mathematics

MDPI

Article

Reliability Inference for the Multicomponent System Based on Progressively Type II Censored Samples from Generalized Pareto Distributions

Lauren Sauer [1], Yuhlong Lio [1] and Tzong-Ru Tsai [2,*]

1 Department of Mathematical Sciences, University of South Dakota, Vermillion, SD 57069, USA; lauren.jo.sauer@gmail.com (L.S.); Yuhlong.Lio@usd.edu (Y.L.)

2 Department of Statistics, Tamkang University, Tamsui District, New Taipei City 251301, Taiwan

* Correspondence: tzongru@gms.tku.edu.tw

Received: 22 June 2020; Accepted: 15 July 2020; Published: 17 July 2020

Abstract: In this paper, the reliability of a k-component system, in which all components are subject to common stress, is considered. The multicomponent system will continue to survive if at least s out of k components' strength exceed the common stress. The system reliability is investigated by utilizing the maximum likelihood estimator based on progressively type II censored samples from generalized Pareto distributions. The confidence interval of the system reliability can be obtained by using asymptotic normality with Fisher information matrix or bootstrap method approximation. An intensive simulation study is conducted to evaluate the performance of maximum likelihood estimators of the model parameters and system reliability for a variety of cases. For the confidence interval of the system reliability, simulation results indicate the bootstrap method approximation outperforms over the asymptotic normality approximation in terms of coverage probability.

Keywords: bootstrap procedure; delta method; maximum likelihood estimation; stress-strength

1. Introduction

The system of a stress-strength model has been used for various purposes across many disciplines, such as engineering, medicine, psychology, and others. Due to the diverse spread of possible application, the system reliability has become a popular topic in research. The concept of stress and strength has been well-defined in the past few decades. Today, numerous researchers have considered the reliability assessment of the stress-strength system by using various inference methods and assumptions for different distributions in literature. The system with strength, X, subjected to stress, Y, is defined as failure when the applied stress is greater than the strength of system. The performance of system is referred to reliability, which can be measured by $\delta = P(Y < X)$. Numerous authors have studied the point and interval estimation methods of δ. For example, Kundu and Gupta [1] studied δ based on random samples from two independent Weibull distributions. Rezaei et al. [2] investigated δ utilizing random samples from two independent generalized Pareto (GP) distributions. Kohansal and Rezakhah [3] worked on δ with progressively censored samples from the independent two-parameter Rayleigh distribution, and Akgül and Şenğlu [4] considered δ based on two type II right censoring ranked set samples from two independent Weibull distributions.

Meanwhile the aforementioned stress-strength model can be extended to a multicomponent stress-strength (MSS) system that consists of k independent and identical strength components and all components are subject to common stress. One example of the MSS systems is the suspension bridge, and its reliability can be measured by the probability that the strengths of a certain number of cables outweigh the stress. The MSS system will continue to survive if at least s $(1 \leq s \leq k)$ of the components survive. It is also referred to as the s-out-of-k system. The reliability of the MSS system is evaluated by the probability, $\delta_{s,k} = P$ (at least s out of k components' strength exceed the common stress). Various researchers have studied the inference of $\delta_{s,k}$ for the MSS system using different distributions. For example, Rao [5] studied $\delta_{s,k}$ for the generalized exponential distribution; Rao et al. [6] investigated $\delta_{s,k}$ for the Burr-XII distribution; Hassan and Alohali [7] studied $\delta_{s,k}$ for the generalized linear failure rate distribution; and Pak et al. [8] investigated $\delta_{s,k}$ for the power Lindley model. Readers can refer Kotz, et al. [9] for a comprehensive review of the MSS.

Progressive Type II Censoring Scheme

In order to collect lifetime data, a life test is completed once a specified time has been reached or all components have failed. Depending on the exact type of life test, censoring schemes may be considered to save the testing time and cost. Censoring refers to the value of an observation is only partially known. Type I censoring is used to terminate the life test when the preassigned schedule time is up and type II censoring is used to terminate the life test when a specified number of lifetimes of components are observed. If components are removed during the life test, the specific censoring scheme is called progressive censoring, see Mann et al. [10].

In this paper, the progressive type II censoring scheme is considered for evaluating the reliability of an MSS system. The progressive type II censoring scheme can be implemented as follows: n identical units are placed on a life test at the initial time. Once one unit fails, R_1 units are removed from the surviving components. After the next unit fails, R_2 units are removed from the surviving components. This process continues until the m-th unit failure is observed and then all R_m units in the life test are removed. It can be shown that $n = m + R_1 + R_2 + ... + R_m$. The progressive type II censoring scheme is denoted by $(R_1, R_2, ..., R_m)$ hear and after. The stress-strength parameter, denoted by $\delta = P(Y < X)$, of a single-component system has been well studied. Kohansal and Rezakhah [3] investigated δ based on progressively type II censored samples. Lio and Tsai [11] studied δ based on progressive first failure-censoring samples from the Burr-XII distributions. Rezaei et al. [12] investigated δ based on progressively censored samples from the GP distributions. Based on our best knowledge, no research working on $\delta_{s,k}$ for an MSS system based on progressively type II censored samples is found in literature. In this paper, the strength of the components and the common stress applied to the MSS system are assumed to follow GP distributions, which have different shape parameters and the same scale parameter.

The rest of this paper is organized as follows: the analytic form of $\delta_{s,k}$ for an MSS system based on progressively type II censored samples from the GP distributions is presented in Section 2. Section 3 provides the maximum likelihood estimates (MLEs) of the model parameters and $\delta_{s,k}$. In Section 4, two procedures for computing the confidence interval of the MSS system reliability are addressed via using the delta method with Fisher information matrix and a bootstrap procedure, respectively. Section 5 presents the design of Monte Carlo simulation and numerical results. Moreover, comprehensive discussions are presented to evaluate the performance of the proposed inference procedures. Conclusions and remarks are given in Section 6.

