mathematics

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Multiparametric Contractions and Related Hardy-Roger Type Fixed Point Theorems

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Received: 18 May 2020; Accepted: 8 June 2020; Published: 11 June 2020

Abstract: In this paper we present some novel fixed point theorems for a family of contractions depending on two functions (that are not defined on $t = 0$) and on some parameters that we have called multiparametric contractions. We develop our study in the setting of b-metric spaces because they allow to consider some families of functions endowed with b-metrics deriving from similarity measures that are more general than norms. Taking into account that the contractivity condition we will employ is very general (of Hardy-Rogers type), we will discuss the validation and usage of this novel condition. After that, we introduce the main results of this paper and, finally, we deduce some consequences of them which illustrates the wide applicability of the main results.

Keywords: b-metric space; multiparametric contraction; fixed point; contractivity condition; Hardy-Rogers contractivity condition

1. Introduction

The field of Fixed Point Theory has very recently undergone a great development, mainly due to the great number of contractivity conditions, especially in two directions: by considering new terms and by involving auxiliary functions. Let us briefly describe respective examples. Let (X, d) be a metric space and let $\mathsf{f} : X \to X$ be a mapping from X into itself. Starting from the celebrated Banach's contractivity condition [1]:

$$d\left(\mathsf{f}x, \mathsf{f}y\right) \leq \lambda\, d\left(x, y\right) \quad \text{for all } x, y \in X,$$

where $\lambda \in [0, 1)$, an initial extension of the previous assumption was due to Kannan [2]:

$$d\left(\mathsf{f}x, \mathsf{f}y\right) \leq \lambda' \left[\, d\left(\mathsf{f}x, y\right), d\left(x, \mathsf{f}y\right) \right] \quad \text{for all } x, y \in X \qquad (\lambda' \in [0, 1/2)\,).$$

This result allowed to extend Banach's principle to a family of self-mappings that did not need to be continuous. Later, other terms were involved in the contractivity condition, as in the following examples:

$$d\left(\mathsf{f}x, \mathsf{f}y\right) \leq \lambda \max\left\{ d\left(x, y\right), d\left(x, \mathsf{f}x\right), d\left(y, \mathsf{f}y\right), \frac{d\left(\mathsf{f}x, y\right) + d\left(x, \mathsf{f}y\right)}{2} \right\} \quad \text{for all } x, y \in X,$$

or

$$d(\mathsf{f}x,\mathsf{f}y) \leq \lambda \max\{d(x,y),d(x,\mathsf{f}x),d(y,\mathsf{f}y),d(\mathsf{f}x,y),d(x,\mathsf{f}y)\} \quad \text{for all } x,y \in X.$$

Independently, contractivity conditions evolved towards the inclusion of auxiliary functions. A first example in this direction was the Boyd and Wang's contractivity condition [3]:

$$d(\mathsf{f}x,\mathsf{f}y) \leq \phi(d(x,y)) \quad \text{for all } x,y \in X,$$

where $\phi : [0,\infty) \to [0,\infty)$ was a function satisfying key properties (in that case, $\phi(t) < t$ and $\limsup_{r\to t^+} \phi(r) \leq \phi(t)$ for all $t > 0$; this last condition is verified, for instance, by any upper semicontinuous from the right on $(0,\infty)$ function). In recent times, several classes of auxiliary functions have enriched this theory a lot (altering functions [4,5], simulation functions [6,7], R-functions [8–10], etc.).

As a mixture of both lines of research, in 1977 Jaggi [11] introduced the following kind of rational type contractivity condition (where $\alpha,\beta \in (0,1)$ satisfy $\alpha+\beta < 1$):

$$d(\mathsf{f}x,\mathsf{f}y) \leq \alpha\, d(x,y) + \beta\, \frac{d(x,\mathsf{f}x)\; d(y,\mathsf{f}y)}{d(x,y)} \quad \text{for all } x,y \in X \text{ such that } x \neq y.$$

Obviously, such kind of contractivity conditions can only be verified by pairs of distinct points of the metric space (see [12]). This new family of hypotheses allowed the researcher to realize that, in many cases, contractivity conditions became trivial when the pair of points are equal, that is, $x = y$. As a consequence, a lot of results were introduced by assuming that the contractivity condition must be only verified for distinct points. Hence, auxiliary functions $\phi : [0,\infty) \to [0,\infty)$ did not need to be defined in $t = 0$, which led to the fact that recent results only use functions such as $\phi : (0,\infty) \to (0,\infty)$, where $\phi(0)$ does not necessarily exist. However, when we combine several restrictions, it is possible to pose a contractivity condition of type

$$d(\mathsf{f}x,\mathsf{f}y) \leq \phi(M(x,y))$$

in which, for some distinct points $x_0, y_0 \in X$, we can deduce $M(x_0,y_0) = 0$ but ϕ is not defined in $t = 0$. As a consequence, when we apply the contractivity condition, we must take care about the fact that $M(x,y) > 0$.

In this paper, we present some novel fixed point theorems for a family of contractions depending on two functions (that are not defined on $t = 0$) and on some parameters that we have called multiparametric contractions. We develop our study in the setting of b-metric spaces because they are, in our opinion, a very successful context because they allow to consider some important families of functions endowed with b-metrics deriving from similarity measures that are more general than norms. Taking into account that the contractivity condition we will employ is very general and it makes use of functions that are not defined on $t = 0$, we will discuss the validation and use of this condition in Section 3. After that, we introduce the main results of this paper and, finally, we deduce some consequences of them which illustrates the wide applicability of the main results.

2. Background on *b*-Metric Spaces and Fixed Point Theory

Let $\mathbb{N} = \{1,2,3,\ldots\}$ the family of all positive integers. Henceforth, let X be a non-empty set and let $\mathsf{s} \in [1,\infty)$ be a real number.

A b-metric on X is a function $b : X \times X \to [0,\infty)$ satisfying null self-distance ($b(\mathsf{u},\mathsf{u}) = 0$), indistinguishability of indiscernibles (if $b(\mathsf{u},\mathsf{v}) = 0$, then $\mathsf{u} = \mathsf{v}$), symmetry ($b(\mathsf{v},\mathsf{u}) = b(\mathsf{u},\mathsf{v})$) and the following generalized version, involving the number s, of the triangle inequality:

$$b(\mathsf{u},\mathsf{w}) \leq \mathsf{s}\,[\,b(\mathsf{u},\mathsf{v}) + b(\mathsf{v},\mathsf{w})\,] \quad \text{for all } \mathsf{u},\mathsf{v},\mathsf{w} \in X.$$

When $\mathsf{s} = 1$, we recover the notion of metric space. However, the notion of b-metric is more general than the concept of metric (see [13–15]). For instance, in general, a b-metric is not necessarily continuous.

Example 1 ([16–19])**.** *Let (X, d) be a metric space and let $\mathsf{r} > 1$. If we consider the function $d_{\mathsf{r}} : X \times X \to [0, \infty)$ defined by $d_{\mathsf{r}}(x, y) = d(x, y)^{\mathsf{r}}$ for all $x, y \in X$, then $(X, d_{\mathsf{r}}, \mathsf{s})$ forms a b-metric space with $\mathsf{s} = 2^{\mathsf{r}-1}$.*

In a b-metric space (X, b, s), a sequence $\{\mathsf{u}_n\}$ is b-convergent to $u \in X$ if $\lim_{n\to\infty} b(\mathsf{u}_n, \mathsf{u}) = 0$, and it is b-Cauchy if $\lim_{n,m\to\infty} b(x_n, x_m) = 0$. The reader can check that each b-convergent sequence is b-Cauchy. The b-metric space (X, b, s) is complete if each b-Cauchy sequence is b-convergent to a point in X.

