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Pointwise Rectangular Lipschitz Regularities for Fractional Brownian Sheets and Some Sierpinski Selfsimilar Functions

Mourad Ben Slimane [1,*,†], Moez Ben Abid [2,†], Ines Ben Omrane [3,†] and Mohamad Maamoun Turkawi [1,†]

1 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; mturkawi@ksu.edu.sa
2 Ecole Supérieure des Sciences et Technologie de Hammam Sousse, Université de Sousse, Sousse 4011, Tunisia; moezbenabid@yahoo.fr
3 Department of Mathematics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, Riyadh 11623, Saudi Arabia; imbenomrane@imamu.edu.sa
* Correspondence: mbenslimane@ksu.edu.sa
† These authors contributed equally to this work.

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Abstract: We consider pointwise rectangular Lipschitz regularity and pointwise level coordinate axes Lipschitz regularities for continuous functions f on the unit cube I^2 in $\mathbb{R}^2$. Firstly, we provide characterizations by simple estimates on the decay rate of the coefficients (resp. leaders) of the expansion of f in the rectangular Schauder system, near the point considered. We deduce that pointwise rectangular Lipschitz regularity yields pointwise level coordinate axes Lipschitz regularities. As an application, we refine earlier results in Ayache et al. (Drap brownien fractionnaire. Potential Anal. 2002, *17*, 31–43) and Kamont (On the fractional anisotropic Wiener field. Probab. Math. Statist. **1996**, *16*, 85–98), where uniform rectangular Lipschitz regularity of the trajectories of the fractional Brownian sheet over the total I^2 (or any cube) was considered. Actually, we prove that fractional Brownian sheets are pointwise rectangular and level coordinate axes monofractal. On the opposite, we construct a class of Sierpinski selfsimilar functions that are pointwise rectangular and level coordinate axes multifractal.

Keywords: rectangular Lipschitz regularity; level coordinate axes Lipschitz regularity; expansion in the rectangular Schauder system; monofractal; multifractal; Sierpinski selfsimilar functions; fractional Brownian sheets

1. Introduction

Many authors have studied several properties of anisotropic random fields, see for example [1–13], and the references therein for further information. Some of these fields are models for textures in images (see [11] and the references therein).

For a given two parameter index $\bar{H} = (H_1, H_2) \in (0,1)^2$, the fractional Brownian sheet $\{B^{\bar{H}}(x) : x = (x_1, x_2)\}$, is a real-valued centered Gaussian field, introduced by Kamont in [7], then redefined by Ayache et al. in [2] through the following fractional integration with respect to the standard real-valued Brownian sheet W on $\mathbb{R}^2$

$$\forall x = (x_1, x_2) \in \mathbb{R}^2_+ \quad B^{\bar{H}}(x) = \int_{\mathbb{R}^2} \prod_{i=1}^{2} g_{H_i}(x_i, u_i)\, dW(u)\,, \tag{1}$$

where

$$u = (u_1, u_2) \ , \quad g_a(t,s) = (t-s)_+^{a-1/2} - (-s)_+^{a-1/2} \text{ and } \quad s_+ = \max(s,0) \ .$$

The realizations of the fractional Brownian sheet are continuous. The covariance is given by

$$\mathbb{E}[B^{\bar{H}}(t_1,t_2)B^{\bar{H}}(s_1,s_2)] = \prod_{i=1}^{2} K^{2H_i}(t_i,s_i)$$

where

$$K^{2a}(t,s) = \frac{1}{2}(|t|^{2a} + |s|^{2a} - |t-s|^{2a})$$

is the covariance kernel of the fractional Brownian motion B_a in $\mathbb{R}$ with Hurst index a.
Recall that a corresponds to the critical uniform Lipschitz exponent of the sample paths of B_a over any arbitrary compact interval I; one has, almost surely

$$\sup\{\alpha \in (0,1) \ : \quad B_a \in Lip^{\alpha}(I)\} = a \ .$$

Actually, (see for example [14]), B_a is monofractal of order a in the sense that

$$\forall \ t \qquad \sup\{\alpha \in (0,1) \ : \quad B_a \in Lip^{\alpha}(t)\} = a \ .$$

Let us recall the notions of $Lip^{\alpha}(I)$ and $Lip^{\alpha}(t_0)$. Without any loss of generality, let I denotes the unit interval $[0,1]$.

Definition 1. *Let u be a continuous function on I (we write $u \in C(I)$). Let $t_0 \in I$. Let $\alpha \in (0,1)$.*
We say that u is Lipschitz of order α at t_0 and we write $u \in Lip^{\alpha}(t_0)$, if there exists $C > 0$ such that

$$\forall \ t \in I \qquad |u(t) - u(t_0)| \le C|t - t_0|^{\alpha} \ . \tag{2}$$

We say that u is Lipschitz of order α on I and we write $u \in Lip^{\alpha}(I)$, if the constant C in (2) is uniform on all $t_0 \in I$.
We say that u is uniform Lipschitz on I if there exists $\delta > 0$ such that $u \in Lip^{\delta}(I)$.

When $H_1 \neq H_2$, the fractional Brownian sheet given in (1) has the following anisotropic operator-selfsimilarity:

$$\forall a_1, a_2 > 0 \quad a_1^{H_1} a_2^{H_2} B^{\bar{H}}(x_1/a_1, x_2/a_2) = B^{\bar{H}}(x) \quad \text{in law} \ . \tag{3}$$

It is proved that the fractional Brownian sheet has stationary rectangular increments. In [2] (resp. [7]), it is also proved that for any cube $\mathcal{Q} \subset \mathbb{R}^2$, the restrictions $B^{\bar{H}}_{\mathcal{Q}}$ of realizations of $B^{\bar{H}}$ to $\mathcal{Q}$ are uniform rectangular Lipschitz with order $\bar{H}' = (H'_1, H'_2)$ for $H'_1 < H_1$ and $H'_2 < H_2$, in the sense that

$$\exists \ C > 0 \ ; \quad \forall \ (x,y) \in \mathcal{Q}^2 \qquad |\Box_y B^{\bar{H}}_{\mathcal{Q}}(x)| \le C \prod_{i=1}^{2} |y_i - x_i|^{H'_i} \ , \tag{4}$$

where rectangular increments of a continuous function f are defined for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ by

$$\Box_y f(x) = f(y) - f(x_1, y_2) - f(y_1, x_2) + f(x) \ . \tag{5}$$

This paper is concerned with pointwise rectangular Lipschitz regularity and pointwise level coordinate axes Lipschitz regularities; without any loss of generality, we will take $\mathcal{Q}$ the unit cube $I^2 = [0,1]^2$.

Definition 2. *(Pointwise rectangular Lipschitz regularity)*

Let f be a continuous function on I^2 (we write $f \in C(I^2)$). Let $x \in I^2$. Let $\bar{\alpha} = (\alpha_1, \alpha_2) \in (0,1)^2$. We say that f is rectangular Lipschitz of order $\bar{\alpha}$ at x and we write $f \in Lip^{\bar{\alpha}}(x)$, if there exists $C > 0$ such that

$$\forall\, y \in I^2 \qquad |\Box_y f(x)| \leq C \prod_{i=1}^{2} |y_i - x_i|^{\alpha_i} \, . \tag{6}$$

We say that f is rectangular Lipschitz of order $\bar{\alpha}$ on I^2 and we write $f \in Lip^{\bar{\alpha}}(I^2)$, if the constant C in (6) is uniform on all $x \in I^2$.

We say that f is uniform Lipschitz on I^2 if there exists $\delta > 0$ such that $f \in Lip^{\delta}(I^2)$ in the sense that there exists $C > 0$ such that

$$\forall\, x, y \in I^2 \qquad |f(y) - f(x)| \leq C|x - y|^{\delta} \, . \tag{7}$$

Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$.