2. The MSS System Based on Pareto Distribution

Vilfredo Pareto introduced the Pareto distribution in 1897 in an economics text after observing the trend in income data. Since then, the Pareto distribution has been used to model population sizes, environmental extrema, and insurance claims. In the past decades, the Pareto distribution also has been used to characterize lifetime data. It is chiefly suited for modeling events with left-skewed, non-negative data. Pickands introduced the GP distribution in 1975, which was a slightly modified version of the Pareto distribution, see Arnold [13]. The probability density function (PDF) of the GP distribution is defined by

$$f(x; \sigma, \eta) = \frac{1}{\sigma} \left(1 - \frac{\eta x}{\sigma}\right)^{\frac{1-\eta}{\eta}}, \; x > 0, \tag{1}$$

where $\sigma > 0$ and $\eta \in (-\infty, 0)$. Considering the re-parameterized transformation process by letting $\sigma = \dfrac{1}{\lambda\alpha}$ and $\eta = -\dfrac{1}{\alpha}$, the PDF of the GP distribution in Equation (1) can be represented by

$$f_X(x;\alpha,\lambda) = \alpha\lambda(1+\lambda x)^{-\alpha-1},\ x > 0, \tag{2}$$

and the cumulative density function (CDF) of the GP distribution can be presented by

$$F_X(x;\alpha,\lambda) = 1-(1+\lambda x)^{-\alpha},\ x > 0, \tag{3}$$

where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The GP distribution with Equations (2) and (3) has been used to investigate δ for a single-component system based on random samples (see Rezaei et al. [2]) and progressively type II censored samples (see Rezaei et al. [12]). In this study, the reliability of an MSS system is investigated based on the GP distribution defined by Equations (2) and (3).

2.1. Reliability of MSS

Let $X_1, X_2, ..., X_k \sim \mathrm{GP}(\alpha_1, \lambda)$ denote the respective strengths of k components in an MSS system and $Y \sim \mathrm{GP}(\alpha_2, \lambda)$ denote the common stress. The reliability of the MSS system is defined by

$$\begin{aligned}
\delta_{s,k} &= P(\text{at least } s \text{ out of } k \text{ components' strength exceed } Y) \\
&= \int_{-\infty}^{\infty} P(\text{at least } s \text{ out of } k \text{ components' strength exceed } y | Y = y) dF_Y(y) \\
&= \int_{-\infty}^{\infty} \sum_{i=s}^{k} \binom{k}{i} (1-F_X(y))^i (F_X(y))^{k-i} dF_Y(y) \\
&= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1-F_X(y))^i (F_X(y))^{k-i} dF_Y(y) \\
&= \sum_{i=s}^{k} \binom{k}{i} \int_{0}^{\infty} (1-(1-(1+\lambda y)^{-\alpha_1}))^i (1-(1+\lambda y)^{-\alpha_1})^{k-i} \alpha_2\lambda(1+\lambda y)^{-\alpha_2-1} dy \\
&= \sum_{i=s}^{k} \binom{k}{i} \int_{0}^{\infty} (1+\lambda y)^{-\alpha_1 i} (1-(1+\lambda y)^{-\alpha_1})^{k-i} \alpha_2\lambda(1+\lambda y)^{-\alpha_2-1} dy \\
&= \alpha_2 \sum_{i=s}^{k} \binom{k}{i} \int_{0}^{\infty} (1+\lambda y)^{-\alpha_1 i-\alpha_2-1} (1-(1+\lambda y))^{-\alpha_1(k-i)} \lambda dy
\end{aligned} \tag{4}$$

Equation (4) can be also represented as the follows,

$$\begin{aligned}
\delta_{s,k} &= \frac{\alpha_2}{\alpha_1} \sum_{i=s}^{k} \binom{k}{i} \int_0^1 u^{(-1/\alpha_1)(-\alpha_1 i-\alpha_2-1)} (1-u)^{k-i} u^{-\frac{1}{\alpha_1}-1} du \\
&= \frac{\alpha_2}{\alpha_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i+\frac{\alpha_2}{\alpha_1}, k-i+1\right)
\end{aligned} \tag{5}$$

where $B\left(i+\frac{\alpha_2}{\alpha_1}, k-i+1\right) = \int_0^1 u^{i+\frac{\alpha_2}{\alpha_1}-1}(1-u)^{k-i} du$. When $k = 1$, $\delta_{1,1} = P(Y < X) = \delta$. Note here that $\delta_{s,k}$ depends on α_1 and α_2 but is free of λ. Equation (5) can be used to derive the maximum likelihood estimator of $\delta_{s,k}$ based on the progressively type II censored samples in Section 3.

3. Maximum Likelihood Estimation

Let $\{x_{(1)}, x_{(2)}, ..., x_{(r_1)}\}$ be the realization of a progressively type II censored sample from $\mathrm{GP}(\alpha_1, \lambda)$ with scheme $R_x = \{R_{x,1}, R_{x,2}, ..., R_{x,r_1}\}$ and $\{y_{(1)}, y_{(2)}, ..., y_{(r_2)}\}$ be a progressively type II censored sample from $\mathrm{GP}(\alpha_2, \lambda)$ with scheme $R_y = \{R_{y,1}, R_{y,2}, ..., R_{y,r_2}\}$. Then, the likelihood function based on $\mathrm{GP}(\alpha_1, \lambda)$, $\mathrm{GP}(\alpha_2, \lambda)$ and schemes R_x and R_y is given as follows,

$$L \equiv L(\Theta) = C_1 \prod_{i=1}^{r_1} f_X(x_{(i)}; \alpha_1, \lambda)(1 - F_X(x_{(i)}; \alpha_1, \lambda))^{R_{x,i}} \times C_2 \prod_{j=1}^{r_2} f_Y(y_{(j)}; \alpha_2, \lambda)(1 - F_Y(y_{(j)}; \alpha_2, \lambda))^{R_{y,j}} \quad (6)$$

where $\Theta = (\alpha_1, \alpha_2, \lambda)$, $C_1 = n(n - R_{x,1} - 1)(n - R_{x,1} - R_{x,2} - 2)...(n - R_{x,1} - ... - R_{x,r_1 - 1} - r_1 + 1)$ and $C_2 = m(m - R_{y,1} - 1)(m - R_{y,1} - R_{y,2} - 2)...(m - R_{y,1} - ... - R_{y,r_1 - 1} - r_2 + 1)$. Additionally, note that in this case, $n = r_1 + \sum_{i=1}^{r_1} R_{x,i}$ and $m = r_2 + \sum_{i=1}^{r_2} R_{y,i}$. The likelihood function in Equation (6) can be represented as follows,

$$\begin{aligned} L = C_1 \prod_{i=1}^{r_1} \lambda\alpha_1(1 + x_{(i)}\lambda)^{-\alpha_1 - 1}(1 - (1 - (1 + \lambda x_{(i)})^{-\alpha_1}))^{R_{x,i}} \times \\ C_2 \prod_{j=1}^{r_2} \lambda\alpha_2(1 + y_{(j)}\lambda)^{-\alpha_2 - 1}(1 - (1 - (1 + \lambda y_{(j)})^{-\alpha_2}))^{R_{y,j}} \end{aligned} \quad (7)$$

and log-likelihood function, ℓ, is given as

$$\begin{aligned} \ell \equiv \ell(\Theta) = \log(C_1) + \log(C_2) + (r_1 + r_2)\log(\lambda) + r_1 \log(\alpha_1) + r_2 \log(\alpha_2) - \\ \sum_{i=1}^{r_1}(1 + \alpha_1(R_{x,i} + 1))\log(1 + \lambda x_{(i)}) - \sum_{j=1}^{r_2}(1 + \alpha_2(R_{y,j} + 1))\log(1 + \lambda y_{(j)}) \end{aligned} \quad (8)$$