Lemma 1 ([20])**.** *Let $\{\mathsf{u}_n\}$ be a sequence of elements in a b-metric space (X, b, s). If there exists $\mathsf{C} \in [0, 1/\mathsf{s})$ such that $b(\mathsf{u}_n, \mathsf{u}_{n+1}) \leq \mathsf{C}b(\mathsf{u}_{n-1}, \mathsf{u}_n)$ for every $n \in \mathbb{N}$, then $\{\mathsf{u}_n\}$ is a b-Cauchy sequence.*

Lemma 2 ([21])**.** *Let $\{\mathsf{u}_n\}$ be a sequence in a b-metric space (X, b, s) such that $\{b(\mathsf{u}_n, \mathsf{u}_{n+1})\} \to 0$ as $n \to \infty$. If the sequence $\{\mathsf{u}_n\}$ is not b-Cauchy, then there exist $\mathsf{e} > 0$ and two partial subsequences $\{\mathsf{u}_{p(r)}\}_{r\in\mathbb{N}}$ and $\{\mathsf{u}_{q(r)}\}_{r\in\mathbb{N}}$ of $\{\mathsf{u}_n\}_{n\in\mathbb{N}}$ such that*

$$p(r) < q(r) < p(r+1) \quad \text{and} \quad \mathsf{e} < b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)+1}) \quad \text{for all } r \in \mathbb{N},$$
$$\lim_{r\to\infty} b(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)+1}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)+1}) = \mathsf{e}.$$

A fixed point of a self-mapping $\mathsf{f} : X \to X$ is an element $\mathsf{u}_0 \in X$ such that $\mathsf{f}\mathsf{u}_0 = \mathsf{u}_0$. We will say that f is fixed-points free if it has not a fixed point. Associated also to the self-mapping f, a sequence $\{\mathsf{u}_n\}$ in X is a Picard sequence of f if $\mathsf{u}_{n+1} = \mathsf{f}\mathsf{u}_n$ for all $n \in \mathbb{N}$.

Following [22], a sequence $\{\mathsf{u}_n\}$ in X is infinite if $\mathsf{u}_n \neq \mathsf{u}_m$ for all $n \neq m$, and $\{\mathsf{u}_n\}$ is almost periodic if there exist $r_0, N \in \mathbb{N}$ such that

$$\mathsf{u}_{r_0+r+Np} = \mathsf{u}_{r_0+r} \quad \text{for all } p \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \ldots, N-1\}.$$

Proposition 1 ([22], Proposition 2.3)**.** *Every Picard sequence is either infinite or almost periodic.*

Proposition 2. *Let $\{\mathsf{u}_r\}$ be a Picard sequence in a b-metric space (X, b, s) such that $\{b(\mathsf{u}_r, \mathsf{u}_{r+1})\} \to 0$. If there are $r_1, r_2 \in \mathbb{N}$ such that $r_1 < r_2$ and $\mathsf{u}_{r_1} = \mathsf{u}_{r_2}$, then there is $r_0 \in \mathbb{N}$ and $\mathsf{u} \in X$ such that $\mathsf{u}_r = \mathsf{u}$ for all $r \geq r_0$ (that is, $\{\mathsf{u}_r\}$ is constant from a term onwards). In such a case, u is a fixed point of the self-mapping for which $\{\mathsf{u}_n\}$ is a Picard sequence.*

Proof. Let $\mathsf{f} : X \to X$ be a mapping for which $\{\mathsf{u}_r\}$ is a Picard sequence. The set

$$\Omega = \{\, k \in \mathbb{N} : \exists\, r_0 \in \mathbb{N} \text{ such that } \mathsf{u}_{r_0} = \mathsf{u}_{r_0+k} \,\}$$

is non-empty because $r_2 - r_1 \in \Omega$, so it has a minimum $k_0 = \min \Omega$. Then $k_0 \geq 1$ and there is $r_0 \in \mathbb{N}$ such that $\mathsf{u}_{r_0} = \mathsf{u}_{r_0+k_0}$. As $\{\mathsf{u}_r\}$ is not infinite, then it must be almost periodic. In fact, it is easy to check, by induction on p, that:

$$\mathsf{u}_{r_0+r+pk_0} = \mathsf{u}_{r_0+r} \quad \text{for all } p \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \ldots, k_0-1\}. \tag{1}$$

If $k_0 = 1$, then $\mathsf{u}_{r_0} = \mathsf{u}_{r_0+1}$. Similarly $\mathsf{u}_{r_0+2} = \mathsf{f}\mathsf{u}_{r_0+1} = \mathsf{f}\mathsf{u}_{r_0} = \mathsf{u}_{r_0+1}$. By induction, $\mathsf{u}_{r_0+r} = \mathsf{u}_{r_0}$ for all $r \geq 0$, which is precisely the conclusion. Next we are going to prove that the case $k_0 \geq 2$ leads to a contradiction.

Assume that $k_0 \geq 2$. Then all two terms in the set $\{ \mathsf{u}_{r_0}, \mathsf{u}_{r_0+1}, \mathsf{u}_{r_0+2}, \ldots, \mathsf{u}_{r_0+k_0-1} \}$ are distinct, that is, $\mathsf{u}_{r_0+i} \neq \mathsf{u}_{r_0+j}$ for all $0 \leq i < j \leq k_0 - 1$ (on the contrary case, k_0 is not the minimum of Ω). Let define

$$\mathsf{e}_0 = \frac{\min\left(\{ b(\mathsf{u}_{r_0+i}, \mathsf{u}_{r_0+i+1}) : 0 \leq i \leq k_0 - 1 \}\right)}{2}.$$

Then $\mathsf{e}_0 > 0$. Since $\{b(\mathsf{u}_r, \mathsf{u}_{r+1})\} \to 0$, there is $m_0 \in \mathbb{N}$ such that $m_0 \geq r_0$ and $b(\mathsf{u}_{m_0}, \mathsf{u}_{m_0+1}) < \mathsf{e}_0$. Let $i_0 \in \{0, 1, 2, \ldots, k_0 - 1\}$ the unique integer number such that the non-negative integer numbers $m_0 - r_0$ and i_0 are congruent modulo k_0, that is, i_0 is the rest of the integer division of $m_0 - r_0$ over k_0. Hence there is a unique integer $p \geq 0$ such that $(m_0 - r_0) - i_0 = pk_0$. Since $m_0 = r_0 + i_0 + pk_0$, property (1) guarantees that

$$\mathsf{u}_{m_0} = \mathsf{u}_{r_0+i_0+pk_0} = \mathsf{u}_{r_0+i_0},$$

where $r_0 + i_0 \in \{r_0, r_0 + 1, r_0 + 2, \ldots, r_0 + k_0 - 1\}$. As a consequence:

$$2\mathsf{e}_0 = \min\left(\{ b(\mathsf{u}_{r_0+i}, \mathsf{u}_{r_0+i+1}) : 0 \leq i \leq k_0 - 1 \}\right) \leq b(\mathsf{u}_{r_0+i_0}, \mathsf{u}_{r_0+i_0+1}) = b(\mathsf{u}_{m_0}, \mathsf{u}_{m_0+1}) < \mathsf{e}_0,$$

which is a contradiction. □

Corollary 1. *Let (X, b, s) be a b-metric space and let $\{\mathsf{u}_r\} \subseteq X$ be a Picard sequence of* f *such that $\{b(\mathsf{u}_r, \mathsf{u}_{r+1})\} \to 0$. If* f *is fixed-points free, then $\{\mathsf{u}_r\}$ is infinite (that is, $\mathsf{u}_r \neq \mathsf{u}_{r'}$ for all $r \neq r'$).*

Remark 1. *If $\psi : (0, \infty) \to \mathbb{R}$ is a non-decreasing function and $t, r \in (0, \infty)$ are such that $\psi(t) < \psi(s)$, then $t < s$.*

Given $t_0 > 0$, we will use the notation $\psi(t_0^+)$ to stand the lateral limit $\lim_{t \to t_0^+} \psi(t)$ (if it exists), that is, a limit taken on values verifying $t > t_0$. We also consider the limit superior $\limsup_{t \to t_0^+} \phi(t)$, which is the greatest limit of the images by ϕ of any strictly decreasing sequence in the interval (t_0, ∞) converging to t_0.

3. Discussion on the Contractivity Condition

As we have pointed out in the introduction, the contractivity condition we will employ is as general that, for the sake of clarity, we have to previously discuss about how it must be correctly applied. We set our study in the context of b-metric spaces. In the following definition, we introduce the algebraic tools we will handle in order to complete this study.

