, and let T be a nonexpansive self-mapping of C with $\mathrm{Fix}(T) \neq \emptyset$. Suppose $S := I - T$ is φ-accretive and (t_n) satisfies the divergence condition*

$$\sum_{n=0}^{\infty} t_n(1-t_n) = \infty. \tag{11}$$

Then (x_n) converges in norm to a fixed point of T.

Proof. First observe that the φ-accretiveness of $S = I - T$ implies that T has a unique fixed point, i.e., $\mathrm{Fix}(T) = \{z\}$ is singleton. Set $r = \|x_0 - z\|$. By Lemma 2(i), $\|x_n - z\| \le r$ for all n. Applying Proposition 1(i), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-t_n)(x_n - z) + t_n(Tx_n - z)\|^2 \\ &\le (1-t_n)\|x_n - z\|^2 + t_n\|Tx_n - q\|^2 - t_n(1-t_n)h(\|x_n - Tx_n\|) \\ &\le \|x_n - q\|^2 - t_n(1-t_n)h(\|x_n - Tx_n\|).\end{aligned}$$

It turns out that

$$t_n(1-t_n)h(\|x_n - Tx_n\|) \le \|x_n - q\|^2 - \|x_{n+1} - z\|^2.$$

Consequently,

$$\sum_{n=0}^{\infty} t_n(1-t_n)h(\|x_n - Tx_n\|) \le \|x_0 - z\|^2 = r^2 < \infty.$$

The divergence condition (11) together with Lemma 2(ii) implies that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. By the demiclosedness principle of $S = I - T$, we have that $\omega_w(x_n) \subset \mathrm{Fix}(T) = \{z\}$. This proves that $x_n \rightharpoonup z$.

Now using the φ-accretiveness of $S = I - T$, we obtain

$$[\varphi(\|x_n\|) - \varphi(\|z\|)] \cdot (\|x_n\| - \|z\|) \le \langle Sx_n, J(x_n - z)\rangle \le r\|x_n - Tx_n\| \to 0. \tag{12}$$

It turns out that $\|x_n\| \to \|z\|$. Therefore, the Kadec-Klee property of the uniform convexity of X implies that $x_n \to z$. The proof is complete. □

Remark 1. *The conclusion of Theorem 1 remains true if the uniform convexity of X is weakened to strict convexity together with the Kadec-Klee property, and if $S = I - T$ is demiclosed.*

Theorem 2. *Let X be a reflexive Banach space satisfying Opial's condition and the Kadec-Klee property, let C be a nonempty closed convex subset of X, and let T be a nonexpansive self-mapping of C with $\mathrm{Fix}(T) \neq \emptyset$. Suppose $S := I - T$ is φ-accretive and (t_n) satisfies the conditions*

$$t_n \le b \text{ (for some } b \in (0,1)) \quad \text{and} \quad \sum_{n=0}^{\infty} t_n = \infty. \tag{13}$$

Then (x_n) converges in norm to a fixed point of T.

Proof. We sketch the proof here. Under the condition (13), we have by ([24], Lemma 2) that $x_n - Tx_n \to 0$ strongly. Now Opial's property together with the fact that $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in \mathrm{Fix}(T)$ [Lemma 2(i)] implies that the sequence $\{x_n\}$ converges weakly to a point z in $\mathrm{Fix}(T)$. In the meanwhile, we still have (12) which implies $\|x_n\| \to \|z\|$. Now the strong convergence of $\{x_n\}$ to z follows from the Kadec-Klee property of X. □

Remark 2. *It is unclear if, in Theorem 2, the divergence condition (13) can be weakened to the divergence condition (11). Alternatively, it is unknown whether $\|x_n - Tx_n\| \to 0$ holds true in a Banach space with Opial's condition, without the assumption that $\sup_{n\geq 0} t_n < 1$. (Note: Ishikawa ([24], Lemma 2) requires this assumption.)*

3.2. Regularization

Finding a fixed point of a nonexpansive mapping T is equivalent to finding a zero of the accretive operator $S = I - T$. Thus regularization methods may apply. In [5], Gwinner introduced the regularization operator $S_\varepsilon := (1-\varepsilon)(I-T) + \varepsilon R$, where $\varepsilon \in (0,1)$ is a regularization parameter and $R : C \to X$ is a mapping which is referred to as regularizer. The following result on this regularization was proved.