The MLEs of α_1, α_2, and λ are denoted as $\hat{\alpha}_1, \hat{\alpha}_2$, and $\hat{\lambda}$, respectively, and can be obtained as the solution to the following system,

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{r_1}{\alpha_1} - \sum_{i=1}^{r_1}(R_{x,i} + 1)\log(1 + \lambda x_{(i)}) = 0 \quad (9)$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2}(R_{y,j} + 1)\log(1 + \lambda y_{(j)}) = 0 \quad (10)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{r_1 + r_2}{\lambda} - \sum_{i=1}^{r_1}\frac{(1 + \alpha_1(R_{x,i} + 1))}{1 + \lambda x_{(i)}} - \sum_{j=1}^{r_2}\frac{(1 + \alpha_2(R_{y,j} + 1))}{1 + \lambda y_{(j)}} = 0. \quad (11)$$

From Equations (9)–(11), we obtain

$$\hat{\alpha}_1 = \frac{r_1}{\sum_{i=1}^{r_1}(R_{x,i} + 1)\log(1 + \hat{\lambda} x_{(i)})}, \quad (12)$$

$$\hat{\alpha}_2 = \frac{r_2}{\sum_{j=1}^{r_2}(1 + R_{y,j})\log(1 + \hat{\lambda} y_{(j)})}. \quad (13)$$

It is important to mention that $\hat{\lambda}$ is needed to calculate both $\hat{\alpha}_1$ and $\hat{\alpha}_2$. For this purpose, Equations (12) and (13) are used to transform Equation (11) into the following equation

$$\begin{aligned} \frac{r_1 + r_2}{\lambda} = \sum_{i=1}^{r_1} \frac{\left(1 + \frac{r_1}{\sum_{i=1}^{r_1}(R_{x,i} + 1)\log(1 + \lambda x_{(i)})}(R_{x,i} + 1)\right)}{1 + \lambda x_{(i)}} \\ + \sum_{j=1}^{r_2} \frac{\left(1 + \frac{r_2}{\sum_{j=1}^{r_2}(1 + R_{y,j})\log(1 + \lambda y_{(j)})}(R_{y,j} + 1)\right)}{1 + \lambda y_{(j)}}. \end{aligned} \quad (14)$$

Let

$$\frac{1}{g(\lambda)} = \sum_{i=1}^{r_1} \frac{\left(1 + \frac{r_1}{\sum_{i=1}^{r_1}(R_{x,i}+1)\log(1+\lambda x_{(i)})}(R_{x,i}+1)\right)}{(1+\lambda x_{(i)})(r_1+r_2)} + \sum_{j=1}^{r_2} \frac{\left(1 + \frac{r_2}{\sum_{j=1}^{r_2}(1+R_{y,j})\log(1+\lambda y_{(j)})}(R_{y,j}+1)\right)}{(1+\lambda y_{(j)})(r_1+r_2)} \tag{15}$$

Therefore, $\hat{\lambda}$ can be obtained through using the following nonlinear equation

$$g(\lambda) = \lambda. \tag{16}$$

Because $\hat{\lambda}$ is a fixed point solution to Equation (16), it can be obtained by using the following iterative procedure:

$$g(\lambda_j) = \lambda_{j+1}, \tag{17}$$

where λ_j is the obtained value of $\hat{\lambda}$ at the jth iteration. The iteration procedure will stop when the difference between λ_j and λ_{j+1} is sufficiently small. Then, the average of λ_j and λ_{j+1} is used as the value of $\hat{\lambda}$. An alternative procedure to solve $\hat{\lambda}$ can be obtained through using **R** procedure uniroot to search the unit root for equation $g(\lambda) - \lambda = 0$. The MLE of $\delta_{s,k}$, denoted as $\hat{\delta}_{s,k}$, is then obtained by replacing α_1 and α_2 with their respective MLEs into Equation (5):

$$\hat{\delta}_{s,k} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1} \sum_{i=s}^{k} \binom{k}{i} B\left(i + \frac{\hat{\alpha}_2}{\hat{\alpha}_1}, k-i+1\right) \tag{18}$$

4. Confidence Intervals

4.1. Confidence Interval Based on Fisher Information

Rezaei et al. [12] presented the asymptotic distribution of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda}$ and $\hat{\delta}_{1,1}$ based on the progressively type II censored samples. The procedure can be extended to derive an approximate confidence interval of $\delta_{s,k}$. Denote the Fisher information matrix by

$$I(\Theta) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}. \tag{19}$$

Given a progressively type II censored sample, $x_{(1)} < x_{(2)} < x_{(3)} < \cdots < x_{(r)}$, from $GP(\alpha, \lambda)$ with censoring scheme $R = \{R_1, R_2, ..., R_r\}$, Kamps and Cramer [14] had provided the PDF of $X_{(i)}$ as

$$f_{X_{(i)}}(x) = c_{i-1} \sum_{l=1}^{i} a_{l,i} \alpha_1 \lambda (1+\lambda x)^{-(\alpha_1 \gamma_l + 1)}, \tag{20}$$

where $\gamma_l = n - l + 1 + \sum_{j=l}^{r} R_j$, $c_{i-1} = \prod_{j=1}^{i} \gamma_j$ and $a_{l,i} = \prod_{j=1; j\neq l}^{i} \frac{1}{\gamma_j - \gamma_i}$ for $i = 1, 2, 3, \cdots, r$. Following the same procedure presented by Rezaei et al. [12], I_{ij} for $i, j = 1, 2, 3$ can be obtained by

$$I_{11} = -E\left(\frac{\partial^2 \ell}{\partial \alpha_1^2}\right) = \frac{r_1}{\alpha_1^2} \tag{21}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 \ell}{\partial \alpha_1 \alpha_2}\right) = 0 \tag{22}$$