Definition 1. *Let (X, b, s) be a b-metric space, let $\mathsf{f} : X \to X$ be a self-mapping and let $\varkappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ be a set of five non-negative real numbers.We will denote by*

$$\mathsf{A}_\mathsf{f} : X \times X \to [0, \infty)$$

to the function defined, for all $x, y \in X$, by:

$$\mathsf{A}_\mathsf{f}(x, y) = \kappa_1 b(x, y) + \kappa_2 b(x, \mathsf{f}x) + \kappa_3 b(y, \mathsf{f}y) + \kappa_4 b(x, \mathsf{f}y) + \kappa_5 b(y, \mathsf{f}x). \quad (2)$$

Given two auxiliary functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ and a real number $q \in [1, \infty)$, we will say that f *is a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on (X, b, s) if*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \phi(\mathsf{A}_\mathsf{f}(x, y)) \quad \textit{for all } x, y \in X \textit{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0. \quad (3)$$

On the one hand, notice that function A_f depends on the b-metric b, on the function f and on the constants of the set $\varkappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$. However, we center our attention on the dependence w.r.t.

f because the main aim of fixed point theory is to introduce fixed point result for an operator $\mathsf{f} : X \to X$ (if we have removed f from A_f, then reader would not been able to appreciate the dependence on f on the right-hand side of the contractivity condition (3)). Furthermore, the function A_f makes that (3) is known as a Hardy-Rogers type contractivity condition. In addition to this, this function is not necessarily symmetric, so some results can be optimized later. Indeed, our contractions satisfy:

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \min\left\{\phi(\mathsf{A}_\mathsf{f}(x,y)), \phi(\mathsf{A}_\mathsf{f}(x,y))\right\}.$$

On the other hand, the contractivity condition (3) depends on a function $\phi : (0,\infty) \to \mathbb{R}$ which is not defined for $t = 0$, so its applicability needs to only consider pairs of points x and y for which $\mathsf{A}_\mathsf{f}(x,y) > 0$. Is the condition $b(\mathsf{f}x, \mathsf{f}y) > 0$ strong enough in order to guarantee that $\mathsf{A}_\mathsf{f}(x,y) > 0$? The response is not. The condition $b(\mathsf{f}x, \mathsf{f}y) > 0$ guarantees that x and y are distinct because $\mathsf{f}x \neq \mathsf{f}y$. However, we cannot guarantee that $\mathsf{A}_\mathsf{f}(x,y) > 0$ when $x \neq y$. For instance, when $\kappa_i = 0$ for all $i \in \{1,2,3,4,5\}$ then $\mathsf{A}_\mathsf{f}(x,y) = 0$. In such a case, we cannot apply assumption (3) because the domain of function $\phi : (0,\infty) \to \mathbb{R}$ is the family of all strictly positive real numbers, and the evaluation $\phi(\mathsf{A}_\mathsf{f}(x,y))$ is meaningless. Furthermore, although $\kappa_1 = \kappa_2 = \kappa_3 = 0$ and $\kappa_4, \kappa_5 > 0$, it is possible that $\mathsf{A}_\mathsf{f}(x,y) = 0$, as we show in the following result.

Proposition 3. *Let (X, b, s) be a b-metric space, let $\mathsf{f} : X \to X$ be a mapping and let $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \geq 0$ be five non-negative real numbers. Suppose that there are two distinct points $x_0, y_0 \in X$ such that $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$, where A_f is defined in (2). Then $\kappa_1 = 0$ and at least one of the following four statements hold.*

1. *$\kappa_i = 0$ for all $i \in \{1,2,3,4,5\}$. In this case, A_f is constantly 0.*
2. *$\kappa_2 \neq 0$ and x_0 is a fixed point of f.*
3. *$\kappa_3 \neq 0$ and y_0 is a fixed point of f.*
4. *$\kappa_1 = \kappa_2 = \kappa_3 = 0$ and at least one of κ_4 and κ_5 is strictly positive. In such case, if $\kappa_4 \neq 0$ then $\mathsf{f}x_0 = y_0$, and if $\kappa_5 \neq 0$ then $\mathsf{f}y_0 = x_0$. As a consequence, if κ_4 and κ_5 are strictly positive at the same time, then x_0 and y_0 are distinct fixed points of f^2.*

Proof. If $\kappa_i = 0$ for all $i \in \{1,2,3,4,5\}$, then the first case holds. For the contrary case, assume that some κ_i is distinct to zero. Since $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$ and $\kappa_i \geq 0$ for all $i \in \{1,2,3,4,5\}$, then

$$\kappa_1 b(x_0, y_0) = \kappa_2 b(x_0, \mathsf{f}x_0) = \kappa_3 b(y_0, \mathsf{f}y_0) = \kappa_4 b(x_0, \mathsf{f}y_0) = \kappa_5 b(y_0, \mathsf{f}x_0) = 0.$$

Since $b(x_0, y_0) > 0$, then necessarily $\kappa_1 = 0$. If $\kappa_2 \neq 0$, then $b(x_0, \mathsf{f}x_0) = 0$, so x_0 is a fixed point of f and the second case holds. Next assume that $\kappa_2 = 0$. Similarly, if $\kappa_3 \neq 0$, then $b(y_0, \mathsf{f}y_0) = 0$, so y_0 is a fixed point of f and the third case holds. Next assume that $\kappa_3 = 0$. Since $\kappa_1 = \kappa_2 = \kappa_3 = 0$, then either κ_4 or κ_5 does not vanish. If $\kappa_4 \neq 0$, then $b(x_0, \mathsf{f}y_0) = 0$, so $\mathsf{f}y_0 = x_0$. Similarly, if $\kappa_5 \neq 0$, then $b(y_0, \mathsf{f}x_0) = 0$, so $\mathsf{f}x_0 = y_0$. Finally, if κ_4 and κ_5 are strictly positive at the same time, then $\mathsf{f}^2(x_0) = \mathsf{f}(\mathsf{f}x_0) = \mathsf{f}y_0 = x_0$ and $\mathsf{f}^2(y_0) = \mathsf{f}(\mathsf{f}y_0) = \mathsf{f}x_0 = y_0$, so x_0 and y_0 are distinct fixed points of f^2, and the fourth case holds. □

The previous proposition let us to imagine a case in which f is fixed-points free although it satisfies the contractivity condition (3).

Example 2. *Let $X = \{x_0, y_0\}$, where $x_0 \neq y_0$, and let define $\mathsf{f} : X \to X$ by $\mathsf{f}x_0 = y_0$ and $\mathsf{f}y_0 = x_0$. Then f is fixed-points free. However, if $\kappa_1 = \kappa_2 = \kappa_3 = 0$, then $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$ whatever the values of κ_4 and κ_5. Hence the contractivity condition (3) is empty, so it is not useful in order to guarantee the existence of fixed points of f.*

A simple way to guarantee that $\mathsf{A}_\mathsf{f}(x,y) > 0$ for all $x, y \in X$ such that $b(\mathsf{f}x, \mathsf{f}y) > 0$ follows from the assumption that $\kappa_1 \neq 0$. Anyway, although $\kappa_1 = 0$, the equality $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$ implies that x_0 or y_0

is a fixed point of f when $\kappa_2 \neq 0$ or $\kappa_3 \neq 0$, respectively. Therefore, in such a case, the existence of a fixed point of f is guaranteed.

Corollary 2. *Let (X, b, s) be a b-metric space, let $\mathsf{f} : X \to X$ be a mapping, let $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \geq 0$ be five non-negative real numbers and let A_f be defined as in (2).*

- *If $\kappa_1 > 0$, then $\mathsf{A}_\mathsf{f}(x, y) > 0$ for all distinct points $x, y \in X$.*
- *If $\kappa_2 > 0$ and there are $x_0, y_0 \in X$ such that $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$, then x_0 is a fixed point of f.*
- *If $\kappa_3 > 0$ and there are $x_0, y_0 \in X$ such that $\mathsf{A}_\mathsf{f}(x_0, y_0) = 0$, then y_0 is a fixed point of f.*
- *If $\kappa_1 + \kappa_2 + \kappa_3 > 0$, then either f admits a fixed point or $\mathsf{A}_\mathsf{f}(x, y) > 0$ for all distinct points $x, y \in X$.*

Corollary 3. *Let (X, b, s) be a b-metric space, let $\mathsf{f} : X \to X$ be a mapping, let $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \geq 0$ be five non-negative real numbers and let A_f be defined as in (2). Suppose that f is fixed-points free. If $\kappa_1 + \kappa_2 + \kappa_3 > 0$, then $\mathsf{A}_\mathsf{f}(x, y) > 0$ for all distinct points $x, y \in X$.*

4. Fixed Point Theory for Multiparametric Contractions in the Setting in *b*-Metric Spaces

In the previous section, we have described the cautions we must observe when applying the contractivity condition (3). In this section, we introduce the main results of this paper. To reach this objective, we need to impose some appropriate conditions on the auxiliary functions $\psi, \phi : (0, \infty) \to \mathbb{R}$. Inspired by some results in [21], the restrictions we will consider are the following:

(c_0) $\phi(t) < \psi(t)$ for any $t > 0$;
(c_1) ψ is nondecreasing;
(c_2) $\limsup_{t \to t_0^+} \phi(t) < \psi(t_0^+)$ for any $t_0 > 0$.

We start this study by introducing a common result in which we describe sufficient conditions in order to guarantee that the fixed point, if it exists, it is unique.

Theorem 1. *Let $\mathsf{f} : X \to X$ be a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space (X, b, s). If the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0) and (c_1), and*

$$0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q, \tag{4}$$

then f admits, at most, a unique fixed point.