Theorem 3 ([5] Theorem 2)**.** *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and T a nonexpansive self-mapping of C with $\mathrm{Fix}(T) \neq \emptyset$. Assume $R : C \to X$ is a continuous, bounded, uniformly φ-accretive operator. Assume the normalized duality map $J : X \to X^*$ is weakly sequentially continuous. Choose positive real numbers δ_n and $\varepsilon_n \in (0,1)$ such that*

(i) $\lim_{n\to\infty} \varepsilon_n = 0$,
(ii) $\lim_{n\to\infty}(\delta_n/\varepsilon_n) = 0$.

If the approximate solutions $\tilde{y}_n \in C$ satisfy

$$\|(1-\varepsilon_n)(I-T)(\tilde{y}_n) + \varepsilon_n R(\tilde{y}_n)\| \leq \delta_n, \tag{14}$$

then the sequence $\{\tilde{y}_n\}$ converges strongly to a fixed point $\hat{z}$ of T, which is uniquely determined by the variational inequality:

$$\langle R(\hat{z}), J(\hat{z} - z)\rangle \leq 0 \quad \forall z \in \mathrm{Fix}(T). \tag{15}$$

Remark 3. *The condition in Theorem 3 that the normalized duality map $J : X \to X^*$ be weakly continuous is quite restrictive because this condition rules out the applicability of Theorem 3 to the uniformly convex sequence space ℓ_p ($1 < p < \infty$, $p \neq 2$) since the normalized duality map J of ℓ_p ($1 < p < \infty$, $p \neq 2$) fails to be weakly continuous ([16], Proposition 3.2).Below we use the condition that X admits a weakly continuous duality map J_μ for some gauge μ; as a result, our theorem applies to every ℓ_p space ($1 < p < \infty$) which has a weakly continuous duality map J_μ with gauge $\mu(t) = t^{p-1}$ for $t \geq 0$.*

Our next result improves Theorem 3 (i.e., [5], Theorem 2) twofold: first we remove the uniform convexity of the space X and secondly, we replace the normalized duality map J with a general duality map J_μ for some gauge μ, the latter being more flexible in applications (such as ℓ_p spaces for $1 < p < \infty$).

Theorem 4. *Assume X is a reflexive Banach space with a weakly sequentially continuous duality map $J_\mu : X \to X^*$ for some gauge μ, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume $R : C \to X$ is a continuous, bounded, uniformly φ-accretive operator. Assume δ_n and ε_n satisfy the conditions (i) and (ii) in Theorem 3. Then the sequence $\{\tilde{y}_n\}$ of approximate solutions determined by (14) converges strongly to the unique solution $\hat{z}$ of the variational inequality (15).*

Proof. First observe that, since the duality maps J_μ and J satisfy the relation

$$\|x\| J_\mu(x) = \mu(\|x\|) J(x), \quad x \in X,$$

the uniform φ-accretiveness of R, (8), can equivalently be rephrased as

$$\langle Sx - Sy, J_\mu(x-y)\rangle \geq \varphi(\|x-y\|) \cdot \mu(\|x-y\|) \quad \forall x, y \in C. \tag{16}$$

Now set

$$\beta_n := \langle (1-\varepsilon_n)(I-T)(\tilde{y}_n) + \varepsilon_n R(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle,$$

where $z \in \mathrm{Fix}(T)$. It turns out from (14) that

$$|\beta_n| \leq \delta_n \mu(\|\tilde{y}_n - z\|).$$

Also, since T is nonexpansive (so that $I-T$ is accretive), we have for $z \in \mathrm{Fix}(T)$

$$\begin{aligned}\beta_n &= (1-\varepsilon_n)\langle (I-T)(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle + \varepsilon_n \langle R(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle \\ &\geq \varepsilon_n \langle R(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle\end{aligned}$$

for $\langle (I-T)(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle \geq 0$. It follows that

$$\langle R(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle \leq \frac{\beta_n}{\varepsilon_n}. \tag{17}$$

Using (16) and (17), we get, for $z \in \mathrm{Fix}(T)$,

$$\begin{aligned}\mu(\|\tilde{y}_n - z\|)\varphi(\|\tilde{y}_n - z\|) &\leq \langle R(\tilde{y}_n) - R(z), J_\mu(\tilde{y}_n - z)\rangle \\ &= \langle R(\tilde{y}_n), J_\mu(\tilde{y}_n - z)\rangle - \langle R(z), J_\mu(\tilde{y}_n - z)\rangle \\ &\leq (\beta_n/\varepsilon_n) + \|R(z)\|\mu(\|\tilde{y}_n - z\|) \\ &\leq \gamma + \|R(z)\|\mu(\|\tilde{y}_n - z\|), \quad \gamma := \sup_{n\geq 0}(\beta_n/\varepsilon_n).\end{aligned} \tag{18}$$