$$I_{22} = -E\left(\frac{\partial^2 \ell}{\partial \alpha_2^2}\right) = \frac{r_2}{\alpha_2^2} \tag{23}$$

$$I_{13} = I_{31} = -E\left(\frac{\partial^2 \ell}{\partial \alpha_1 \partial \lambda}\right) = \frac{\alpha_1}{\lambda}\sum_{i=1}^{r_1}(1+R_{x,i})\sum_{l=1}^{i} a_{l,i}c_{i-1}B(2,\alpha_1\gamma_l) \tag{24}$$

$$I_{23} = I_{32} = -E\left(\frac{\partial^2 \ell}{\partial \alpha_2 \partial \lambda}\right) = \frac{\alpha_2}{\lambda}\sum_{j=1}^{r_2}(1+R_{y,j})\sum_{l=1}^{j} a_{l,j}c_{j-1}B(2,\alpha_2\gamma_l) \tag{25}$$

$$\begin{aligned} I_{33} = -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) = \frac{1}{\lambda^2}\{(n+m) - \alpha_1\sum_{i=1}^{n}[\alpha_1(1+r_i)+1]c_{i-1}\sum_{k=1}^{i} a_{k,i}B(3,\alpha_1\gamma_k) \\ -\alpha_2\sum_{j=1}^{m}[\alpha_2(1+r_j)+1]c_{j-1}\sum_{k=1}^{j} a_{k,j}B(3,\alpha_2\gamma_k)\} \end{aligned} \tag{26}$$

Using the delta method, the variance of $\hat{\delta}_{s,k}$ can be approximated by $\sigma_\delta^2 = \nabla\mathbf{\Psi}_{\mathbf{s,k}}^{\mathbf{T}} I(\Theta)^{-1}\nabla\mathbf{\Psi}_{\mathbf{s,k}}$, where $\nabla\mathbf{\Psi}_{\mathbf{s,k}}$ is the gradient of $\delta_{s,k}$ with respective to Θ. A symmetric $1-p$ confidence interval can be approximated by

$$(\hat{\delta}_{s,k} - z_{p/2}\hat{\sigma}_\delta, \hat{\delta}_{s,k} + z_{p/2}\hat{\sigma}_\delta), \tag{27}$$

where $\hat{\sigma}_\delta^2$ is the value of σ_δ^2 but replacing Θ by $\hat{\Theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\lambda})$. $z_{p/2}$ is the $(p/2)$-th quantile of the standard normal distribution, $N(0,1)$; that is $P(Z > z_{p/2}) = 1 - \Phi(z_{p/2}) = p/2$ for $0 < p < 1$, where $\Phi(.)$ is the CDF of $N(0,1)$. The computed confidence interval of $\delta_{s,k}$ using the delta method with Fisher information will be denoted by CI-D.

4.2. Bootstrap Confidence Interval

Since the exact sampling distribution of $\hat{\delta}_{s,k}$ is unavailable, bootstrapping methodology can be an alternative to develop an approximated confidence interval of $\delta_{s,k}$ besides the aforementioned delta method approximation. Bootstrap methodology is implemented based on the resampling procedure. Dr. Bradley Efron is the pioneer to connect the basic statistical concepts and ideas of the bootstrap method in 1979 [15]. The bootstrap procedure can be a nonparametric or parametric procedure. In this study, the parametric percentile bootstrap method is applied to find the confidence interval of $\delta_{s,k}$. The readers can refer to Efron and Tibshirani [16] for more information about the use of bootstrap methods. In this study, progressively type II censored samples from the GP distributions are generated based on the algorithm established by Balakrishnan and Sandhu [17]. The following algorithm is used to implement the parametric percentile bootstrap method:

Step 1 Given a progressively type II censored sample $\{x_{(1)}, x_{(2)}, \cdots, x_{(r_1)}\}$ from the $GP(\alpha_1, \lambda)$ with the censoring scheme of $R_x = \{R_{x,1}, R_{x,2}, ..., R_{x,r_1}\}$ and a progressively type II censored sample $\{y_{(1)}, y_{(2)}, \cdots, y_{(r_2)}\}$ from the $GP(\alpha_2, \lambda)$ with the censoring scheme of $R_y = \{R_{y,1}, R_{y,2}, ..., R_{y,r_2}\}$. Obtain MLEs $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\lambda}$ using the procedure described in Section 3.

Step 2 A bootstrap progressively type II censored sample, denoted by $\{x^*_{(j)}, j = 1, 2, \cdots, r_1\}$, is generated from the $GP(\hat{\alpha}_1, \hat{\lambda})$ based on the censoring scheme of $R_x = \{R_{x,1}, R_{x,2}, ..., R_{x,r_1}\}$. A bootstrap progressively type II censored sample, denoted by $\{y^*_{(j)}, j = 1, 2, \cdots, r_2\}$, is generated from the $GP(\hat{\alpha}_2, \hat{\lambda})$ based on the censoring scheme of $R_y = \{R_{y,1}, R_{y,2}, ..., R_{y,r_2}\}$.

Step 3 The bootstrap estimates, $\hat{\alpha}_1^*, \hat{\alpha}_2^*$ and $\hat{\lambda}^*$, for α_1, α_2 and λ are obtained by using the procedure described in Section 3 with the generated bootstrap progressively type II censored samples in Step 2. Then, the bootstrap estimate of $\delta_{s,k}$ is obtained by using Equation (5) and replacing α_1 and α_2 by $\hat{\alpha}_1^*$ and $\hat{\alpha}_2^*$, respectively. Denote the obtained bootstrap estimate by $\hat{\delta}^*_{s,k}$.

Step 4 Repeat steps 2 and 3 N times, where N is a given huge number. The bootstrap sample, $\{\hat{\delta}^*_{s,k_j}, j = 1, 2, \cdots, N\}$, is collected.

Step 5 The empirical distribution function, denoted by $\hat{G}$, based on the bootstrap sample, $\{\hat{\delta}^*_{s,k_j}, j = 1, 2, \cdots, N\}$, is obtained. Let $\hat{\delta}_{s,k_{Bp}}(x) = \hat{G}^{-1}(x)$ for $0 < x < 1$. The $100(1-p)\%$ confidence interval of $\delta_{s,k}$ is given by

$$(\hat{\delta}_{s,k_{Bp}}(p/2), \hat{\delta}_{s,k_{Bp}}(1-p/2)). \tag{28}$$

We denote the obtained confidence interval of $\delta_{s,k}$ via using the parametric percentile bootstrap method by CI-B.