Proof. To prove the uniqueness, suppose that f admits two distinct fixed points, that is, there are $x_*, x_{**} \in X$ such that $\mathsf{f}x_* = x_* \neq x_{**} = \mathsf{f}x_{**}$. Then $b(x_*, x_{**}) > 0$ and

$$\begin{aligned} \mathsf{A}_\mathsf{f}(x_*, x_{**}) &= \kappa_1 b(x_*, x_{**}) + \kappa_2 b(x_*, \mathsf{f}x_*) + \kappa_3 b(x_{**}, \mathsf{f}x_{**}) + \kappa_4 b(x_*, \mathsf{f}x_{**}) + \kappa_5 b(x_{**}, \mathsf{f}x_*) \\ &= \kappa_1 b(x_*, x_{**}) + \kappa_4 b(x_*, x_{**}) + \kappa_5 b(x_{**}, x_*) \\ &= (\kappa_1 + \kappa_4 + \kappa_5)\, b(x_*, x_{**}). \end{aligned}$$

Therefore $\mathsf{A}_\mathsf{f}(x_*, x_{**}) > 0$ because $\kappa_1 + \kappa_4 + \kappa_5 > 0$ and $b(x_*, x_{**}) > 0$. Hence the contractivity condition (3) can be applied because $b(\mathsf{f}x_*, \mathsf{f}x_{**}) = b(x_*, x_{**}) > 0$, and it guarantees that

$$\psi(\mathsf{s}^q b(\mathsf{f}x_*, \mathsf{f}x_{**})) \leq \phi(\mathsf{A}_\mathsf{f}(x_*, x_{**})).$$

As a consequence, assumptions (4), (c_0) and (c_1) lead to

$$\begin{aligned} \psi(\mathsf{s}^q\, b(x_*, x_{**})) &= \psi(\mathsf{s}^q b(\mathsf{f}x_*, \mathsf{f}x_{**})) \leq \phi(\mathsf{A}_\mathsf{f}(x_*, x_{**})) \\ &= \phi((\kappa_1 + \kappa_4 + \kappa_5)\, b(x_*, x_{**})) \\ &< \psi((\kappa_1 + \kappa_4 + \kappa_5)\, b(x_*, x_{**})) \leq \psi(\mathsf{s}^q\, b(x_*, x_{**})), \end{aligned}$$

which is a contradiction. Hence we can conclude that f admits, at most, a unique fixed point. □

In the following results, the uniqueness of the fixed point will be deduced from Theorem 1 after firstly proving the existence of such kind of points. In this sense, we introduce now our first main theorem.

Theorem 2. *Let $\mathsf{f}: X \to X$ be a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space (X, b, s). If the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0) and (c_1), and the numbers in $\varkappa$ verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3 \quad \text{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < 1,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

As some arguments of the following proof can be repeated under distinct global hypotheses, we divide the proof into some steps in order to recall them later (in particular, steps 1 and 2 only depend on the notion of $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space).

Proof. We reason by reductio ad absurdum assuming that f is fixed-points free and getting a contradiction.

Step 1. $\mathsf{A}_\mathsf{f}(x, y) > 0$ for all distinct points $x, y \in X$.

If follows from Corollary 3 taking into account that $\kappa_1 + \kappa_2 + \kappa_3 > 0$ and f is fixed-points free.

Let ω be an arbitrary point in X and let $\{\mathsf{u}_n\}$ be a sequence defined as follows:

$$\mathsf{u}_1 = \mathsf{f}\omega, \ \mathsf{u}_2 = \mathsf{f}\mathsf{u}_1, \ \ldots, \ \mathsf{u}_n = \mathsf{f}\mathsf{u}_{n-1}, \ldots$$

for any $n \in \mathbb{N}$.

Step 2. For all $n \geq 2$, $\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1}, \mathsf{u}_n) > 0$ and

$$0 < \mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1}) < (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}). \tag{5}$$

To prove it, observe that $\mathsf{u}_n \neq \mathsf{u}_{n+1}$ for all $n \in \mathbb{N}$ because we assume that f is fixed-points free, and also $\mathsf{A}_\mathsf{f}(\mathsf{u}_n, \mathsf{u}_{n+1}) > 0$ for all $n \in \mathbb{N}$ because of Step 1. Notice that

$$\begin{aligned} 0 < \mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1}, \mathsf{u}_n) &= \kappa_1 b(\mathsf{u}_{n-1}, \mathsf{u}_n) + \kappa_2 b(\mathsf{u}_{n-1}, \mathsf{f}\mathsf{u}_{n-1}) + \kappa_3 b(\mathsf{u}_n, \mathsf{f}\mathsf{u}_n) + \\ &\quad + \kappa_4 b(\mathsf{u}_{n-1}, \mathsf{f}\mathsf{u}_n) + \kappa_5 b(\mathsf{u}_n, \mathsf{f}\mathsf{u}_{n-1}) \\ &= \kappa_1 b(\mathsf{u}_{n-1}, \mathsf{u}_n) + \kappa_2 b(\mathsf{u}_{n-1}, \mathsf{u}_n) + \kappa_3 b(\mathsf{u}_n, \mathsf{u}_{n+1}) + \\ &\quad + \kappa_4 b(\mathsf{u}_{n-1}, \mathsf{u}_{n+1}) + \kappa_5 b(\mathsf{u}_n, \mathsf{u}_n) \qquad (6) \\ &\leq \kappa_1 b(\mathsf{u}_{n-1}, \mathsf{u}_n) + \kappa_2 b(\mathsf{u}_{n-1}, \mathsf{u}_n) + \kappa_3 b(\mathsf{u}_n, \mathsf{u}_{n+1}) + \\ &\quad + \mathsf{s}\kappa_4 \left[b(\mathsf{u}_{n-1}, \mathsf{u}_n) + b(\mathsf{u}_n, \mathsf{u}_{n+1}) \right] \\ &= (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}). \qquad (7) \end{aligned}$$

Letting $x = \mathsf{u}_{n-1}$ and $y = \mathsf{u}_n$ in (3) for some $n \geq 2$, and taking into account that $b(\mathsf{f}\mathsf{u}_{n-1}, \mathsf{f}\mathsf{u}_n) = b(\mathsf{u}_n, \mathsf{u}_{n+1}) > 0$,

$$\psi(\mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1})) = \psi(\mathsf{s}^q b(\mathsf{f}\mathsf{u}_{n-1}, \mathsf{f}\mathsf{u}_n)) \leq \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1}, \mathsf{u}_n)).$$

As the argument of ϕ in the right-hand term is strictly positive, then (c_0) and the nondecreasing character of ψ lead to

$$\begin{aligned} \psi(\mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1})) &\leq \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1}, \mathsf{u}_n)) \\ &< \psi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1}, \mathsf{u}_n)) \\ &\leq \psi\left((\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}) \right). \end{aligned} \tag{8}$$

Since ψ is nondecreasing by (c_1), then we deduce that

$$0 < \mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1}) < (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}),$$

so Step 2 is completed.

Step 3. We claim that $\kappa_3 + \mathsf{s}\kappa_4 < 1$ and

$$b(\mathsf{u}_n, \mathsf{u}_{n+1}) < \mathsf{C}_0\, b(\mathsf{u}_{n-1}, \mathsf{u}_n) \quad \text{for all } n \geq 2, \tag{9}$$

where

$$\mathsf{C}_0 = \frac{\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4}{s^q - \kappa_3 - \mathsf{s}\kappa_4} \in \left(0, \frac{1}{\mathsf{s}}\right).$$

At this moment of the proof, we use that $\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < 1$ for the first time. This inequality is equivalent to

$$0 \leq \mathsf{s}\ (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) < 1 - \kappa_3 - \mathsf{s}\kappa_4,$$

which means that $1 - \kappa_3 - \mathsf{s}\kappa_4 > 0$. Furthermore, from (5) and $\kappa_3 + \mathsf{s}\kappa_4 < 1 \leq \mathsf{s}^q$ we deduce that

$$0 < (\mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4)\, b(\mathsf{u}_n, \mathsf{u}_{n+1}) < (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n), \tag{10}$$

which leads to (9). Notice that $\mathsf{C}_0 > 0$ becuase the inequality (10) is strict. Furthermore:

$$\begin{aligned} \mathsf{C}_0 < \tfrac{1}{\mathsf{s}} \quad &\Leftrightarrow \quad \tfrac{\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4}{\mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4} < \tfrac{1}{\mathsf{s}} \quad \Leftrightarrow \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \mathsf{s}^2\kappa_4 < \mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4 \\ &\Leftrightarrow \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < \mathsf{s}^q, \end{aligned} \tag{11}$$

which holds because we assume that $\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < 1$.