Definition 3. *(Pointwise level coordinate axes Lipschitz regularities)*

Let $s \in (0,1)$, $x = (x_1, x_2) \in I^2$ and $f \in C(I^2)$. We say that $f \in N^s(x, e_1)$ if there exists $C > 0$ such that

$$\forall y = (y_1, y_2) \in I^2 \qquad |f(y) - f(x_1, y_2)| \leq C\, |y_1 - x_1|^s \, .$$

We say that $f \in N^s(x, e_2)$ if there exists $C > 0$ such that

$$\forall y = (y_1, y_2) \in I^2 \qquad |f(y) - f(y_1, x_2)| \leq C\, |y_2 - x_2|^s \, .$$

Clearly

$$\forall\, \sigma \in (0,1) \quad \bigcap_{i=1}^{2} N^{\alpha_i}(x, e_i) \subset Lip^{(\sigma\alpha_1, (1-\sigma)\alpha_2)}(x) \, . \tag{8}$$

Let us mention that pointwise level coordinate axes Lipschitz regularities $N^s(x, e_i)$ are stronger then pointwise directional Lipschitz regularities $C^s(x, e_i)$ that have been studied in [15]. Let $f \in C(I^2), x = (x_1, x_2) \in I^2$ and $0 < s < 1$. Recall that $f \in C^s(x, e_1)$ (resp. $f \in C^s(x, e_2)$) if there exists a positive constant C such that $|f(y_1, x_2) - f(x)| \leq C|y_1 - x_1|^s \quad \forall\, y_1 \in I$ (resp. $|f(x_1, y_2) - f(x)| \leq C|y_2 - x_2|^s \quad \forall\, y_2 \in I$). Actually $N^s(x, e_i) \subset C^s(x, e_i)$. Of course there is no converse embedding between $C^s(x, e_i)$ and both pointwise rectangular Lipschitz regularity and pointwise level coordinate axes Lipschitz regularities.

In the next section (respectively the third section), we characterize pointwise rectangular Lipschitz regularity (respectively pointwise level coordinate axes Lipschitz regularities) by simple estimates on the decay rate of the coefficients/leaders of the expansion of the function in the basis of tensor products of Schauder functions, near the point considered (see Theorem 1/Theorem 2 (respectively Theorem 3/Theorem 4)). We deduce that pointwise rectangular Lipschitz regularity yields pointwise level coordinate axes Lipschitz regularities (see Theorem 5).

In the fourth section, as an application, we refine result (4) by proving that fractional Brownian sheets are pointwise rectangular and level coordinate axes monofractal (see Theorem 6). A second application will be done to more general anisotropic deterministic selfsimilar functions that can modelize anisotropic turbulence or cascades. We construct a class of Sierpinski selfsimilar functions that are pointwise rectangular and level coordinate axes multifractal (see Theorem 7).

Finally, a short conclusion section is given.

2. Characterization of $Lip^{\bar{\alpha}}(x)$ in Rectangular Schauder Bases

2.1. Characterization with Rectangular Schauder Coefficients

The rectangular Schauder system $\{\Phi_{\mathbf{m}=(m_1,m_2)}\}_{\mathbf{m}\in\mathbb{N}_0^2}$ of $C(I^2)$ is obtained by tensor products $\Phi_{\mathbf{m}}(y) = \Phi_{\mathbf{m}}(y_1, y_2) = \prod_{i=1}^2 \phi_{m_i}(y_i)$, of classical (1-variable) Schauder functions $\{\phi_m, m \geq 0\}$ on I, normed in L^∞. Recall that $\phi_0 = 1$, $\phi_1(t) = t$, and for $m \geq 2$, $m = 2^j + n$ with $j \geq 0$ and $1 \leq n \leq 2^j$, $\phi_m(t) = \phi(2^{j+1}t - 2n + 1)$ with support $[(n-1)2^{-j}, n2^{-j}]$, where $\phi(t) = \max(0, 1 - |t|)$.

Let $M = \mathbb{N}_0 \cup \{-2, -1\}$. For $j \in M$, let

$$\tilde{N}_{-2} = \{0\},\ \tilde{N}_{-1} = \{1\},\ \text{and}\ \tilde{N}_j = \{2^j + n : n = 1, \cdots, 2^j\}\ \text{ for } j \geq 0\ . \tag{9}$$

It is known that if $u \in C(I)$, then $u = \sum_{j\in M} \sum_{m\in\tilde{N}_j} b_m(u)\phi_m$ with

$$b_0(u) = u(0)\ \ ,\ \ b_1(u) = u(1) - u(0)$$

and

$$\forall\, j \geq 0\ \ \forall\, m = 2^j + n \in \tilde{N}_j \quad b_m(u) = u(\frac{2n-1}{2^{j+1}}) - \frac{1}{2}\left(u(\frac{n-1}{2^j}) + u(\frac{n}{2^j})\right)\ . \tag{10}$$

For $\mathbf{j} = (j_1, j_2) \in M^2$, we put

$$\tilde{N}_{\mathbf{j}} = \tilde{N}_{j_1} \times \tilde{N}_{j_2}\ . \tag{11}$$

Denote by $\mathbf{0}$ and $\mathbf{1}$ respectively the vectors $(0,0)$ and $(1,1)$. If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ belong to $\mathbb{R}^2$, we will write $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \{1,2\}$, $\mathbf{a} < \mathbf{b}$ (resp. $\mathbf{a} > \mathbf{b}$) if $a_i < b_i$ (respectively $a_i > b_i$) for all $i \in \{1,2\}$, and $\mathbf{a} \not\leq \mathbf{b}$ if either $a_1 > b_1$ or $a_2 > b_2$.

Any $f \in C(I^2)$ can be written as

$$\begin{aligned}
f(y) &= \sum_{\mathbf{j}\in M^2} \sum_{\mathbf{m}\in\tilde{N}_{\mathbf{j}}} C_{\mathbf{m}}\Phi_{\mathbf{m}}(y) \\
&= C_{(0,0)} + C_{(0,1)}y_2 + C_{(1,0)}y_1 + C_{(1,1)}y_1y_2 \\
&+ \sum_{j\geq 0} \sum_{m\in\tilde{N}_j} C_{(0,m)}\phi_m(y_2) + \sum_{j\geq 0} \sum_{m\in\tilde{N}_j} C_{(m,0)}\phi_m(y_1) \\
&+ y_1 \sum_{j\geq 0} \sum_{m\in\tilde{N}_j} C_{(1,m)}\phi_m(y_2) + y_2 \sum_{j\geq 0} \sum_{m\in\tilde{N}_j} C_{(m,1)}\phi_m(y_1) \\
&+ \sum_{\mathbf{j}\geq\mathbf{0}} \sum_{\mathbf{m}=(m_1,m_2)\in\tilde{N}_{\mathbf{j}}} C_{\mathbf{m}}\phi_{m_1}(y_1)\phi_{m_2}(y_2)\ ,
\end{aligned}$$

where

$$C_{(0,0)} = f(0,0),\ C_{(0,1)} = f(0,1) - f(0,0),\ C_{(1,0)} = f(1,0) - f(0,0)\ ,$$

$$C_{(1,1)} = f(1,1) + f(0,0) - f(1,0) - f(0,1)\ ,$$

$$\forall\, j \geq 0\ \ \forall\, m \in \tilde{N}_j \quad C_{(0,m)} = f_m(0)$$

with

$$f_m(t) = f(t, \frac{2n-1}{2^{j+1}}) - \frac{1}{2}\left(f(t, \frac{n-1}{2^j}) + f(t, \frac{n}{2^j})\right)\ , \tag{12}$$

$$\forall\, j \geq 0\ \ \forall\, m \in \tilde{N}_j \quad C_{(m,0)} = g_m(0)$$

with

$$g_m(t) = f(\frac{2n-1}{2^{j+1}}, t) - \frac{1}{2}\left(f(\frac{n-1}{2^j}, t) + f(\frac{n}{2^j}, t)\right)\ , \tag{13}$$

$$\forall\, j \geq 0\ \ \forall\, m \in \tilde{N}_j \quad C_{(1,m)} = f_m(1) - f_m(0)\ ,$$

$$\forall\, j \geq 0 \;\; \forall\, m \in \tilde{N}_j \qquad C_{(m,1)} = g_m(1) - g_m(0)\;,$$

and

$$\forall\, \mathbf{j} \geq \mathbf{0} \;\; \forall\, \mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}} \qquad C_{\mathbf{m}} = f_{m_2}(\frac{2n_1 - 1}{2^{j_1+1}}) - \frac{1}{2}\left[f_{m_2}(\frac{n_1 - 1}{2^{j_1}}) + f_{m_2}(\frac{n_1}{2^{j_1}})\right]\,. \tag{14}$$