If $\mu(\|\tilde{y}_n - z\|) \geq \gamma$, i.e., $\|\tilde{y}_n - z\| \geq \mu^{-1}(\gamma)$, then it follows from the last relation that

$$\varphi(\|\tilde{y}_n - z\|) \leq \frac{\gamma}{\mu(\|\tilde{y}_n - z\|)} + \|R(z)\| \leq 1 + \|R(z)\|.$$

Consequently, we obtain

$$\|\tilde{y}_n - z\| \leq \max\left\{\mu^{-1}(\gamma), \varphi^{-1}(1 + \|R(z)\|)\right\} =: c_z.$$

Hence, $\{\tilde{y}_n\}$ is bounded, and $|\beta_n| \leq \delta_n \mu(c_z)$ for all n. Now since R is a bounded operator, $\{R(\tilde{y}_n)\}$ is also bounded. It then turns out from (14) that $\lim_{n\to\infty} \|\tilde{y}_n - T\tilde{y}_n\| = 0$. Since $I-T$ is demiclosed, $\omega_w(\tilde{y}_n) \subset \mathrm{Fix}(T)$. Let $\tilde{y} \in \omega_w(\tilde{y}_n)$ and $\{\tilde{y}_{n_k}\}$ be a subsequence of $\{\tilde{y}_n\}$ weakly converging to $\tilde{y} \in \mathrm{Fix}(T)$. Substituting $\tilde{y}$ for z in (18) yields

$$\mu(\|\tilde{y}_{n_k} - \tilde{y}\|)\varphi(\|\tilde{y}_{n_k} - \tilde{y}\|) \leq (\delta_{n_k}/\varepsilon_{n_k})\mu(c_{\tilde{y}}) - \langle R(\tilde{y}), J_\mu(\tilde{y}_{n_k} - \tilde{y})\rangle.$$

Since $(\delta_{n_k}/\varepsilon_{n_k}) \to 0$ and $J_\mu(\tilde{y}_{n_k} - \tilde{y}) \to 0$ weakly, due to the weak continuity of the duality map J_μ, the last relation ensures $\mu(\|\tilde{y}_{n_k} - \tilde{y}\|)\varphi(\|\tilde{y}_{n_k} - \tilde{y}\|) \to 0$, which in turns implies that $\tilde{y}_{n_k} \to \tilde{y}$ strongly. Returning to (17) via the subsequence $\{\tilde{y}_{n_k}\}$ and using the continuity of R gives that $\langle R(\tilde{y}), J_\mu(\tilde{y} - z)\rangle \leq 0$ for all $z \in \mathrm{Fix}(T)$. That is, $\tilde{y}$ is a solution to the variational inequality (15).

It remains to show that (15) has a unique solution. To see this, we assume $\hat{z}$ and $\tilde{z}$ both are solutions of (15). We then have

$$\langle R(\hat{z}), J(\hat{z}-\tilde{z})\rangle \le 0 \quad \text{and} \quad \langle R(\tilde{z}), J(\tilde{z}-\hat{z})\rangle \le 0.$$

Adding them up and making use of the uniform φ-accretiveness of R yields

$$0 \ge \langle R(\hat{z}) - R(\tilde{z}), J(\hat{z}-\tilde{z})\rangle \ge \varphi(\|\hat{z}-\tilde{z}\|)\cdot\|\hat{z}-\tilde{z}\|.$$

It follows immediately that $\hat{z} = \tilde{z}$ and the solution of the variational inequality (15) is unique. Hence, we have verified that the sequence $\{\tilde{y}_n\}$ converges strongly to the unique solution of (15). □

Take R to be of the form

$$R(x) = x - f(x), \quad x \in C, \tag{19}$$

where $f : C \to C$ is an α-contraction, with $\alpha \in [0,1)$. Namely,

$$\|f(x) - f(y)\| \le \alpha\|x-y\| \quad \forall x, y \in C.$$

Then it is easy to find that R is uniformly φ-accretive with $\varphi(t) = (1-\alpha)t$ for $t \ge 0$ (namely, $I - T$ is strongly accretive):

$$\langle x-y, R(x) - R(y)\rangle \ge (1-\alpha)\|x-y\|^2, \quad x, y \in C.$$

Consequently, the following result is a direct consequence of Theorem 4.