5. Simulation Study

An intensive simulation study is conducted in this section to evaluate the performance of the MLE and two aforementioned confidence interval procedures for $\delta_{s,k}$, $1 \leq s \leq k$ and $1 \leq k \leq 5$, under different progressive type II censoring schemes, respectively. In order to generate progressively type II censored samples, some progressive censoring schemes are adopted. Those selected progressive censoring schemes are similar to those considered by Wu and Kuş [18] and Lio and Tsai [11]. The surviving items can be removed at the initial or end stages, or the removal can be done during the life test. A pair of the GP distributions with $\Theta = (\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$ are considered to implement the simulation study. The other parameters in the simulation study are set by $n_1 = n_2 = n = 20$, 30, 50, 60 and $r_1 = r_2 = m = 5, 15, 20, 30$. In the simulation study, we consider $R_x = R_y = (0,0,0,0,15)$, $(15,0,0,0,0)$, $(3,3,3,3,3)$ for $m = 5$; $R_x = R_y = (0,0,...,0,15)$, $(15,0,0,...,0)$, $(3,0,0,3,0,0,...,3,0,0)$ for $m = 15$; $R_x = R_y = (0,0,...,0,30)$, $(30,0,...,0,0)$, $(3,0,3,0,...,3,0)$ for $m = 20$; and $R_x = R_y = (0,0,...,0,20)$, $(20,0,0,...,0)$, $(2,0,0,2,0,0,...,2,0,0)$ for $m = 30$. The algorithm proposed by Balakrishnan and Sandhu [17] is used to generate progressively type II censored samples for $N = 10,000$ simulation runs. During the ith ($1 \leq i \leq N$) simulation run, the MLE, $\hat{\delta}_{s,k_i}$, and the correspondent 95% CI-D using Fisher information matrix are obtained, and additionally, $N = 10,000$ bootstrap sample observations, $\hat{\delta}^*_{s,k_j}$, $j = 1, 2, \cdots, N$ are generated to develop a 95% bootstrap approximate CI-B. Refer to Section 4.2 for more details regarding Bootstrap approximate CI-B.

Based on $\hat{\delta}_{s,k_i}$, $i = 1, 2, \cdots N$, the bias and mean square error (MSE) of $\hat{\delta}_{s,k}$ are evaluated by

$$\text{Bias} = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\delta}_{s,k_i} - \delta_{s,k}) \tag{29}$$

and

$$\text{MSE} = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\delta}_{s,k_i} - \delta_{s,k})^2, \tag{30}$$

respectively. To evaluate the performance of each confidence interval procedure, the coverage probability (CP) that is defined as the percentage among 10,000 simulated confidence intervals covering the true parameter $\delta_{s,k}$ is used. Moreover, the average of lower limit and average of upper limit are respectively calculated from the simulated 10,000 lower confidence limits and upper confidence limits for each confidence interval procedure.

Because the exact forms of the MLEs, $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\lambda}$, do not exist, **R** procedure uniroot is used to search the solution, labeled by $\hat{\lambda}$, of λ to Equation (16). Then the MLEs, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be obtained by plugging $\hat{\lambda}$ into Equations (12) and (13), respectively. After the values of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are obtained, the MLE, $\hat{\delta}_{s,k}$, of $\delta_{s,k}$ can be obtained by plugging $\hat{\alpha}_1$ and $\hat{\alpha}_2$ into Equation (5). Using the delta method with the Fisher information matrix described in Section 4.1, the CI-D of $\delta_{s,k}$ can be obtained for each combination of $(n, m) = (20, 5)$, (30,15), (50,30) and (63,30) and different censoring schemes. The above procedure has been implemented for 10,000 simulation runs. The CP of the CI-D is obtained as the percentage of these 10,000 repetitions of simulated confidence intervals that cover the true $\delta_{s,k}$.

The parts of the simulation results from using the delta method are reported in Table 1, which displays the general pattern of all the simulated CI-D. In view of Table 1, we can find that the delta method is too conservative and produces almost all the CPs of CI-D uniformly larger than the nominal level. All the simulated values of CP in Table 1 were higher than 0.99 except for the cell with $(n, m) = (50, 30)$ and the removal scheme of (0,0,0,...,0,30). The simulation results in Table 1 indicate that the delta method cannot perform well to obtain a reliable confidence interval of $\delta_{s,k}$. Actually, the proposed maximum likelihood estimation method in Section 3 produces a good point estimate of $\delta_{s,k}$ in terms of small bias and MSE. The proposed parametric percentile bootstrap method described in Section 4.2 can replace the delta method to obtain a reliable confidence interval of $\delta_{s,k}$. We will use the simulation results provided in Tables 2–9 to show good quality of the proposed maximum likelihood estimation method and the parametric percentile bootstrap method.

Table 1. The confidence interval of $\delta_{s,k}$ (CI-D) of $\delta_{s,k}$ and the coverage probability (CP) of the CI-D for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 3$ and $k = 5$.

n	m	Removal Scheme	CP *	Confidence Interval	
				Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	0.9924	0.0091	0.9504
		(15, 0, 0, 0, 0)	0.9931	0.0097	0.9479
		(3, 3, 3, 3, 3)	0.9927	0.0091	0.9503
30	15	(0, 0,...,15)	0.9988	0.0364	0.9292
		(15, 0,...,0)	0.9987	0.0389	0.9236
		(3, 0,0,...,3,0, 0)	0.9987	0.0376	0.9263
50	30	(0, 0, 0,...,0, 30)	0.9335	0.1939	0.7815
		(30, 0, 0,...,0, 0)	0.9995	0.0852	0.8832
		(3, 0, 3,0,...,3, 0)	0.9995	0.0832	0.8874
60	30	(3,0,0,...,3, 0, 0)	0.9995	0.0826	0.8884

Note: * The coverage probability of 95% confidence interval based on the normal approximation.

All the simulation results for CI-B show similar numerical behavior. Tables 2–9 only report a portion of simulation results of the bias, MSE and CI-B for discussion. First, we would like to evaluate the quality of the proposed MLEs. In view of Tables 2–9, we find that the bias of of $\hat{\delta}_{s,k}$ is close to 0 and the MSE of $\hat{\delta}_{s,k}$ are also small for all simulation settings. The findings indicate the proposed maximum likelihood estimation method is reliable to obtain the point estimate of the model parameters; and then, the reliable point estimate, $\hat{\delta}_{s,k}$, can be obtained by plugging $\hat{\alpha}_1$ and $\hat{\alpha}_2$ into Equation (5).