It is known that both uniform and pointwise Lipschitz regularities $Lip^{\alpha}(I)$ and $Lip^{\alpha}(t_0)$ (given in Definition 1) are characterized in Schauder bases (see [16] for example).

Proposition 1. *Let $0 < \alpha < 1$. Let $u \in C(I)$.*

1.
$$u \in Lip^{\alpha}(I) \quad \Leftrightarrow \quad \exists\, C > 0 \;\; \forall\, j \geq 0 \;\; \forall\, m \in \tilde{N}_j \quad |b_m(u)| \leq C 2^{-\alpha j}\,. \tag{15}$$

2. *Let $t_0 \in I$. If $u \in Lip^{\alpha}(t_0)$ then there exists C such that*
$$\forall\, j \geq 0 \;\; \forall\, m = 2^j + n \in \tilde{N}_j \quad |b_m(u)| \leq C(2^{-\alpha j} + |t_0 - n2^{-j}|^{\alpha})\,. \tag{16}$$

Conversely, if u is uniform Lipschitz on I and (16) holds then $u \in Lip^{\alpha'}(t_0)$ for all $\alpha' < \alpha$.

Remark 1. *Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Clearly if $W(y) = u(y_1)v(y_2)$, $U(y) = u(y_1)$ and $V(y) = v(y_2)$ then*

$$\Box_y W(x) = (u(y_1) - u(x_1))(v(y_2) - v(x_2))\;,$$
$$\Box_y U(x) = 0 \qquad \text{and} \qquad \Box_y V(x) = 0\,.$$

Thanks to Remark 1

- function
$$g_0(y) := C_{(0,0)} + C_{(0,1)} y_2 + C_{(1,0)} y_1 + C_{(1,1)} y_1 y_2 \tag{17}$$
belongs to $Lip^{\bar{\alpha}}(I^2)$ for all $\bar{\alpha} = (\alpha_1, \alpha_2) < \mathbf{1}$,
- function
$$g_1(y) := \sum_{j\geq 0} \sum_{m \in \tilde{N}_j} C_{(0,m)} \phi_m(y_2) = f(0, y_2) - f(0,0) - C_{(0,1)} y_2 \tag{18}$$
belongs to $Lip^{\bar{\alpha}}(x)$ for all $\bar{\alpha} = (\alpha_1, \alpha_2) < \mathbf{1}$,
- function
$$g_2(y) := \sum_{j\geq 0} \sum_{m \in \tilde{N}_j} C_{(m,0)} \phi_m(y_1) = f(y_1, 0) - f(0,0) - C_{(1,0)} y_1 \tag{19}$$
belongs to $Lip^{\bar{\alpha}}(I^2)$ for all $\bar{\alpha} = (\alpha_1, \alpha_2) < \mathbf{1}$,
- function
$$g_3(y) := y_1 \sum_{j\geq 0} \sum_{m \in \tilde{N}_j} C_{(1,m)} \phi_m(y_2) = y_1\,[f(1, y_2) - f(0, y_2)] - y_1 C_{(1,0)} - y_1 y_2 C_{(1,1)} \tag{20}$$
belongs to $Lip^{\bar{\alpha}}(x)$ if and only if the one variable function $f(1,t) - f(0,t)$ belongs to $Lip^{\alpha_2}(x_2)$,
- function
$$g_4(y) := y_2 \sum_{j\geq 0} \sum_{m \in \tilde{N}_j} C_{(m,1)} \phi_m(y_1) = y_2\,[f(y_1, 1) - f(y_1, 0)] - y_2 C_{(0,1)} - y_1 y_2 C_{(1,1)} \tag{21}$$
belongs to $Lip^{\bar{\alpha}}(x)$ if and only if the one variable function $f(t,1) - f(t,0)$ belongs to $Lip^{\alpha_1}(x_1)$.

If f is uniform Lipschitz on I^2, then functions $f(1,t) - f(0,t)$ and $f(t,1) - f(t,0)$ are uniform Lipschitz on I, consequently g_3 and g_4 can be sharply characterized in $Lip^{\bar{\alpha}}(x)$ using the second result of Proposition 1. More precisely

Proposition 2. *Let $f \in C(I^2)$, $\mathbf{0} < \bar{\alpha} < \mathbf{1}$ and $x \in I^2$.*

1. *If g_3 and g_4 belong to $Lip^{\bar{\alpha}}(x)$ then there exists $C > 0$ such that*

$$\forall\, j \geq 0 \ \ \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(1,m)}| \leq C(2^{-j} + \left|n2^{-j} - x_2\right|)^{\alpha_2} \tag{22}$$

and

$$\forall\, j \geq 0 \ \ \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(m,1)}| \leq C(2^{-j} + \left|n2^{-j} - x_1\right|)^{\alpha_1} . \tag{23}$$

2. *Conversely, if f is uniform Lipschitz on I^2 and both (22) and (23) are satisfied then*

$$\forall\, i \in \{3,4\} \ \ \forall\, \bar{\alpha}' < \bar{\alpha} \qquad g_i \in Lip^{\bar{\alpha}'}(x) . \tag{24}$$

Remark 2. *Note that if $f \in Lip^{\bar{\alpha}}(x)$ then g_3 and g_4 belong to $Lip^{\bar{\alpha}}(x)$. In fact*

$$\begin{aligned} |f(1,t) - f(0,t) - f(1,x_2) + f(0,x_2)| &= |\square_{(1,t)} f(x) - \square_{(0,t)} f(x)| \\ &\leq |\square_{(1,t)} f(x)| + |\square_{(0,t)} f(x)| \\ &\leq C|t - x_2|^{\alpha_2} \end{aligned}$$

and

$$\begin{aligned} |f(t,1) - f(t,0) - f(x_1,1) + f(x_1,0)| &= |\square_{(t,1)} f(x) - \square_{(t,0)} f(x)| \\ &\leq |\square_{(t,1)} f(x)| + |\square_{(t,0)} f(x)| \\ &\leq C|t - x_1|^{\alpha_1} . \end{aligned}$$

To achieve the sharp characterization of $Lip^{\bar{\alpha}}(x)$ by Schauder coefficients, it remains to deal with the series

$$F(y) = \sum_{\mathbf{j} \geq \mathbf{0}} \ \sum_{\mathbf{m}=(m_1,m_2) \in \tilde{N}_{\mathbf{j}}} C_{\mathbf{m}} \phi_{m_1}(y_1) \phi_{m_2}(y_2) . \tag{25}$$

Proposition 3. *Let $f \in C(I^2)$, $\mathbf{0} < \bar{\alpha} < \mathbf{1}$ and $x \in I^2$.*

1. *If $f \in Lip^{\bar{\alpha}}(x)$ then there exists $C > 0$ such that*

$$\forall\, \mathbf{j} \geq \mathbf{0} \ \ \forall\, \mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}} \qquad |C_{\mathbf{m}}| \leq C \prod_{i=1}^{2} (2^{-j_i} + \left|n_i 2^{-j_i} - x_i\right|)^{\alpha_i} . \tag{26}$$

2. *Conversely, if f is uniform Lipschitz on I^2 and (26) is satisfied, then the function F given in (25) satisfies*

$$\forall\, \bar{\alpha}' < \bar{\alpha} \qquad F \in Lip^{\bar{\alpha}'}(x) . \tag{27}$$

Proof of Proposition 3.