Step 4. The sequence $\{\mathsf{u}_n\}$ converges to a point of $x_* \in X$ such that $b(x_*, \mathsf{f}x_*) \leq \limsup_{n\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*) < b(x_*, \mathsf{f}x_*)$ (which is a contradiction).

Step 3 and Lemma 1 ensure that $\{\mathsf{u}_n\}$ is a Cauchy sequence in (X, b, s) and, as it is complete, there is $x_* \in X$ such that $\{b(\mathsf{u}_n, x_*)\} \to 0$. In particular, $\{b(\mathsf{u}_n, \mathsf{u}_{n+1})\} \to 0$. Since we suppose that f is fixed-points free, then $b(x_*, \mathsf{f}x_*) > 0$. If the cardinal of the set

$$\{\, n \in \mathbb{N} : \mathsf{u}_n = x_* \,\}$$

is infinite, then there is a partial subsequence $\{\mathsf{u}_{n(k)}\}_{k\in\mathbb{N}}$ of $\{\mathsf{u}_n\}$ such that $\mathsf{u}_{n(k)+1} = \mathsf{f}\mathsf{u}_{n(k)} = \mathsf{f}x_*$ for all $k \in \mathbb{N}$, so $\{\mathsf{u}_{n(k)+1}\}$ converges, at the same time, to x_* and $\mathsf{f}x_*$, which is impossible because $x_* \neq \mathsf{f}x_*$. As a consequence, there is $n_0 \in \mathbb{N}$ such that $\mathsf{u}_n \neq x_*$ for all $n \geq n_0$. In order not to complicate the notation, without loss of generality, suppose that $\mathsf{u}_n \neq x_*$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 < \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*) &= \kappa_1 b(\mathsf{u}_n, x_*) + \kappa_2 b(\mathsf{u}_n, \mathsf{f}\mathsf{u}_n) + \kappa_3 b(x_*, \mathsf{f}x_*) + \\ &\qquad + \kappa_4 b(\mathsf{u}_n, \mathsf{f}x_*) + \kappa_5 b(\mathsf{f}\mathsf{u}_n, x_*) \\ &= \kappa_1 b(\mathsf{u}_n, x_*) + \kappa_2 b(\mathsf{u}_n, \mathsf{u}_{n+1}) + \kappa_3 b(x_*, \mathsf{f}x_*) + \\ &\qquad + \kappa_4 b(\mathsf{u}_n, \mathsf{f}x_*) + \kappa_5 b(\mathsf{u}_{n+1}, x_*) \\ &\leq \kappa_1 b(\mathsf{u}_n, x_*) + \kappa_2 b(\mathsf{u}_n, \mathsf{u}_{n+1}) + \kappa_3 b(x_*, \mathsf{f}x_*) + \\ &\qquad + \mathsf{s}\kappa_4 b(\mathsf{u}_n, x_*) + \mathsf{s}\kappa_4 b(x_*, \mathsf{f}x_*) + \kappa_5 b(\mathsf{u}_{n+1}, x_*) \\ &= (\kappa_1 + \mathsf{s}\kappa_4)\, b(\mathsf{u}_n, x_*) + \kappa_2 b(\mathsf{u}_n, \mathsf{u}_{n+1}) + \kappa_5 b(\mathsf{u}_{n+1}, x_*) \\ &\qquad + (\kappa_3 + \mathsf{s}\kappa_4)\, b(x_*, \mathsf{f}x_*). \end{aligned}$$

In particular, the limit superior $\limsup_{n\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*)$ exits, and it satisfies:

$$\limsup_{n\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*) \leq (\kappa_3 + \mathsf{s}\kappa_4)\, b(x_*, \mathsf{f}x_*) < b(x_*, \mathsf{f}x_*).$$

On the other hand, by (8),

$$\psi\left(\mathsf{s}b(\mathsf{u}_{n+1}, \mathsf{f}x_*)\right) \leq \psi(\mathsf{s}^q b(\mathsf{f}\mathsf{u}_n, \mathsf{f}x_*)) \leq \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*)) < \psi(\mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*)).$$

Hence

$$\mathsf{s}b(\mathsf{u}_{n+1}, \mathsf{f}x_*) < \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*).$$

Since

$$b(x_*, \mathsf{f}x_*) \leq \mathsf{s}b(x_*, \mathsf{u}_{n+1}) + \mathsf{s}b(\mathsf{u}_{n+1}, \mathsf{f}x_*) < \mathsf{s}b(x_*, \mathsf{u}_{n+1}) + \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*),$$

then

$$b(x_*, \mathsf{f}x_*) \leq \limsup_{n\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_n, x_*) < b(x_*, \mathsf{f}x_*),$$

which is a contradiction.

This general contradiction proves that f necessarily admits a fixed point. The uniqueness of the fixed point follows from Theorem 1. □

There is a particularly simple case that we want to highlight in the following result.

Corollary 4. *Let $\mathsf{f} : X \to X$ be a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space (X, b, s). If the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0) and (c_1), and the numbers in $\varkappa$ verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3 \quad \textit{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 < \frac{1}{\mathsf{s}},$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

Proof. Under these assumptions,

$$\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 \leq \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \mathsf{s}\kappa_3 + 2\mathsf{s}^2\kappa_4 = \mathsf{s}\,(\kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4) < \mathsf{s}\,\frac{1}{\mathsf{s}} = 1,$$

so Theorem 2 is applicable. □

Next we relax the inequality $\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + (\mathsf{s} + \mathsf{s}^2)\,\kappa_4 < 1$ by the weaker one

$$\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < \mathsf{s}^q.$$

However, we additionally need to assume that $\kappa_3 + \mathsf{s}\kappa_4 < 1$. As a consequence, although their proofs employ the same arguments, the following result is independent from Theorem 2.

Theorem 3. *Let $\mathsf{f} : X \to X$ be a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space (X, b, s). If the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0) and (c_1), and the numbers in $\varkappa$ verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1 \quad \textit{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < \mathsf{s}^q,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

Proof. We also reason by contradiction. Assume that f is fixed-points free. In such a case, Steps 1 and 2 of the proof of Theorem 2 also hold, so $\mathsf{A}_{\mathsf{f}}(x,y) > 0$ for all distinct points $x, y \in X$ and

$$0 < \mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1}) < (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}).$$

As we are now supposing that $\kappa_3 + \mathsf{s}\kappa_4 < 1$, then $0 < 1 - \kappa_3 - \mathsf{s}\kappa_4 \leq \mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4$, so the last inequality also lead to

$$b(\mathsf{u}_n, \mathsf{u}_{n+1}) < \mathsf{C}_0\, b(\mathsf{u}_{n-1}, \mathsf{u}_n) \quad \text{for all } n \geq 2$$

where $\mathsf{C}_0 = (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) \,/\, (\mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4)$. Furthermore, inequality $\mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < \mathsf{s}^q$ is equivalent to $\mathsf{C}_0 \in (0, 1/\mathsf{s})$ as we demonstrated in (11). Therefore, Steps 3 and 4 of the proof of Theorem 2 can be identically repeated, so we get a contradiction. Hence f has at least one fixed point. □

In the next result we accept the equality in an inequality inspired in Corollary 4. This fact leads to $\mathsf{C}_0 \in (0,1)$, which is not strong enough to guarantee that the sequence $\{\mathsf{u}_n\}$ is Cauchy in (X, b, s). Hence we need to include an additional assumption on the auxiliary functions ψ and ϕ.