Corollary 1. *Assume X is a reflexive Banach space with a weakly sequentially continuous duality map $J_\mu : X \to X^*$ for some gauge μ, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. Assume $f : C \to C$ is an α-contraction for some $\alpha \in [0,1)$. Assume δ_n and ε_n satisfy the conditions (i) and (ii) in Theorem 3. Then the sequence $\{\tilde{y}_n\}$ of approximate solutions determined by*

$$\|\tilde{y}_n - [(1-\varepsilon_n)T(\tilde{y}_n) + \varepsilon_n f(\tilde{y}_n)]\| \le \delta_n \tag{20}$$

converges strongly to the unique solution $\hat{z}$ of the variational inequality:

$$\langle \hat{z} - f(\hat{z}), J(\hat{z}-z)\rangle \le 0 \quad \forall z \in \mathrm{Fix}(T). \tag{21}$$

Remark 4. *Theorem 4 and Corollary 1 both are applicable to ℓ_p spaces for each $1 < p < \infty$ since the generalized duality map J_μ of ℓ_p with gauge $\mu(t) = t^{p-1}$ is weakly continuous; Gwinner's theorem ([5], Theorem 2) however is not applicable since the normalized duality map J of ℓ_p for $1 < p < \infty$, $p \neq 2$ is not weakly continuous.*

Applying Mann's iteration method to the regularized operator $(1-\varepsilon)(I-T) + \varepsilon R$ with the regularizer R of form (19) leads to the iteration process [5], which we call regularized Mann iteration method:

$$z_{n+1} = \alpha_n(1-\varepsilon_n)Tz_n + \alpha_n\varepsilon_n f(z_n) + (1-\alpha_n)z_n, \tag{22}$$

where the initial point $z_0 \in C$ is arbitrary, $f : C \to C$ is an α-contraction with $\alpha \in [0,1)$, and α_n, $\varepsilon_n \in (0,1)$ for $n \ge 0$.

The convergence of this algorithm depends on the convergence of the approximate scheme (20) as shown below.

Theorem 5. *Assume X is a Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. Assume $R : C \to C$ is an α-contraction with $\alpha \in [0,1)$. Assume (α_n), (ε_n), and (δ_n) satisfy the conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n\varepsilon_n = \infty$,

(C2) $\lim_{n\to\infty} \frac{|\varepsilon_n - \varepsilon_{n-1}|}{\alpha_n\varepsilon_n^2} = 0$,

(C3) $\lim_{n\to\infty}(\delta_n/\varepsilon_n) = 0$,
(C4) $\lim_{n\to\infty} \frac{\delta_n+\delta_{n-1}}{\alpha_n\varepsilon_n^2} = 0$.

Assume the sequence $\{\tilde{y}_n\}$ determined by the approximate equation (20) converges strongly to the unique solution $\hat{z}$ of the variational inequality (21). Then the sequence $\{z_n\}$ generated by the iteration process (22) also converges strongly to the same solution $\hat{z}$ of (21).

Proof. We rewrite (20) as

$$\tilde{y}_n = (1-\varepsilon_n)T(\tilde{y}_n) + \varepsilon_n f(\tilde{y}_n) + e_n, \tag{23}$$

where e_n is the error such that $\|e_n\| \le \delta_n$ for all n. It turns out from (23) that

$$\begin{aligned}\|\tilde{y}_{n+1} - \tilde{y}_n\| &= \|(1-\varepsilon_{n+1})[T(\tilde{y}_{n+1}) - T(\tilde{y}_n)] + \varepsilon_{n+1}[f(\tilde{y}_{n+1}) - f(\tilde{y}_n)] \\ &\quad + (\varepsilon_{n+1}-\varepsilon_n)[f(\tilde{y}_n) - T(\tilde{y}_n)] + e_{n+1} - e_n\| \\ &\le (1-(1-\alpha)\varepsilon_{n+1})\|\tilde{y}_{n+1} - \tilde{y}_n\| + \beta|\varepsilon_{n+1} - \varepsilon_n| + \delta_{n+1} + \delta_n,\end{aligned}$$

where $\beta > 0$ is a constant such that $\beta \ge \|f(\tilde{y}_n) - T(\tilde{y}_n)\|$ for all n. This implies that

$$\|\tilde{y}_{n+1} - \tilde{y}_n\| \le \frac{\beta|\varepsilon_{n+1} - \varepsilon_n| + \delta_{n+1} + \delta_n}{(1-\alpha)\varepsilon_{n+1}}. \tag{24}$$