1. Assume that $f \in Lip^{\bar{\alpha}}(x)$. Recall that coefficients $C_{\mathbf{m}}$ are given by (14) where f_{m_2} is as in (12). Write

$$f_{m_2}(t) = f(t, \frac{2n_2-1}{2^{j_2+1}}) - f(x_1, \frac{2n_2-1}{2^{j_2+1}}) - f(t, x_2) + f(x)$$
$$- \frac{1}{2}\left[f(t, \frac{n_2-1}{2^{j_2}}) - f(x_1, \frac{n_2-1}{2^{j_2}}) - f(t, x_2) + f(x)\right.$$
$$\left. + f(t, \frac{n_2}{2^{j_2}}) - f(x_1, \frac{n_2}{2^{j_2}}) - f(t, x_2) + f(x)\right]$$
$$+ f_{m_2}(x_1)$$
$$= \Box_{(t, \frac{2n_2-1}{2^{j_2+1}})} f(x) - \frac{1}{2}\left[\Box_{(t, \frac{n_2-1}{2^{j_2}})} f(x) + \Box_{(t, \frac{n_2}{2^{j_2}})} f(x)\right] + f_{m_2}(x_1) .$$

Thus

$$C_{\mathbf{m}} = f_{m_2}(\frac{2n_1-1}{2^{j_1+1}}) - \frac{1}{2}\left[f_{m_2}(\frac{n_1-1}{2^{j_1}}) + f_{m_2}(\frac{n_1}{2^{j_1}})\right]$$
$$= \Box_{(\frac{2n_1-1}{2^{j_1+1}}, \frac{2n_2-1}{2^{j_2+1}})} f(x) - \frac{1}{2}\left(\Box_{(\frac{2n_1-1}{2^{j_1+1}}, \frac{n_2-1}{2^{j_2}})} f(x) + \Box_{(\frac{2n_1-1}{2^{j_1+1}}, \frac{n_2}{2^{j_2}})} f(x)\right)$$
$$- \frac{1}{2}\left[\Box_{(\frac{n_1-1}{2^{j_1}}, \frac{2n_2-1}{2^{j_2+1}})} f(x) - \frac{1}{2}\left(\Box_{(\frac{n_1-1}{2^{j_1}}, \frac{n_2-1}{2^{j_2}})} f(x) + \Box_{(\frac{n_1-1}{2^{j_1}}, \frac{n_2}{2^{j_2}})} f(x)\right)\right.$$
$$\left. + \Box_{(\frac{n_1}{2^{j_1}}, \frac{2n_2-1}{2^{j_2+1}})} f(x) - \frac{1}{2}\left(\Box_{(\frac{n_1}{2^{j_1}}, \frac{n_2-1}{2^{j_2}})} f(x) + \Box_{(\frac{n_1}{2^{j_1}}, \frac{n_2}{2^{j_2}})} f(x)\right)\right] .$$

Since $f \in Lip^{\tilde{\alpha}}(x)$ then (26) holds.

2. Conversely, assume that f is uniform Lipschitz on I^2 and (26) is satisfied. Using (12)

$$|f_m(t)| \le \frac{1}{2}|f(t, \frac{2n-1}{2^{j+1}}) - f(t, \frac{n-1}{2^j})| + \frac{1}{2}|f(t, \frac{2n-1}{2^{j+1}}) - f(t, \frac{n}{2^j})| \le C2^{-\delta j} .$$

Using (7) and (14)

$$\forall\, \mathbf{j} \ge \mathbf{0}\ \ \forall \mathbf{m} \in \tilde{N}_{\mathbf{j}} \qquad |C_{\mathbf{m}}| \le C2^{-\delta j_1} . \tag{28}$$

Since

$$C_{\mathbf{m}} = f_{m_1}(\frac{2n_2-1}{2^{j_2+1}}) - \frac{1}{2}\left[f_{m_1}(\frac{n_2-1}{2^{j_2}}) + f_{m_1}(\frac{n_2}{2^{j_2}})\right]$$

then similarly

$$\forall\, \mathbf{j} \ge \mathbf{0}\ \ \forall \mathbf{m} \in \tilde{N}_{\mathbf{j}} \qquad |C_{\mathbf{m}}| \le C2^{-\delta j_2} . \tag{29}$$

It follows that

$$\forall\, \mathbf{j} \ge \mathbf{0}\ \ \forall \mathbf{m} \in \tilde{N}_{\mathbf{j}}\ \ \forall \theta \in (0,1) \qquad |C_{\mathbf{m}}| \le C2^{-\theta\delta j_1} 2^{-(1-\theta)\delta j_2} . \tag{30}$$

Put $\delta_1 = \delta\theta$ and $\delta_2 = \delta(1-\theta)$. By (26) there exists $C > 0$ such that

$$\forall \sigma \in [0,1]\ \forall \mathbf{j} \ge \mathbf{0}\ \forall \mathbf{m} \in \tilde{N}_{\mathbf{j}} \quad |C_{\mathbf{m}}| \le C\prod_{i=1}^{2} 2^{-(1-\sigma)\delta_i j_i}(2^{-j_i} + \left|n_i 2^{-j_i} - x_i\right|)^{\sigma\alpha_i} .$$

Then

$$|C_{\mathbf{m}}| \le C\prod_{i=1}^{2} \mu_{\delta_i,\alpha_i,j_i,m_i,x_i} \tag{31}$$

where for $0 < \delta < 1$, $0 < \alpha < 1$, $j \ge 0$, $m = 2^j + n \in \tilde{N}_j$ and $t \in \mathbb{R}$

$$\mu_{\delta,\alpha,j,m,t} = 2^{-(1-\sigma)\delta j}(2^{-j} + \left|n2^{-j} - t\right|)^{\sigma\alpha} . \tag{32}$$

For $h \in \mathbb{R}$, put

$$\Delta_h \phi_m(t) = \phi_m(t+h) - \phi_m(t) \tag{33}$$

and

$$R_{\delta,\alpha}(h,t) = \sum_{j=0}^{\infty} \sum_{m \in \tilde{N}_j} \mu_{\delta,\alpha,j,m,t} \left| \Delta_h \phi_m(t) \right| . \tag{34}$$

Clearly, if $\mathbf{h} = (h_1, h_2)$ then

$$\square_{x+\mathbf{h}} \Phi_{\mathbf{m}}(x) = \prod_{i=1}^{2} \Delta_{h_i} \phi_{m_i}(x_i) \tag{35}$$

and the function F given in (25) satisfies

$$|\square_{x+\mathbf{h}} F(x)| \le C \prod_{i=1}^{2} R_{\delta_i,\alpha_i}(h_i, x_i) . \tag{36}$$

Relation (36) together with the following lemma yield (27).
□

Lemma 1. *There exists $C > 0$ such that*

$$\forall\, 0 < |h| \le 1 \;\; \forall\, t \qquad R_{\delta,\alpha}(h,t) \le C |h|^{\sigma\alpha + (1-\sigma)\delta} . \tag{37}$$