Theorem 4. *Let $\mathsf{f} : X \to X$ be a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on a b-metric space (X, b, s). If the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0), (c_1) and (c_2), and the numbers in $\varkappa$ verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\textit{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

Proof. As in the proof of Theorem 3, we also reason by contradiction. Assume that f is fixed-points free. In such a case, Steps 1 and 2 of the proof of Theorem 2 also hold, so

$$\mathsf{A}_{\mathsf{f}}(x,y) > 0 \text{ for all distinct points } x, y \in X \tag{12}$$

and

$$0 < \mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1}) < (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}).$$

As we are now supposing that $\kappa_3 + \mathsf{s}\kappa_4 < 1$, then $0 < 1 - \kappa_3 - \mathsf{s}\kappa_4 \leq \mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4$, so the last inequality also lead to

$$b(\mathsf{u}_n, \mathsf{u}_{n+1}) < \mathsf{C}_0\, b(\mathsf{u}_{n-1}, \mathsf{u}_n) \quad \text{for all } n \geq 2$$

where $\mathsf{C}_0 = (\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) \,/\, (\mathsf{s}^q - \kappa_3 - \mathsf{s}\kappa_4)$. Inequality $\kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q$ is equivalent to say that $\mathsf{C}_0 \in (0,1)$, so the last property becomes

$$b(\mathsf{u}_n, \mathsf{u}_{n+1}) < b(\mathsf{u}_{n-1}, \mathsf{u}_n) \quad \text{for all } n \geq 2.$$

Let $\gamma \geq 0$ the limit of the strictly decreasing sequence $\{b(\mathsf{u}_n, \mathsf{u}_{n+1})\}$. To prove that $\gamma = 0$, suppose that $\gamma > 0$. Let $t_0 = \mathsf{s}\gamma > 0$. As Steps 1 and 2 of Theorem 2 are now valid, recall that (8) assures that

$$\begin{aligned} \psi(\mathsf{s}^q b(\mathsf{u}_n, \mathsf{u}_{n+1})) &\leq \phi(\mathsf{A}_{\mathsf{f}}(\mathsf{u}_{n-1}, \mathsf{u}_n)) < \psi(\mathsf{A}_{\mathsf{f}}(\mathsf{u}_{n-1}, \mathsf{u}_n)) \\ &\leq \psi\left((\kappa_1 + \kappa_2 + \mathsf{s}\kappa_4) b(\mathsf{u}_{n-1}, \mathsf{u}_n) + (\kappa_3 + \mathsf{s}\kappa_4) b(\mathsf{u}_n, \mathsf{u}_{n+1}) \right), \end{aligned}$$

which leads, by (6) and (7), to

$$\begin{aligned}
\mathsf{s}^q\, b(\mathsf{u}_n,\mathsf{u}_{n+1}) &< \mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)\\
&\leq (\kappa_1+\kappa_2+\mathsf{s}\kappa_4)b(\mathsf{u}_{n-1},\mathsf{u}_n)+(\kappa_3+\mathsf{s}\kappa_4)b(\mathsf{u}_n,\mathsf{u}_{n+1})\\
&< (\kappa_1+\kappa_2+\mathsf{s}\kappa_4)b(\mathsf{u}_{n-1},\mathsf{u}_n)+(\kappa_3+\mathsf{s}\kappa_4)b(\mathsf{u}_{n-1},\mathsf{u}_n)\\
&= (\kappa_1+\kappa_2+\kappa_3+2\mathsf{s}\kappa_4)b(\mathsf{u}_{n-1},\mathsf{u}_n)\\
&\leq \mathsf{s}^q\, b(\mathsf{u}_{n-1},\mathsf{u}_n)
\end{aligned}$$

As the sequences $\{b(\mathsf{u}_n,\mathsf{u}_{n+1})\}$ and $\{b(\mathsf{u}_{n-1},\mathsf{u}_n)\}$ are strictly decreasing and converging to $\gamma > 0$, then the sequence $\{\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)\}$ satisfies $t_0 = \mathsf{s}\gamma < \mathsf{s}^q\, b(\mathsf{u}_n,\mathsf{u}_{n+1}) < \mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)$ for all $n \in \mathbb{N}$ and also

$$\lim_{n\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n) = \mathsf{s}\gamma = t_0 > 0.$$

Letting $n \to \infty$ in

$$\begin{aligned}
\psi(\mathsf{s}^q b(\mathsf{u}_n,\mathsf{u}_{n+1})) &\leq \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)) < \psi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n))\\
&\leq \psi\left((\kappa_1+\kappa_2+\mathsf{s}\kappa_4)b(\mathsf{u}_{n-1},\mathsf{u}_n)+(\kappa_3+\mathsf{s}\kappa_4)b(\mathsf{u}_n,\mathsf{u}_{n+1})\right)\\
&\leq \psi\left(\mathsf{s}^q\, b(\mathsf{u}_{n-1},\mathsf{u}_n)\right),
\end{aligned}$$

we deduce that

$$\lim_{n\to\infty} \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)) = \lim_{n\to\infty} \psi\left(\mathsf{s}^q\, b(\mathsf{u}_{n-1},\mathsf{u}_n)\right) = \lim_{t\to t_0^+} \psi\left(t\right) = \psi\left(t_0^+\right).$$

However, condition (c_2) means that

$$\lim_{n\to\infty} \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{n-1},\mathsf{u}_n)) \leq \limsup_{t\to t_0^+} \phi(t) < \psi(t_0^+),$$

which is a contradiction. This contradiction permit us tu deduce that $\gamma = 0$, so $\{b(\mathsf{u}_n,\mathsf{u}_{n+1})\} \to 0$.

Next, let show that $\{\mathsf{u}_n\}$ is a Cauchy sequence in (X, b, s) by contradiction. If it is not Cauchy, Lemma 2 demonstrates that there exist $\mathsf{e} > 0$ and subsequences $\{\mathsf{u}_{p(r)}\}_{r\in\mathbb{N}}$ and $\{\mathsf{u}_{q(r)}\}_{r\in\mathbb{N}}$ of $\{\mathsf{u}_n\}_{n\in\mathbb{N}}$ such that

$$p\,(r) < q\,(r) < p\,(r+1) \quad \text{and} \quad \mathsf{e} < b(\mathsf{u}_{p(r)+1},\mathsf{u}_{q(r)+1}) \quad \text{for all } r \in \mathbb{N}, \tag{13}$$

$$\lim_{r\to\infty} b(\mathsf{u}_{p(r)},\mathsf{u}_{q(r)}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)+1},\mathsf{u}_{q(r)}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)},\mathsf{u}_{q(r)+1}) = \lim_{r\to\infty} b(\mathsf{u}_{p(r)+1},\mathsf{u}_{q(r)+1}) = \mathsf{e}. \tag{14}$$

Let $t_0 = \mathsf{s}^q \mathsf{e} > 0$. Corollary 1 ensures that $\mathsf{u}_{r_1} \neq \mathsf{u}_{r_2}$ for all $r_1 \neq r_2$. Since $\mathsf{u}_{p(r)} \neq \mathsf{u}_{q(r)}$, (12) implies that $\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)},\mathsf{u}_{q(r)}) > 0$ for all $r \in \mathbb{N}$. Applying (c_0) and the contractivity condition (3) to $x = \mathsf{u}_{p(r)}$ and $y = \mathsf{u}_{q(r)}$, we deduce that

$$\psi(\mathsf{s}^q b(\mathsf{u}_{p(r)+1},\mathsf{u}_{q(r)+1})) = \psi(\mathsf{s}^q b(\mathsf{f}\mathsf{u}_{p(r)},\mathsf{f}\mathsf{u}_{q(r)})) \leq \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)},\mathsf{u}_{q(r)})) < \psi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)},\mathsf{u}_{q(r)})), \tag{15}$$

where

$$\begin{aligned} t_0 = \mathsf{s}^q\,\mathsf{e} < \mathsf{s}^q b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)+1}) &\le \mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) \\ &= \kappa_1 b(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) + \kappa_2 b(\mathsf{u}_{p(r)}, \mathsf{f}\mathsf{u}_{p(r)}) + \kappa_3 b(\mathsf{u}_{q(r)}, \mathsf{f}\mathsf{u}_{q(r)}) + \\ &\quad + \kappa_4 b(\mathsf{u}_{p(r)}, \mathsf{f}\mathsf{u}_{q(r)}) + \kappa_5 b(\mathsf{f}\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) \\ &= \kappa_1 b(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) + \kappa_2 b(\mathsf{u}_{p(r)}, \mathsf{u}_{p(r)+1}) + \kappa_3 b(\mathsf{u}_{q(r)}, \mathsf{u}_{q(r)+1}) + \\ &\quad + \kappa_4 b(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)+1}) + \kappa_5 b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)}). \end{aligned}$$

Therefore

$$t_0 = \mathsf{s}^q\,\mathsf{e} < \mathsf{s}^q b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)+1}) < \mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}).$$

Letting $r \to \infty$, we deduce from (14) that

$$t_0 = \mathsf{s}^q\,\mathsf{e} \le \lim_{r\to\infty} \mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)}) = \kappa_1\,\mathsf{e} + \kappa_2\,\mathsf{e} + \kappa_3\,\mathsf{e} + \kappa_4\,\mathsf{e} + \kappa_5\,\mathsf{e} \le \mathsf{s}^q\,\mathsf{e} = t_0.$$

This means that $\{\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)})\}$ is a sequence whose terms, by (13), are strictly greater than t_0 and converging to t_0. Letting $r \to \infty$ in (15), we observe that

$$\lim_{r\to\infty} \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)})) = \lim_{r\to\infty} \psi(\mathsf{s}^q b(\mathsf{u}_{p(r)+1}, \mathsf{u}_{q(r)+1})) = \lim_{t\to t_0^+} \psi(t) = \psi(t_0^+).$$

However, condition (c_2) means that

$$\lim_{r\to\infty} \phi(\mathsf{A}_\mathsf{f}(\mathsf{u}_{p(r)}, \mathsf{u}_{q(r)})) \le \limsup_{t\to t_0^+} \phi(t) < \psi(t_0^+),$$

which is a contradiction. This contradiction proves that $\{\mathsf{u}_n\}$ is a Cauchy sequence in (X, b, s). The rest of the proof is similar to Step 4 in the proof of Theorem 2, where we demostrated that the sequence $\{\mathsf{u}_n\}$ converges to a point of $x_* \in X$ such that $b(x_*, fx_*) \le \limsup_{n\to\infty} \mathsf{A}_\mathsf{f}(u_n, x_*) < b(x_*, fx_*)$, which is a contradiction. This contradiction finishes the proof. □

5. Consequences and Comparative Results

The first three consequences are particularizations of the three main Theorems 2, 3 and 4 to the case in which $q = 1$. The reader can check that, indeed, they are equivalent to their corresponding general results.