Second, we would like to check if the proposed parametric percentile bootstrap method produces a more reliable confidence interval than the delta method does? Tables 2–9 show that the values of CP in all simulation cases are uniformly closer to the nominal value of 0.95 than that in Table 1. Overall, the simulation results of CI-B for all cases under investigation show that the width of the approximated confidence interval of $\delta_{s,k}$ is shorter as the number of components in the life testing increases and the MSE of $\hat{\delta}_{s,k}$ decreases as the number of components in the life testing increases. Hence, using more sample resources for live testing can result in a more reliable point and interval estimation results.

In summary, the proposed maximum likelihood estimation procedure is reliable to obtain the MLEs of the model parameters and $\delta_{s,k}$. The delta method is easy to be used to obtain a confidence interval of $\delta_{s,k}$ with less computation time than using the parametric percentile bootstrap method. However, the CP of the CI-D could seriously overestimate the nominal confidence level. This fact indicates that the confidence interval based on the delta method is too conservative. The simulation results in Tables 2–9 show that the CP of the parametric percentile bootstrap confidence interval of $\delta_{s,k}$ is close to its nominal confidence level. Hence, the parametric percentile bootstrap method outperforms the delta method and can provide a reliable confidence interval for $\delta_{s,k}$. On the basis of our findings, we recommend obtaining an approximated confidence interval of $\delta_{s,k}$ for the MSS system by using the parametric percentile bootstrap method based on progressively type II censored samples from the GP distributions.

Table 2. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 1$ and $k = 1$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
		(0, 0, 0, 0, 15)	–0.0037	0.0072	0.9635	0.3205	0.6588
20	5	(15, 0, 0, 0, 0)	–0.0201	0.0197	0.9618	0.2045	0.7334
		(3, 3, 3, 3, 3)	–0.0091	0.0211	0.9573	0.2240	0.7498
		(0, 0,...,15)	–0.0068	0.0215	0.9551	0.2285	0.7541
30	15	(15, 0,...,0)	–0.0122	0.0067	0.9663	0.3055	0.6482
		(3,0,0,...,3,0, 0)	–0.0078	0.0070	0.9651	0.3136	0.6542
		(0, 0, 0,...,0, 30)	–0.0041	0.0055	0.9584	0.3433	0.6400
50	20	(30, 0, 0,...,0, 0)	–0.0131	0.0050	0.9585	0.3274	0.6285
		(3, 0, 3,0,...,3, 0)	–0.0071	0.0053	0.9602	0.3379	0.6364
		(0, 0, 0, ...,0, 20)	–0.0042	0.0035	0.9611	0.3677	0.6139
	30	(20, 0, 0, 0,...,0)	–0.0113	0.0034	0.9567	0.3574	0.6062
		(2, 0, 0,...,2, 0,0)	–0.0078	0.0035	0.9595	0.3628	0.6105
60	30	(3,0,0,...,3, 0, 0)	–0.0068	0.0035	0.9598	0.3644	0.6118

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 3. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 1$ and $k = 2$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	–0.0205	0.0267	0.9551	0.3242	0.8912
		(15, 0, 0, 0, 0)	–0.0338	0.0255	0.9618	0.2922	0.8772
		(3, 3, 3, 3, 3)	–0.0227	0.0264	0.9573	0.3184	0.8884
30	15	(0, 0,...,15)	–0.0084	0.0090	0.9635	0.4477	0.8216
		(15, 0,...,0)	–0.0174	0.0086	0.9663	0.4285	0.8124
		(3, 0,0,...,3,0, 0)	–0.0128	0.0089	0.9651	0.4389	0.8177
50	20	(0, 0, 0,...,0, 30)	–0.0078	0.0069	0.9584	0.4773	0.8057
		(30, 0, 0,...,0, 0)	–0.0175	0.0065	0.9585	0.4572	0.7953
		(3, 0, 3,0,...,3, 0)	–0.0110	0.0067	0.9602	0.4705	0.8025
	30	(0, 0, 0,...,0, 20)	–0.0068	0.0044	0.9611	0.5087	0.7825
		(20, 0, 0, 0,...,0)	–0.0145	0.0043	0.9567	0.4959	0.7750
		(2, 0, 0,...,2, 0,0)	–0.0107	0.0044	0.9595	0.5025	0.7791
60	30	(3,0,0,...,3, 0, 0)	–0.0096	0.0045	0.9598	0.5046	0.7804

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 4. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 3$ and $k = 3$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	0.0115	0.0137	0.9551	0.0938	0.5257
		(15, 0, 0, 0, 0)	–0.0005	0.0112	0.9618	0.0818	0.4962
		(3, 3, 3, 3, 3)	0.0093	0.0131	0.9573	0.0915	0.5192
30	15	(0, 0,...,15)	0.0271	0.0042	0.9635	0.1386	0.3988
		(15, 0,...,0)	–0.0042	0.0036	0.9663	0.1303	0.3873
		(3, 0,0,...,3,0, 0)	–0.0006	0.0039	0.9651	0.1348	0.3938
50	20	(0, 0, 0,...,0, 30)	0.0010	0.0031	0.9584	0.1507	0.3776
		(30, 0, 0,...,0, 0)	–0.0062	0.0027	0.9585	0.1417	0.3658
		(3, 0, 3,0,...,3, 0)	–0.0013	0.0030	0.9602	0.1470	0.3738
	30	(0, 0, 0,...,0, 20)	–0.0005	0.0020	0.9611	0.1641	0.3500
		(20, 0, 0, 0,...,0)	–0.0060	0.0018	0.9567	0.1580	0.3427
		(2, 0, 0,...,2, 0,0)	–0.0032	0.0019	0.9595	0.1611	0.3467
60	30	(3,0,0,...,3, 0, 0)	–0.0024	0.0019	0.9598	0.1621	0.3480

Note: * The coverage probability of 95% confidence interval based on the percentile Bootstrap method.