Proof of Lemma 1. Clearly

$$\mu_{\delta,\alpha,j,m,t} \le C 2^{-(1-\sigma)\delta j} \left(2^{-j\sigma\alpha} + \left| n 2^{-j} - t \right|^{\sigma\alpha} \right) . \tag{38}$$

Remark 3. *If $m = 2^j + n \in \tilde{N}_j$, with $j \ge 0$ then ϕ_m has support $[(n-1)2^{-j}, n2^{-j}]$. It follows that for $t \in I$ and $j \ge 0$, there exists a unique value of $m = 2^j + n$ for which $t \in [(n-1)2^{-j}, n2^{-j})$.*

On the other hand

$$|\Delta_h \phi_m(t)| \le |\phi_m(t)| + |\phi_m(t+h)| . \tag{39}$$

Relation (38) together with Remark 3 and the triangle inequality $\left| n2^{-j} - (t+h) \right| \le \left| n2^{-j} - t \right| + |h|$ yield

$$\sum_{m \in \tilde{N}_j} \mu_{\delta,\alpha,j,m,t} \left| \Delta_h \phi_m(t) \right| \le C 2^{-(1-\sigma)\delta j} \left(2^{-\sigma\alpha j} + |h|^{\sigma\alpha} \right) . \tag{40}$$

Since $0 < |h| \le 1$, let $J \in \mathbb{N}_0$ such that $2^{-J} \le |h| < 2.\, 2^{-J}$. Split $R_{\delta,\alpha}(h,t)$ as

$$R_{\delta,\alpha}(h,t) = \sum_{j=0}^{J} \sum_{m \in \tilde{N}_j} \mu_{\delta,\alpha,j,m,t} \left| \Delta_h \phi_m(t) \right| + \sum_{j=J+1}^{\infty} \sum_{m \in \tilde{N}_j} \mu_{\delta,\alpha,j,m,t} \left| \Delta_h \phi_m(t) \right| . \tag{41}$$

Since $\alpha > 0, 0 < \sigma < 1$ and $\delta > 0$ then relation (40) yields

$$\sum_{j=J+1}^{\infty} \sum_{m \in \tilde{N}_j} \mu_{\delta,\alpha,j,m,t} \left| \Delta_h \phi_m(t) \right| \le C |h|^{\sigma\alpha + (1-\sigma)\delta} . \tag{42}$$

Let us bound the first sum in (41). Since ϕ is Lipschitz then

$$\left| \Delta_h \phi_m(t) \right| \le C 2^j \, |h| .$$

An argument similar to that of (40) yields

$$\sum_{j=0}^{J}\sum_{m\in\tilde{N}_j}\mu_{\delta,\alpha,j,m,t}\,|\Delta_h\phi_m(t)|\le C\sum_{j=0}^{J}2^{-(1-\sigma)\delta j}(2^{-\sigma\alpha j}+|h|^{\sigma\alpha})2^j\,|h|\le C|h|^{\sigma\alpha+(1-\sigma)\delta}\,. \tag{43}$$

Both (42) and (43) yield (37). □

Proposition 2 together with Remark 2 and Proposition 3 yield the following full characterization of $Lip^{\bar{\alpha}}(x)$.

Theorem 1. *Let $f\in C(I^2)$, $\mathbf{0}<\bar{\alpha}<\mathbf{1}$ and $x\in I^2$.*

1. *If $f\in Lip^{\bar{\alpha}}(x)$ then (22) together with (23) and (26) hold.*
2. *Conversely, if f is uniform Lipschitz on I^2 and (22) together with (23) and (26) hold, then*

$$\forall\ \bar{\alpha}'<\bar{\alpha}\qquad f\in Lip^{\bar{\alpha}'}(x)\,. \tag{44}$$

2.2. Characterization of $Lip^{\bar{\alpha}}(x)$ by Decay Conditions of Schauder Leaders

An equivalent characterization of $Lip^{\bar{\alpha}}(x)$ by decay conditions of Schauder leaders can also be obtained.

If $j\ge 0$ and $m=n+2^j\in\tilde{N}_j$, we will denote by λ the dyadic interval

$$\lambda_m=[(n-1)2^{-j},n2^{-j})\,. \tag{45}$$

Set

$$3\lambda_m=[(n-2)2^{-j},(n+1)2^{-j})\,. \tag{46}$$

If $t\in I$, denote by $\lambda_j(t)$ the dyadic interval at scale j that contains t.

For $\mathbf{j}=(j_1,j_2)\ge\mathbf{0}$ and $\mathbf{m}=(m_1,m_2)\in\tilde{N}_{\mathbf{j}}$, with $m_i=n_i+2^{j_i}$, define the Schauder leader of f at $\mathbf{m}$ by

$$d_{\mathbf{m}}=d_{\lambda_{m_1}\times\lambda_{m_2}}=\sup_{\lambda_{m_1'}\times\lambda_{m_2'}\subset\lambda_{m_1}\times\lambda_{m_2}}|C_{\mathbf{m}'}|\,, \tag{47}$$

where $\lambda_{m_i'}=[(n_i'-1)2^{-j_i},n_i'2^{-j_i'})$ for $m_i'=n_i'+2^{j_i'}\in\tilde{N}_{j'}$.

If $\mathbf{j}=(-1,j_2\ge 0)$ and $\mathbf{m}=(1,m_2)$, with $m_2=n_2+2^{j_2}\in\tilde{N}_{j_2}$, define the Schauder leader of f at $\mathbf{m}$ by

$$d_{\mathbf{m}}=\sup_{\lambda_{m_2'}\subset\lambda_{m_2}}|C_{(1,m_2')}|\,. \tag{48}$$

If $\mathbf{j}=(j_1\ge 0,-1)$ and $\mathbf{m}=(m_1,1)$, with $m_1=n_1+2^{j_1}\in\tilde{N}_{j_1}$, define the Schauder leader of f at $\mathbf{m}$ by

$$d_{\mathbf{m}}=\sup_{\lambda_{m_1'}\subset\lambda_{m_1}}|C_{(m_1',1)}|\,. \tag{49}$$

If $\mathbf{j}=(j_1,j_2)\ge\mathbf{0}$, set

$$d_{\mathbf{j}}(x)=\max_{\lambda_{m_1'}\times\lambda_{m_2'}\subset 3\lambda_{j_1(x_1)}\times 3\lambda_{j_2(x_2)}}d_{\mathbf{m}'}\,. \tag{50}$$

If $\mathbf{j}=(-1,j_2\ge 0)$, set

$$d_{\mathbf{j}}(x)=\max_{\lambda_{m_2'}\subset 3\lambda_{j_2(x_2)}}d_{(1,m_2')}\,. \tag{51}$$

If $\mathbf{j}=(j_1\ge 0,-1)$, set

$$d_{\mathbf{j}}(x)=\max_{\lambda_{m_1'}\subset 3\lambda_{j_1(x_1)}}d_{(m_1',1)}\,. \tag{52}$$

Theorem 2. *Let $f \in C(I^2)$, $\mathbf{0} < \bar{\alpha} < \mathbf{1}$ and $x \in I^2$.*

1. *If $f \in Lip^{\bar{\alpha}}(x)$ then there exists $C > 0$ such that*

$$\forall\ \mathbf{j} = (-1, j_2 \geq 0) \qquad d_{\mathbf{j}}(x) \leq C2^{-j_2\alpha_2}\ , \tag{53}$$

$$\forall\ \mathbf{j} = (j_1 \geq 0, -1) \qquad d_{\mathbf{j}}(x) \leq C2^{-j_1\alpha_1} \tag{54}$$

and

$$\forall\ \mathbf{j} \geq \mathbf{0} \qquad d_{\mathbf{j}}(x) \leq C\prod_{i=1}^{2} 2^{-j_i\alpha_i}\ . \tag{55}$$

2. *Conversely, if f is uniform Lipschitz on I^2 and (53) together with (54) and (55) hold, then*

$$\forall\ \bar{\alpha}' < \bar{\alpha} \qquad f \in Lip^{\bar{\alpha}'}(x)\ . \tag{56}$$

Proof of Theorem 2.