Using the definition (22) of z_{n+1}, we obtain

$$\begin{aligned}\|z_{n+1} - \tilde{y}_n\| &= \|\alpha_n[(1-\varepsilon_n)Tz_n + \varepsilon_n f(z_n) - \tilde{y}_n] + (1-\alpha_n)(z_n - \tilde{y}_n)\| \\ &= \|\alpha_n[(1-\varepsilon_n)(Tz_n - T\tilde{y}_n) + \varepsilon_n(f(z_n) - f(\tilde{y}_n))] + (1-\alpha_n)(z_n - \tilde{y}_n) - \alpha_n e_n\| \\ &\le \alpha_n[(1-\varepsilon_n)\|z_n - \tilde{y}_n\| + \alpha\varepsilon_n\|z_n - \tilde{y}_n\|] + (1-\alpha_n)\|z_n - \tilde{y}_n\| + \alpha_n\|e_n\| \\ &\le (1-(1-\alpha)\alpha_n\varepsilon_n)\|z_n - \tilde{y}_n\| + \alpha_n\delta_n \\ &\le (1-(1-\alpha)\alpha_n\varepsilon_n)\|z_n - \tilde{y}_{n-1}\| + \|\tilde{y}_n - \tilde{y}_{n-1}\| + \alpha_n\delta_n.\end{aligned} \tag{25}$$

Substituting (24) into (25) yields, for $n \ge 1$,

$$\|z_{n+1} - \tilde{y}_n\| \le (1-(1-\alpha)\alpha_n\varepsilon_n)\|z_n - \tilde{y}_{n-1}\| + \gamma_n, \tag{26}$$

where

$$\gamma_n = \frac{\beta|\varepsilon_n - \varepsilon_{n-1}| + \delta_n + \delta_{n-1}}{(1-\alpha)\varepsilon_n} + \alpha_n\delta_n.$$

The conditions (C2)–(C4) assert that $\gamma_n = o(\alpha_n\varepsilon_n)$. Noticing condition (C1), we can apply Lemma 1 to Equation (26) to conclude $\|z_{n+1} - \tilde{y}_n\| \to 0$. Now since it is assumed that $\tilde{y}_n \to \tilde{z}$, the unique solution of the variational inequality (21), it is also that $z_n \to \tilde{z}$. This finishes the proof. □

Remark 5. *It is not hard to find that the four conditions (C1)–(C4) in Theorem 5 are satisfied for the choices of $\alpha_n = (n+1)^{-p}$, $\varepsilon_n = (n+1)^{-q}$, $\delta_n = (n+1)^{-r}$ for $n \ge 0$, where $p, q, r > 0$ are such that $p + q < 1$ and $r > p + 2q$.*

Remark 6. *To the best of our knowledge, it looks to be the first time in the literature to use the approximate solutions $\tilde{y}_n$ to study strong convergence of the regularized Mann's iteration process (22). Previously, exact solutions y_n were used (see, e.g., [5]). Here y_n is the exact fixed point of the contraction $(1-\varepsilon_n)T(\cdot) + \varepsilon_n f(\cdot)$:*

$$y_n = (1-\varepsilon_n)T(y_n) + \varepsilon_n f(y_n).$$

A key assumption in Theorem 5 is that the sequence $\{\tilde{y}_n\}$ be strongly convergent, which has been proved in a reflexive Banach space with a weakly continuous duality map in Theorem 4. Consequently, we have the following result.

Corollary 2. *Assume X is a reflexive Banach space with a weakly sequentially continuous duality map $J_\mu : X \to X^*$ for some gauge μ (in particular, $X = \ell_p$ for $1 < p < \infty$), C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume $f : C \to C$ is an α-contraction for some $\alpha \in [0, 1)$. Assume the sequence $\{\tilde{y}_n\}$ satisfies (20) and let $\{z_n\}$ be generated by the regularized iteration process (22). Suppose α_n, ε_n, δ_n satisfy the conditions (C1)-(C4) in Theorem 5. Then $\{z_n\}$ converges strongly to the unique solution $\hat{z}$ of the variational inequality (21).*

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