Corollary 5. *Let (X, b, s) be a b-metric space and let $\mathsf{f} : X \to X$ be a self-mapping satisfying*

$$\psi(\mathsf{s}\, b(\mathsf{f}x, \mathsf{f}y)) \le \phi(\mathsf{A}_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where A_f is defined in (2) and the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ satisfy (c_0) and (c_1). If the numbers in $\varkappa$ verify

$$0 < \kappa_1 + \kappa_2 + \kappa_3 \quad \text{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < 1,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \le \mathsf{s}$, then f admits a unique fixed point.

Corollary 6. *Let (X, b, s) be a b-metric space and let $\mathsf{f} : X \to X$ be a self-mapping satisfying*

$$\psi(\mathsf{s}\, b(\mathsf{f}x, \mathsf{f}y)) \le \phi(\mathsf{A}_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where A_f *is defined in (2) and the functions* $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy* (c_0) *and* (c_1)*. If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1 \quad \text{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 + \left(\mathsf{s} + \mathsf{s}^2\right)\kappa_4 < \mathsf{s},$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}$*, then* f *admits a unique fixed point.*

Corollary 7. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\psi(\mathsf{s}\,\mathsf{b}(\mathsf{f}x, \mathsf{f}y)) \leq \phi(\mathsf{A}_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where A_f *is defined in (2) and the functions* $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy* (c_0)*,* (c_1) *and* (c_2)*. If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s},$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}$*, then* f *admits a unique fixed point.*

The case $\mathsf{s} = 1$ leads to metric spaces, and we deduce the following consequence.

Corollary 8. *Let* (X, d) *be a metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\psi(d(\mathsf{f}x, \mathsf{f}y)) \leq \phi(\mathsf{A}_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } d(\mathsf{f}x, \mathsf{f}y) > 0,$$

where A_f *is defined in (2) and the functions* $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy* (c_0) *and* (c_1)*. If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3 \quad \text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\kappa_4 < 1,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq 1$*, then* f *admits a unique fixed point.*

When we include in A_f less terms than in the original definition (2), we are able to conclude many particularizations. For instance, the following ones (where we present the case in which $\kappa_4 = \kappa_5 = 0$), whose proofs make use of the same arguments of the general Theorems 2, 3 and 4.

Corollary 9. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \phi(b_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$*,* b_f *is defined by*

$$b_\mathsf{f}(x, y) = \kappa_1 b(x, y) + \kappa_2 b(x, \mathsf{f}x) + \kappa_3 b(y, \mathsf{f}y) \quad \text{for all } x, y \in X$$

and the functions $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy* (c_0) *and* (c_1)*. If the numbers* $\kappa_1, \kappa_2, \kappa_3 \geq 0$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3 \quad \text{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 < 1,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 \leq \mathsf{s}^q$*, then* f *admits a unique fixed point.*

Corollary 10. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \phi(b_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, b_f *is defined by*

$$b_\mathsf{f}(x, y) = \kappa_1 b(x, y) + \kappa_2 b(x, \mathsf{f}x) + \kappa_3 b(y, \mathsf{f}y) \quad \text{for all } x, y \in X$$

and the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfy* (c_0) *and* (c_1). *If the numbers* $\kappa_1, \kappa_2, \kappa_3 \geq 0$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 < 1 \quad \text{and} \quad \mathsf{s}\kappa_1 + \mathsf{s}\kappa_2 + \kappa_3 < \mathsf{s}^q,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 \leq \mathsf{s}^q$, *then* f *admits a unique fixed point.*

Corollary 11. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \phi(b_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, b_f *is defined by*

$$b_\mathsf{f}(x, y) = \kappa_1 b(x, y) + \kappa_2 b(x, \mathsf{f}x) + \kappa_3 b(y, \mathsf{f}y) \quad \text{for all } x, y \in X$$

and the functions $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfy* (c_0), (c_1) *and* (c_2). *If the numbers* $\kappa_1, \kappa_2, \kappa_3 \geq 0$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 < 1, \quad \text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 \leq \mathsf{s}^q,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, *then* f *admits a unique fixed point.*

The reader can also imagine other combinations as:

$$b'_\mathsf{f}(x, y) = \kappa_1 b(x, y) + \kappa_4 \frac{b(x, \mathsf{f}y) + b(y, \mathsf{f}x)}{2} \quad \text{for all } x, y \in X.$$

In order not to extend this papers, we will only enunciate the main consequences that we can derive from Theorem 4 (we left to the reader to particularize Theorems 2 and 3).

If we take $\tau > 0$ and $\phi(t) = \psi(t) - \tau$ for all $t > 0$, then we can deduce the following $\mathtt{F}$-contraction type corollary of the introduced Hardy-Rogers type results.

Corollary 12. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\tau + \psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \psi(\mathsf{A}_\mathsf{f}(x, y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, $\tau > 0$, A_f *is defined in* (2) *and the function* $\psi : (0, \infty) \to \mathbb{R}$ *is nondecreasing. If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, *then* f *admits a unique fixed point.*

In a similar way, it is also interesting the case in which $\phi(t) = \beta(t)\,\psi(t)$ for all $t > 0$, where $\beta : (0,\infty) \to \mathbb{R}$ satisfies appropriate properties.

Corollary 13. *Let (X, b, s) be a b-metric space and let $\mathsf{f} : X \to X$ be a self-mapping satisfying*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \beta(\mathsf{A_f}(x,y))\,\psi(\mathsf{A_f}(x,y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, $\mathsf{A_f}$ is defined in (2), the function $\psi : (0,\infty) \to (0,\infty)$ is nondecreasing and the function $\beta : (0,\infty) \to (0,1)$ verifies

$$\limsup_{t \to t_0^+} \beta(t) < 1 \quad \text{for any } t_0 > 0. \tag{16}$$

If the numbers in $\varkappa$ verify

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

Proof. Notice that we assume that $\psi(t) > 0$ and $\beta(t) < 1$ for all $t > 0$, so $\phi(t) = \beta(t)\,\psi(t) < \psi(t)$ for all $t > 0$. Furthermore, condition (16) implies (c_2), so Theorem 4 is applicable. □

Remark 2. *Notice that condition (16) does not guarantee that $\beta(t) < 1$ for all $t > 0$. For instance, let consider $\beta : (0,\infty) \to \{0.5, 2\}$ defined by $\beta(1) = 2$ and $\beta(t) = 0.5$ if $t \in (0,\infty) \setminus \{1\}$.*

If we use $\psi(t) = t$ for all $t > 0$ in Corollary 13, we obtain the following consequence.

Corollary 14. *Let (X, b, s) be a b-metric space and let $\mathsf{f} : X \to X$ be a self-mapping satisfying*

$$\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y) \leq \beta(\mathsf{A_f}(x,y))\,\mathsf{A_f}(x,y) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, $\mathsf{A_f}$ is defined in (2) and the function $\beta : (0,\infty) \to (0,1)$ verifies

$$\limsup_{t \to t_0^+} \beta(t) < 1 \quad \text{for any } t_0 > 0.$$

If the numbers in $\varkappa$ verify

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f has at least one fixed point. Furthermore, if we additionally assume that $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, then f admits a unique fixed point.

Other interesting consequence occurs when $\phi(t) = \mathsf{k}\,\psi(t)$ for all $t > 0$, where $\mathsf{k} \in (0,1)$.