1. Let $f \in Lip^{\bar{\alpha}}(x)$.
   - Since g_3 and g_4 belong to $Lip^{\bar{\alpha}}(x)$ then by Proposition 2

$$\forall\ j' \geq 0\ \ \forall\ m' = 2^{j'} + n' \in \tilde{N}_{j'} \quad |C_{(1,m')}| \leq C(2^{-j'} + \left|n'2^{-j'} - x_2\right|)^{\alpha_2} \tag{57}$$

     and

$$\forall\ j' \geq 0\ \ \forall\ m' = 2^{j'} + n' \in \tilde{N}_{j'} \quad |C_{(m',1)}| \leq C(2^{-j'} + \left|n'2^{-j'} - x_1\right|)^{\alpha_1}\ . \tag{58}$$

     Let $j \geq 1$, if $\lambda' \subset 3\lambda$ then $j' \geq j-1$ and $\left|n'2^{-j'} - x_i\right| \leq C2^{-j}$ for all $i \in \{1,2\}$. Hence (57) and (58) yield (53) and (54).
   - Thanks to Proposition 3

$$\forall\ \mathbf{j}' \geq \mathbf{0}\ \ \forall\ \mathbf{m}' = (m_1', m_2') \in \tilde{N}_{\mathbf{j}'} \qquad |C_{\mathbf{m}'}| \leq C\prod_{i=1}^{2}(2^{-j_i'} + \left|n_i'2^{-j_i'} - x_i\right|)^{\alpha_i}\ . \tag{59}$$

     Let $\mathbf{j} \geq \mathbf{1}$, if $\lambda_i' \subset 3\lambda_i$ then $j_i' \geq j_i - 1$ and $\left|n_i'2^{-j_i'} - x_i\right| \leq C2^{-j_i}$ for all $i \in \{1,2\}$. Hence (59) yields (55).
2. The converse part is reminiscent of that of [17] page 17. Assume that f is uniform Lipschitz on I^2 and (53) together with (54) and (55) hold.
   - Let $t \in I$. Let $j' \geq 0$ be given. If $\lambda' = [(n'-1)2^{-j'}, n'2^{-j'})$, denote by $\lambda = [(n-1)2^{-j}, n2^{-j})$ the dyadic interval defined by
     - if $\lambda' \subset 3\lambda_{j'}(t)$, then $\lambda = \lambda_{j'}(t)$ and $j = j'$,
     - else, if $j = \sup\{l : \lambda' \subset 3\lambda_l(t)\}$, then $\lambda = \lambda_j(t)$ and it follows that there exists $C > 0$ such that $\frac{1}{C}2^{-j} \leq |n'2^{-j'} - t| \leq C2^{-j}$.

     Set $m' = n' + 2^{j'}$. In the first case, relation (53) implies that

$$|C_{(1,m')}| \leq d_{(1,j)}(x) \leq C2^{-j\alpha_2} = C2^{-j'\alpha_2}\ .$$

     Similarly, relation (54) implies that

$$|C_{(m',1)}| \leq d_{(j,1)}(x) \leq C2^{-j\alpha_1} = C2^{-j'\alpha_1}\ .$$

In the second case,

$$|C_{(1,m')}| \leq d_{(1,j)}(x) \leq C2^{-j_2\alpha_2} \leq C|n'2^{-j'} - x_2|^{\alpha_2}$$

and

$$|C_{(m',1)}| \leq d_{(j,1)}(x) \leq C2^{-j_1\alpha_1} \leq C|n'2^{-j'} - x_1|^{\alpha_1} .$$

The conclusion of the converse part of Proposition 2 holds.
If $\mathbf{j}' \geq \mathbf{0}$. With the same notations as above

$$|C_{(m_1',m_2')}| \leq C \prod_{i=1}^{2} (2^{-j_i'\alpha_i} + |n_i'2^{-j_i'} - x_i|^{\alpha_i}) .$$

The conclusion of the converse part of Theorem 3 holds.

□

3. Pointwise Level Coordinate Axes Lipschitz Regularities

Remark 4. *Clearly if* $W(y) = u(y_1)v(y_2)$ *then* $W(y) - W(x_1, y_2) = (u(y_1) - u(x_1))v(y_2)$ *and* $W(y) - W(y_1, x_2) = u(y_1)(v(y_2) - v(x_2))$.

Thanks to Remark 4, we have the following results.

- Function g_0 given in (17) belongs to $\bigcap_{i=1}^{2} N^s(x, e_i)$ for all $0 < s < 1$.
- Function g_1 given in (18) belongs to $N^s(x, e_1)$ for all $0 < s < 1$. It belongs to $N^s(x, e_2)$ if and only if the one variable function $f(0, t)$ belongs to $Lip^s(x_2)$.
- Function g_2 given in (19) belongs to $N^s(x, e_2)$ for all $0 < s < 1$. It belongs to $N^s(x, e_1)$ if and only if the one variable function $f(t, 0)$ belongs to $Lip^s(x_1)$.
- Function g_3 given in (20) belongs to $N^s(x, e_1)$ for all $0 < s < 1$. It belongs to $N^s(x, e_2)$ if and only if the one variable function $f(1, t) - f(0, t)$ belongs to $Lip^s(x_2)$.
- Function g_4 given in (21) belongs to $N^s(x, e_2)$ for all $s < 1$. It belongs to $N^s(x, e_1)$ if and only if the one variable function $f(t, 1) - f(t, 0)$ belongs to $Lip^s(x_1)$.

3.1. Characterization of $N^s(x, e_i)$ by Decay Conditions of Schauder Coefficients

If f is uniform Lipschitz on I^2, then g_2 and g_4 (resp. g_1 and g_3) can be sharply characterized in $N^s(x, e_1)$ using the second result of Proposition 1. More precisely

Proposition 4. *Let* $f \in C(I^2)$, $s \in (0, 1)$ *and* $x \in I^2$.

1. *If g_2 and g_4 belong to $N^s(x, e_1)$ then there exists $C > 0$ such that*

$$\forall\, j \geq 0 \;\; \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(m,0)}| \leq C(2^{-j} + \left|n2^{-j} - x_1\right|)^s \tag{60}$$

and

$$\forall\, j \geq 0 \;\; \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(m,1)}| \leq C(2^{-j} + \left|n2^{-j} - x_1\right|)^s . \tag{61}$$

2. *Conversely, if f is uniform Lipschitz on I^2 and both (60) and (61) are satisfied then*

$$\forall\, i \in \{2, 4\} \;\; \forall\, s' < s \qquad g_i \in N^{s'}(x, e_1) . \tag{62}$$

3. *If g_1 and g_3 belong to $N^s(x, e_2)$ then there exists $C > 0$ such that*

$$\forall\, j \geq 0 \;\; \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(0,m)}| \leq C(2^{-j} + \left|n2^{-j} - x_2\right|)^s \tag{63}$$

and

$$\forall\, j \geq 0 \ \ \forall\, m = 2^j + n \in \tilde{N}_j \quad |C_{(1,m)}| \leq C(2^{-j} + \left|n2^{-j} - x_2\right|)^s \,. \tag{64}$$

4. *Conversely, if f is uniform Lipschitz on I^2 and both (63) and (64) are satisfied then*

$$\forall\, i \in \{1,3\} \ \ \forall\, s' < s \qquad g_i \in N^{s'}(x, e_2) \,. \tag{65}$$

Remark 5. *Note that if $f \in N^s(x, e_1)$ (resp. $f \in N^s(x, e_2)$) then g_2 and g_4 belong to $N^s(x, e_1)$ (resp. g_1 and g_3 belong to $N^s(x, e_2)$).*

To achieve the sharp characterization of both $N^s(x, e_1)$ and $N^s(x, e_2)$, it remains to deal with the series F given in (25).