Table 5. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 2$ and $k = 4$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	–0.0125	0.0315	0.9551	0.2637	0.8750
		(15, 0, 0, 0, 0)	–0.0279	0.0294	0.9618	0.2347	0.8575
		(3, 3, 3, 3, 3)	–0.0151	0.0310	0.9573	0.258	0.8715
30	15	(0, 0,...,15)	–0.0057	0.0111	0.9635	0.3754	0.786
		(15, 0,...,0)	–0.0161	0.0104	0.9663	0.3569	0.7754
		(3, 0,0,...,3,0, 0)	–0.0108	0.0108	0.9651	0.3669	0.7819
50	20	(0, 0, 0,...,0, 30)	–0.0059	0.0085	0.9584	0.4035	0.7668
		(30, 0, 0,...,0, 0)	–0.0171	0.0078	0.9585	0.3838	0.7539
		(3, 0, 3,0,...,3, 0)	–0.0096	0.0083	0.9602	0.3968	0.7628
	30	(0, 0, 0,...,0, 20)	–0.0058	0.0055	0.9611	0.4338	0.7377
		(20, 0, 0, 0,...,0)	–0.0146	0.0053	0.9567	0.4209	0.7287
		(2, 0, 0,...,2, 0,0)	–0.0102	0.0054	0.9595	0.4276	0.7337
60	30	(3,0,0,...,3, 0, 0)	–0.0090	0.0055	0.9598	0.4297	0.7352

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 6. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 4$ and $k = 4$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	0.0131	0.0107	0.9551	0.0726	0.4595
		(15, 0, 0, 0, 0)	0.0021	0.0085	0.9618	0.0630	0.4295
		(3, 3, 3, 3, 3)	0.0100	0.0102	0.9573	0.0707	0.4529
30	15	(0, 0,...,15)	0.0032	0.0031	0.9635	0.1080	0.3339
		(15, 0,...,0)	−0.0027	0.0026	0.9663	0.1013	0.3231
		(3, 0,0,...,3,0, 0)	0.0003	0.0029	0.9651	0.1050	0.3292
50	20	(0, 0, 0,...,0, 30)	0.0016	0.0023	0.9584	0.1178	0.3139
		(30, 0, 0,...,0, 0)	−0.0046	0.0019	0.9585	0.1104	0.3030
		(3, 0, 3,0,...,3, 0)	−0.0004	0.0022	0.9602	0.1153	0.3105
	30	(0, 0, 0,...,0, 20)	<0.0001	0.0014	0.9611	0.1286	0.2884
		(20, 0, 0, 0,...,0)	−0.0047	0.0013	0.9567	0.1236	0.2818
		(2, 0, 0,...,2, 0,0)	−0.0023	0.0014	0.9595	0.1262	0.2855
60	30	(3,0,0,...,3, 0, 0)	−0.0016	0.0014	0.9598	0.1270	0.2866

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 7. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 1$ and $k = 5$.

n	*m*	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	−0.0387	0.0242	0.9551	0.4494	0.9689
		(15, 0, 0, 0, 0)	−0.0488	0.0244	0.9618	0.4097	0.9635
		(3, 3, 3, 3, 3)	−0.0403	0.0241	0.9573	0.4423	0.9679
30	15	(0, 0,...,15)	−0.0145	0.0074	0.9635	0.6034	0.9404
		(15, 0,...,0)	−0.0220	0.00742	0.966	0.5814	0.9352
		(3, 0,0,...,3,0, 0)	−0.0182	0.0074	0.9651	0.5934	0.9382
50	20	(0, 0, 0,...,0, 30)	−0.0124	0.0056	0.9584	0.6382	0.9321
		30, 0, 0,...,0, 0)	−0.0204	0.0055	0.9585	0.6158	0.9258
		(3, 0, 3,0,...,3, 0)	−0.0150	0.0056	0.9602	0.6308	0.9302
50	30	(0, 0, 0,...,0, 20)	−0.0095	0.0036	0.9611	0.6744	0.9188
		(20, 0, 0, 0,...,0)	−0.0161	0.0036	0.9567	0.6605	0.9138
		(2, 0, 0,...,2, 0,0)	−0.0129	0.0036	0.9595	0.6677	0.9166
60	30	(3,0,0,...,3, 0, 0)	−0.0120	0.0036	0.9598	0.6700	0.9174

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 8. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 3$ and $k = 5$.

n	m	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	–0.0006	0.0309	0.9551	0.2045	0.8179
		(15, 0, 0, 0, 0)	–0.0171	0.0277	0.9618	0.1804	0.7945
		(3, 3, 3, 3, 3)	–0.0035	0.03024	0.9573	0.2000	0.8131
30	15	(0, 0,...,15)	–0.0016	0.0108	0.9635	0.2965	0.7036
		(15, 0,...,0)	–0.0122	0.0098	0.9663	0.2805	0.6900
		(3, 0, 0,...,3, 0,0)	–0.0067	0.0104	0.9651	0.2891	0.6978
50	20	(0, 0, 0,...,0, 30)	–0.0028	0.0083	0.9584	0.3204	0.6794
		(30, 0, 0, 0,...,0)	–0.0141	0.0074	0.9585	0.3032	0.6647
		(3, 0, 3, 0,...,3, 0)	–0.0065	0.0079	0.9602	0.3146	0.6748
50	30	(0, 0, 0,...,0, 20)	–0.0037	0.0053	0.9611	0.3465	0.6457
		(20, 0, 0, 0,...,0)	–0.0125	0.0050	0.9567	0.3351	0.6358
		(2, 0, 0,...,2, 0, 0)	–0.0085	0.0052	0.9595	0.3410	0.6413
60	30	(3, 0, 0,...,3, 0, 0)	–0.0069	0.00532	0.9598	0.3429	0.6430

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

Table 9. Estimation results of $\delta_{s,k}$ using the parametric percentile bootstrap method for $(\alpha_1, \alpha_2, \lambda) = (2.5, 2.5, 1.0)$, $s = 5$ and $k = 5$.

n	m	Removal Scheme	Bias	MSE	CP *	Confidence Interval	
						Average Lower Limit	Average Upper Limit
20	5	(0, 0, 0, 0, 15)	0.0134	0.0086	0.9551	0.0592	0.4089
		(15, 0, 0, 0, 0)	0.0034	0.0067	0.9618	0.05128	0.3793
		(3, 3, 3, 3, 3)	0.0115	0.0081	0.9573	0.05770	0.4023
30	15	(0, 0,...,15)	0.0034	0.0024	0.9635	0.0885	0.2873
		(15, 0,...,0)	–0.0019	0.0020	0.9663	0.0829	0.2773
		(3, 0,0,...,3,0, 0)	0.0008	0.0022	0.9651	0.0859	0.2830
50	20	(0, 0, 0,...,0, 30)	0.0018	0.0017	0.9584	0.0967	0.2687
		(30, 0, 0,...,0, 0)	–0.0036	0.0014	0.9585	0.0905	0.2588
		(3, 0, 3,0,...,3, 0)	<0.0001	0.0016	0.9602	0.0946	0.2656
	30	(0, 0, 0,...,0, 20)	0.0002	0.0011	0.9611	0.1057	0.2453
		(20, 0, 0, 0,...,0)	–0.0038	0.0010	0.9567	0.1015	0.23942
		(2, 0, 0,...,2, 0,0)	–0.0017	0.0010	0.9595	0.1037	0.2427
60	30	(3,0,0,...,3, 0, 0)	–0.0011	0.0010	0.9598	0.1043	0.2437

Note: * The coverage probability of 95% confidence interval based on the percentile bootstrap method.