Corollary 15. *Let (X, b, s) be a b-metric space and let $\mathsf{f} : X \to X$ be a self-mapping satisfying*

$$\psi(\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y)) \leq \mathsf{k}\,\psi(\mathsf{A_f}(x,y)) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, $\mathsf{k} \in (0,1)$, $\mathsf{A_f}$ *is defined in (2) and the function* $\psi : (0,\infty) \to (0,\infty)$ *is nondecreasing. If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, *then* f *admits a unique fixed point.*

Finally, letting $\psi(t) = t$ for all $t > 0$ in the previous result, we derive the following consequence.

Corollary 16. *Let* (X, b, s) *be a b-metric space and let* $\mathsf{f} : X \to X$ *be a self-mapping satisfying*

$$\mathsf{s}^q b(\mathsf{f}x, \mathsf{f}y) \leq \mathsf{k}\, \mathsf{A_f}(x,y) \quad \text{for all } x, y \in X \text{ such that } b(\mathsf{f}x, \mathsf{f}y) > 0,$$

where $q \geq 1$, $\mathsf{k} \in (0,1)$ *and* $\mathsf{A_f}$ *is defined in (2). If the numbers in* $\varkappa$ *verify*

$$0 < \kappa_1 + \kappa_2 + \kappa_3, \quad \kappa_3 + \mathsf{s}\kappa_4 < 1, \quad \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$$
$$\text{and} \quad \kappa_1 + \kappa_2 + \kappa_3 + 2\mathsf{s}\kappa_4 \leq \mathsf{s}^q,$$

then f *has at least one fixed point. Furthermore, if we additionally assume that* $0 < \kappa_1 + \kappa_4 + \kappa_5 \leq \mathsf{s}^q$, *then* f *admits a unique fixed point.*

Example 3. *Let* $X = [0,1] \cup \{2,3,4\}$ *and let* $b : X \times X \to [0,\infty)$ *be defined, for all* $\mathsf{u}, \mathsf{v} \in X$, *as:*

$$b(\mathsf{u},\mathsf{v}) = \begin{cases} 15, & \text{if } \{\mathsf{u},\mathsf{v}\} = \{3,4\}, \\ |\mathsf{u} - \mathsf{v}|, & \text{in any other case.} \end{cases}$$

Clearly b *is not a metric on* X *because* $b(3,4) = 15 > 3 = 1 + 2 = b(3,2) + b(2,4)$. *However,* b *is a b-metric on* X *with constant* $\mathsf{s} = 5$ *because, for each* $\mathsf{u} \in X \setminus \{3,4\}$,

$$b(3,4) = 15 = 5 \cdot 3 = 5(1+2) = 5\,[\,b(3,2) + b(2,4)\,] \leq 5\,[\,b(3,\mathsf{u}) + b(\mathsf{u},4)\,]$$

(if we consider other points, the Euclidean triangle inequality is applicable). Let $\mathsf{f} : X \to X$ *be the self-mapping defined, for all* $\mathsf{u} \in X$, *as:*

$$\mathsf{fu} = \begin{cases} \dfrac{0.5 + \mathsf{u}}{2}, & \text{if } \mathsf{u} \in [0,1], \\ 0, & \text{if } \mathsf{u} \in \{2,4\}, \\ 1, & \text{if } \mathsf{u} = 3. \end{cases}$$

Notice that $b(2,3) = b(\mathsf{f}2, \mathsf{f}3) = 1$. *This means that other previous theorems in the setting of metric spaces, or even in the setting of b-metric spaces but involving mappings such that* $b(\mathsf{fu}, \mathsf{fv}) < b(\mathsf{u},\mathsf{v})$, *are not applicable to this mapping. In fact, we cannot apply our main theorems by using* $\kappa_1 = \mathsf{s}^q$ *and* $\kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 0$ *because, in this case, using* $\mathsf{u} = 2$ *and* $\mathsf{v} = 3$,

$$\mathsf{A_f}(2,3) = \kappa_1 b(2,3) + \kappa_2 b(2,\mathsf{f}2) + \kappa_3 b(3,\mathsf{f}3) + \kappa_4 b(2,\mathsf{f}3) + \kappa_5 b(3,\mathsf{f}2) = \mathsf{s}^q b(2,3) = \mathsf{s}^q$$

and

$$\psi(\mathsf{s}^q b(\mathsf{f}2,\mathsf{f}3)) = \psi(\mathsf{s}^q) > \phi(\mathsf{s}^q) = \phi(\mathsf{A_f}(2,3)).$$

As a consequence, for this mapping f, *it is necessary to involve other terms (like* $b(\mathsf{u},\mathsf{fu})$ *and* $b(\mathsf{v},\mathsf{fv})$*) in the contractivity condition. Hence, let*

$$q=1,\quad \kappa_1=4,\quad \kappa_2=\kappa_3=\frac{1}{2}\quad \textit{and}\quad \kappa_4=\kappa_5=0,$$
$$\psi,\phi:(0,\infty)\to(0,\infty),\quad \psi(t)=t,\quad \phi(t)=\begin{cases} t-\sin\left(\frac{t}{100}\right), & \textit{if } t\in(0,100),\\ 1, & \textit{if } t\geq 100.\end{cases}$$

The following tables describe the b-metrics $b(\mathsf{u},\mathsf{v})$, $b(\mathsf{fu},\mathsf{fv})$ *and* $b(\mathsf{u},\mathsf{fu})$ *in all possible cases.*

$b(\mathsf{u},\mathsf{v})$	$\mathsf{v}\in[0,1]$	$\mathsf{v}=2$	$\mathsf{v}=3$	$\mathsf{v}=4$
$\mathsf{u}\in[0,1]$	$\lvert\mathsf{u}-\mathsf{v}\rvert$	$2-\mathsf{u}$	$3-\mathsf{u}$	$4-\mathsf{u}$
$\mathsf{u}=2$	$2-\mathsf{v}$	0	1	2
$\mathsf{u}=3$	$3-\mathsf{v}$	1	0	15
$\mathsf{u}=4$	$4-\mathsf{v}$	2	15	0

	$b(\mathsf{u},\mathsf{fu})$
$\mathsf{u}\in[0,1]$	$\frac{\lvert\mathsf{u}-0.5\rvert}{2}$
$\mathsf{u}=2$	2
$\mathsf{u}=3$	2
$\mathsf{u}=4$	4

$b(\mathsf{fu},\mathsf{fv})$	$\mathsf{v}\in[0,1]$	$\mathsf{v}=2$	$\mathsf{v}=3$	$\mathsf{v}=4$
$\mathsf{u}\in[0,1]$	$\frac{\lvert\mathsf{u}-\mathsf{v}\rvert}{2}$	$\frac{0.5+\mathsf{u}}{2}$	$\frac{1.5-\mathsf{u}}{2}$	$\frac{0.5+\mathsf{u}}{2}$
$\mathsf{u}=2$	$\frac{0.5+\mathsf{v}}{2}$	0	1	0
$\mathsf{u}=3$	$\frac{1.5-\mathsf{v}}{2}$	1	0	1
$\mathsf{u}=4$	$\frac{0.5+\mathsf{v}}{2}$	0	1	0

A simple computation considering all possible pairs of points $\mathsf{u},\mathsf{v}\in X$ *show that*

$$\psi(\mathsf{s}^q b(\mathsf{fu},\mathsf{fv}))=\psi(5\,b(\mathsf{fu},\mathsf{fv}))\leq\phi\left(4\,b(\mathsf{u},\mathsf{v})+\frac{1}{2}\,b(\mathsf{u},\mathsf{fu})+\frac{1}{2}\,b(\mathsf{v},\mathsf{fv})\right)=\phi(\mathsf{A_f}(\mathsf{u},\mathsf{v})).$$

For instance, observe that using $\mathsf{u}=2$ *and* $\mathsf{v}=3$,

$$\begin{aligned}\phi(\mathsf{A_f}(2,3)) &= \phi\left(4\,b(2,3)+\frac{1}{2}\,b(2,\mathsf{f}2)+\frac{1}{2}\,b(3,\mathsf{f}3)\right)\\ &= \phi\left(4\cdot 1+\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 2\right)=\phi(6)=6-\sin\left(\frac{6}{100}\right)\\ &> 5.9>5=\psi(5)=\psi(\mathsf{s}^q b(\mathsf{f}2,\mathsf{f}3)).\end{aligned}$$

As all hypotheses of Theorem 4 hold, we conclude that f *has a unique fixed point in X, which is* $\mathsf{u}=0.5$.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah.

Acknowledgments: The authors acknowledge with thanks DSR for financial support. A.F. Roldán López de Hierro is grateful to Junta de Andalucía by project FQM-365 of the Andalusian CICYE and Project TIN2017-89517-P of the Ministerio de Economía, Industria y Competitividad.

Conflicts of Interest: The authors declare no conflict of interest.

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