Proposition 5. *Let $f \in C(I^2)$, $0 < s < 1$ and $x \in I^2$.*

1. (a) *If $f \in N^s(x, e_1)$ then there exists $C > 0$ such that*

$$\forall\, \mathbf{j} \geq \mathbf{0} \ \ \forall\, \mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}} \qquad |C_{\mathbf{m}}| \leq C(2^{-j_1} + \left|n_1 2^{-j_1} - x_1\right|)^s \,. \tag{66}$$

   (b) *Conversely, if f is uniform Lipschitz on I^2 and (66) is satisfied, then*

$$\forall\, s' < s \qquad F \in N^{s'}(x, e_1) \,. \tag{67}$$

2. (a) *If $f \in N^s(x, e_2)$ then there exists $C > 0$ such that*

$$\forall\, \mathbf{j} \geq \mathbf{0} \ \ \forall\, \mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}} \qquad |C_{\mathbf{m}}| \leq C(2^{-j_2} + \left|n_2 2^{-j_2} - x_2\right|)^s \,. \tag{68}$$

   (b) *Conversely, if f is uniform Lipschitz on I^2 and (68) is satisfied, then*

$$\forall\, s' < s \qquad F \in N^{s'}(x, e_2) \,. \tag{69}$$

Proof of Proposition 5.

1. (a) Let $\mathbf{j} \geq \mathbf{0}$ and $\mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}}$. Using (14)

$$\begin{aligned} |C_{\mathbf{m}}| &= |f_{m_2}(\frac{2n_1 - 1}{2^{j_1+1}}) - f_{m_2}(x_1) \\ &- \frac{1}{2}\left(f_{m_2}(\frac{n_1 - 1}{2^{j_1}}) - f_{m_2}(x_1) + f_{m_2}(\frac{n_1}{2^{j_1}}) - f_{m_2}(x_1) \right)| \,. \end{aligned}$$

   Since $f \in N^s(x, e_1)$ then using (12) there exists $C > 0$ such that (66) holds.

   (b) Conversely, assume that f is uniform Lipschitz on I^2 and (66) is satisfied. As in the beginning of the proof of the converse part in Proposition 3

$$|C_{\mathbf{m}}| \leq C\mu_{\delta_1, s, j_1, m_1, x_1} \ 2^{-(1-\sigma)\delta_2 j_2} \,. \tag{70}$$

   So

$$\begin{aligned} |F(y) - F(x_1, y_2)| &\leq CR_{\delta_1, s}(y_1 - x_1, x_1) \sum_{j_2 \geq 0} \left(2^{-(1-\sigma)\delta_2 j_2} \sum_{m_2} |\phi_{m_2}(y_2)| \right) \\ &\leq CR_{\delta_1, s}(y_1 - x_1, x_1) \\ &\leq C|y_1 - x_1|^{\sigma s + (1-\sigma)\delta_1} \,. \end{aligned}$$

2. The proof is similar.

□

Proposition 4 together with both Remark 5 and Proposition 5 yield the following full characterization of both $N^s(x, e_1)$ and $N^s(x, e_2)$.

Theorem 3. *Let $f \in C(I^2)$, $0 < s < 1$ and $x \in I^2$.*

1. (a) *If $f \in N^s(x, e_1)$ then there exists $C > 0$ such that (60), (61) and (66) hold.*
   (b) *Conversely, if f is uniform Lipschitz on I^2 and (60) together with (61) and (66) hold, then*

$$\forall\, s' < s \qquad f \in N^{s'}(x, e_1)\,. \tag{71}$$

2. (a) *If $f \in N^s(x, e_2)$ then there exists $C > 0$ such that (63), (64) and (68) hold.*
   (b) *Conversely, if f is uniform Lipschitz on I^2 and (63) together with (64) and (68) hold, then*

$$\forall\, s' < s \qquad f \in N^{s'}(x, e_2)\,. \tag{72}$$

3.2. Characterization of $N^s(x, e_i)$ by Decay Conditions of Schauder Leaders

If $\mathbf{j} = (-2, j_2 \geq 0)$ and $\mathbf{m} = (0, m_2)$, with $m_2 = n_2 + 2^{j_2} \in \tilde{N}_{j_2}$, then define the Schauder leader of f at $\mathbf{m}$ by

$$d_{\mathbf{m}} = \sup_{\lambda_{m_2'} \subset \lambda_{m_2}} |C_{(0,m_2')}|\,. \tag{73}$$

If $\mathbf{j} = (j_1 \geq 0, -2)$ and $\mathbf{m} = (m_1, 0)$, with $m_1 = n_1 + 2^{j_1} \in \tilde{N}_{j_1}$, then define the Schauder leader of f at $\mathbf{m}$ by

$$d_{\mathbf{m}} = \sup_{\lambda_{m_1'} \subset \lambda_{m_1}} |C_{(m_1',0)}|\,. \tag{74}$$

If $\mathbf{j} = (-2, j_2 \geq 0)$, set

$$d_{\mathbf{j}}(x) = \max_{\lambda_{m_2'} \subset 3\lambda_{j_2}(x_2)} d_{(0,m_2')}\,. \tag{75}$$

If $\mathbf{j} = (j_1 \geq 0, -2)$, set

$$d_{\mathbf{j}}(x) = \max_{\lambda_{m_1'} \subset 3\lambda_{j_1}(x_1)} d_{(m_1',0)}\,. \tag{76}$$

Using the same arguments as in the proof of Theorem 2, we obtain the following results.

Theorem 4. *Let $f \in C(I^2)$, $0 < s < 1$ and $x \in I^2$.*

1. (a) *If $f \in N^s(x, e_1)$ then there exists $C > 0$ such that*

$$\forall\, \mathbf{j} = (j_1 \geq 0, j_2 \in \{-2, -1\}) \qquad d_{\mathbf{j}}(x) \leq C 2^{-j_1 s} \tag{77}$$

   and

$$\forall\, \mathbf{j} \geq \mathbf{0} \qquad d_{\mathbf{j}}(x) \leq C 2^{-j_1 s}\,. \tag{78}$$

   (b) *Conversely, if f is uniform Lipschitz on I^2 and both (77) and (78) hold, then*

$$\forall\, s' < s \qquad f \in N^{s'}(x, e_1)\,. \tag{79}$$

2. (a) *If $f \in N^s(x, e_2)$ then there exists $C > 0$ such that*

$$\forall\, \mathbf{j} = (j_1 \in \{-2, -1\}, j_2 \geq 0) \qquad d_{\mathbf{j}}(x) \leq C 2^{-j_2 s} \tag{80}$$

and

$$\forall\, \mathbf{j} \geq \mathbf{0} \qquad d_{\mathbf{j}}(x) \leq C2^{-j_2 s}\ . \tag{81}$$

(b) *Conversely, if f is uniform Lipschitz on I^2 and both (80) and (81) hold, then*

$$\forall\, s' < s \qquad f \in N^{s'}(x, e_2)\ . \tag{82}$$

3.3. Relationship between $Lip^{\bar{\alpha}}(x)$ and Pointwise Level Coordinate Axes Lipschitz Regularities

We have already relation (8). In addition, both Theorems 2 and 4 imply that rectangular Lipschitz regularity yields pointwise level coordinate axes Lipschitz regularities.