6. Conclusions

In this paper, the reliability of an MSS system with k identical components is investigated based on progressively type II censored samples from GP distributions. The MSS system is functional if at least $s(1 \leq s \leq k)$ components' strength in the system exceed the common stress. The stress distribution and the strength distribution of components are assumed to have different shape parameters and same scale parameter. A maximum likelihood estimation procedure is analytically studied and the Fisher information matrix is obtained. The proposed maximum likelihood estimation procedure is reliable to obtain the MLEs of the model parameters and the stress-strength parameter.

In order to obtain a reliable confidence interval for the stress-strength parameter, the interval inference procedures based on the delta method and the parametric percentile bootstrap method are investigated, and the obtained confidence intervals are denoted by CI-D and CI-B, respectively. The quality of the CI-D and CI-B of the stress-strength parameter are evaluated via using Monte Carlo simulations. Simulation results indicate that the CI-B method outperforms the CI-D method in terms of the coverage probability.

The strength of components and the stress applied to the MSS system are assumed to follow GP distributions, which have different shape parameters and the same scale parameter. The mathematical derivation is very difficult if two GP distributions have different shapes and scale parameters in this study. This topic is beyond the goal of this paper and can be an open question for future study. Moreover, this paper can be a valuable addition to the works of Rezaei, et al. [12] and Rezaei, et al. [2], who considered the Pareto distribution, as well as Pak et al. [8]; Mokhlis and Khames [19]; Rao [5]; and many others who considered reliability inference under various working assumptions. A comprehensive review of all published information regarding this topic should be compiled in the future in hopes of inferring generalized conclusions regarding the reliability of stress-strength systems.

Author Contributions: All authors equally contributed in writing this manuscript. All authors read and approved the final manuscript.

Funding: This study is supported by the grant of Ministry of Science and Technology, Taiwan MOST 108-2221-E-032-018-MY2.

Acknowledgments: The authors would also like to thank for the editing help provided by Dan Van Peursem to improve the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kundu, D.; Gupta, R.D. Estimation of $P(Y < X)$ for Weibull Distributions. *IEEE Trans. Reliab.* **2006**, *55*, 270–280.
2. Rezaei, S.; Tahmasbi, R.; Mahmoodi, M. Estimation of $P(Y < X)$ for generalized Pareto distribution. *J. Stat. Plan. Inference* **2010**, *140*, 480–494.
3. Kohansal, A.; Rezakhah, S. Inference of $R = P(Y < X)$ for Two-parameter Rayleigh Distribution Based on Progressively Censored Samples. *Statistics* **2019**, *53*, 81–100.
4. Akgül, F.G.; Şenğlu, B. Estimation of Stress-Strength Reliability for Weibull Distribution Based on Type-II Right Censored Ranked Set Sampling Data. *Pak. J. Stat. Oper. Res.* **2018**, *14*, 781–806.
5. Rao, G.S. Estimation of Reliability in Multicomponent Stress-strength Based on Generalized Exponential Distribution. *Rev. Colomb. Estadstíca* **2012**, *35*, 67–76.
6. Rao, G.S.; Aslam, M.; Kundu, D. Burr-XII Distribution Parametric Estimation and Estimation of Reliability of Multicomponent Stress-Strength. *Commun. Stat.-Theory Methods* **2015**, *44*, 4953–4961.
7. Hassan, M.K.H.; Alohali, M. I. Estimation of Reliability in a Multicomponent Stress-Strength Model Based on Generalized Linear Failure Rate Distribution. *Hacet. J. Math. Stat.* **2018**, *47*, 1634–1651.
8. Pak, A.; Gupta, A.K.; Khoolenjani, N.B. On Reliability in a Multicomponent Stress-Strength Model with Power Lindley Distribution. *Rev. Colomb. Estadstíca* **2018**, *41*, 251–267.
9. Kotz, S.; Lumelskii, Y.; Pensky, M. *The Stress-Strength Model and Its Generalizations: Theory and Applications*; World Scientific: Singapore, 2003.
10. Mann, N.R.; Schafer, R.E.; Singpurwalla, N.D. *Methods for Statistical Analysis of Reliability and Life Data*; Wiley: New York, NY, USA, 1974.
11. Lio, Y.L.; Tsai, T.-R. Estimation of $\delta = P(X < Y)$ for Burr XII distribution based on the progressively first failure-censored samples. *J. Appl. Stat.* **2011**, *39*, 309–322.
12. Rezaei, S.; Noughabi, R.A.; Nadarajah, S. Estimation of Stress-Strength Reliability of the Generalized Pareto Distribution Based on Progressively Censored Samples. *Ann. Data Sci.* **2015**, *2*, 82–101.
13. Arnold, B.C. *Pareto Distribution*; Wiley: New York, NY, USA, 2015; pp. 1–10.
14. Kamps, U.; Cramer, E. On Distributions of Generalized Order Statistics. *Statistics* **2001**, *35*, 269–280.
15. Chernick, M.R. *Bootstrap Methods: A Practitioner's Guide*; John Wiley and Son: Hoboken, NJ, USA, 1999.

16. Efron, B.; Tibshirani, R.J. *An Introduction to the Bootstrap*; Chapman & Hall: London, UK, 1993.
17. Balakrishnan, N.; Sandhu, R.A. A Simple Simulational Algorithm for Generating Progressive Type-II Censored Samples. *Am. Stat.* **1995**, *49*, 229–230.
18. Wu, S.-J.; Kuş, C. On the Estimation Based on Progressive First Failure-censored Sample. *Comput. Stat. Data Anal.* **2009**, *53*, 3659–3670.
19. Mokhlis, N.A.; Khames, S.K. Reliability of Multi-component Stress-strength Models. *J. Egypt. Math. Soc.* **2011**, *19*, 106–111.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).