Set $g(y) = f(y) - f(y_1, 0)$ and $h(y) = f(y) - f(0, y_2)$. Define

$$N_f(x, e_1) := \sup\{s \in (0,1)\ :\ \ f \in N^s(x, e_1)\} \tag{83}$$

and

$$N_f(x, e_2) := \sup\{s \in (0,1)\ :\ \ f \in N^s(x, e_2)\}\ . \tag{84}$$

Theorem 5. *If $f \in Lip^{\bar{\alpha}}(x)$ and f is uniform Lipschitz on I^2 then $g \in N^{\alpha_1 - \varepsilon}(x, e_1)$ and $h \in N^{\alpha_2 - \varepsilon}(x, e_2)$ for all $\varepsilon > 0$.*

If f is uniform Lipschitz on I^2, then

$$N_g(x, e_1) = \sup\{\alpha_1 \in (0,1)\ :\ \ \exists\, \alpha_2 \in (0,1)\ \ f \in Lip^{\bar{\alpha}}(x)\} \tag{85}$$

and

$$N_h(x, e_2) = \sup\{\alpha_2 \in (0,1)\ :\ \ \exists\, \alpha_1 \in (0,1)\ \ f \in Lip^{\bar{\alpha}}(x)\}\ . \tag{86}$$

4. Applications

4.1. The Fractional Brownian Sheets

The following result refines result (4).

Theorem 6. *The fractional Brownian sheet $B^{\bar{H}}_{I^2}$ is pointwise rectangular and level coordinate axes monofractal. More precisely, with probability 1,*

$$\forall\, \bar{H}' < \bar{H}\ \ \forall\, x \in I^2 \qquad B^{\bar{H}}_{I^2} \in Lip^{\bar{H}'}(x)\ ,$$

$$\forall\, \bar{H}' \nleq \bar{H}\ \ \forall\, x \in I^2 \qquad B^{\bar{H}}_{I^2} \notin Lip^{\bar{H}'}(x)$$

and

$$\forall\, x \in I^2 \quad \forall\, i \in \{1,2\} \qquad N_f(x, e_i) = H_i\ .$$

Proof of Theorem 6. The coefficients of the fractional Brownian sheet $B^{\bar{H}}_{I^2}$ in the tensor product Schauder basis were obtained in [7]; in fact

$$B^{\bar{H}}_{I^2} = \sum_{\mathbf{j} \in M^2} \sum_{\mathbf{m} \in \tilde{N}_{\mathbf{j}}} C_{\mathbf{m}} \Phi_{\mathbf{m}} \tag{87}$$

where $(C_{\mathbf{m}})_{\mathbf{m} \geq \mathbf{0}}$ is a Gaussian sequence, with $E(C_{\mathbf{m}}) = 0$, and the variance given by the formula

$$E(|C_{\mathbf{m}}|^2) = \prod_{i=1}^{2} a_{m_i} \tag{88}$$

with

$$a_0 = 0,\ a_1 = 1 \text{ and } a_{m_i} = (2^{-2H_i} - 2^{-2})2^{-j_i 2H_i} \text{ for } m_i \in \tilde{N}_{j_i} \quad j_i \geq 0\,. \tag{89}$$

As mentioned in [2],

$$\forall\, y_1 \quad B^{\tilde{H}}_{I^2}(y_1, 0) = 0 \text{ and } \forall\, y_2 \quad B^{\tilde{H}}_{I^2}(0, y_2) = 0$$

(this remark follows also from the fact that $a_0 = 0$).

In [7], it is mentioned that if $m_1 = 0$ or $m_2 = 0$ then $C_{\mathbf{m}} = 0$ almost surely. For $\mathbf{m} > \mathbf{0}$, put $g_{\mathbf{m}} = \frac{C_{\mathbf{m}}}{\sqrt{E(|C_{\mathbf{m}}|^2)}}$. The following lemma can be obtained using same arguments as Corollary 4.6 in [7].

Lemma 2. *There exists $C > 0$ such that, with probability 1,*

$$C \leq \sup_{\mathbf{j} > \mathbf{0}\,,\ \mathbf{m} \in \tilde{N}_{\mathbf{j}}} \left(\sup_{\mathbf{j}' \geq \mathbf{j}} \left(\sup_{\mathbf{m}' \in \tilde{N}_{\mathbf{j}'}\,,\ \lambda_{m'_1} \times \lambda_{m'_2} \subset \lambda_{m_1} \times \lambda_{m_2}} \frac{|g_{\mathbf{m}'}|}{\sqrt{1 + |\mathbf{j}' - \mathbf{j}| \ln 2}} \right) \right) < \infty\,.$$

Since $2^{-j' H_i} \leq 2^{-j H_i}$ for all $j' \geq j$, then Theorem 6 follows directly from (88), (89), Theorem 2 and Theorem 5. □

4.2. Sierpinski Selfsimilar Functions

We will construct a class of Sierpinski selfsimilar functions that will be pointwise rectangular and level coordinate axes multifractal. Let s and t be two integers with $s < t$. Choose $A \subset \{0, 1, \ldots, s-1\} \times \{0, 1, \ldots, t-1\}$. For $\omega = (a, b) \in A$, the contraction $S_\omega(x_1, x_2) = \left(\frac{x_1}{s} + \frac{a}{s}, \frac{x_2}{t} + \frac{b}{t} \right)$ maps I^2 into the rectangle

$$\mathfrak{R}_\omega = [\frac{a}{s}, \frac{a+1}{s}] \times [\frac{b}{t}, \frac{b+1}{t}]. \tag{90}$$

The (general) Sierpinski carpet K (see [18–20]) and references therein) is the unique non-empty compact set (see [21,22]) satisfying

$$K = \bigcup_{\omega \in A} S_\omega(K). \tag{91}$$

It is given by

$$\begin{aligned} K &= \{x \in I^2 \ :\ (S_{\omega_1} \circ \cdots \circ S_{\omega_l})^{-1}(x) \in \bigcup_{\omega \in A} \mathfrak{R}_\omega \quad \forall\, \omega = (\omega_1, \ldots, \omega_l) \in A^l \ \ \forall\, l \in \mathbb{N}\} \\ &= \bigcap_{l \in \mathbb{N}} \Big(\bigcup_{\omega \in A^l} \mathfrak{R}_\omega \Big) \end{aligned}$$

where

$$\mathfrak{R}_\omega = (S_{\omega_1} \circ \cdots \circ S_{\omega_l})(I^2) \quad \text{for} \quad \omega = (\omega_1, \ldots, \omega_l)\,.$$

Put $\Gamma(x_1, x_2) = \Lambda(x_1)\Lambda(x_2)$, where $\Lambda(t) = \min(t, 1-t)$ if $t \in [0,1]$ and 0 elsewhere. Clearly $\Lambda(t) = \frac{1}{2}\Phi_2(t)$. The Sierpinski selfsimilar function adapted to the subdivision A satisfies

$$\forall\, x \in I^2 \qquad f(x) = \sum_{\omega \in A} \gamma_\omega f(S_\omega^{-1}(x)) \ + \ \Gamma(x_1, x_2)\,. \tag{92}$$

Define

$$|\gamma|_{max} = \max_{\omega \in A} |\gamma_\omega|\ ,\ \ |\gamma|_{min} = \min_{\omega \in A} |\gamma_\omega|\ ,\ \text{ and } H_{min} = -\frac{\log |\gamma|_{max}}{\log t}\,.$$

The following result was obtained in [18].

Proposition 6. *Suppose that* $\sum_{\omega\in A} |\gamma_\omega| < st$, *then the series*

$$f(x) = \Gamma(x) + \sum_{l=1}^{\infty} \sum_{(\omega_1,\ldots,\omega_l)\in A^l} \gamma_{\omega_1}\cdots\gamma_{\omega_l}\, \Gamma\left(S_{\omega_l}^{-1}\cdots S_{\omega_1}^{-1}(x)\right). \tag{93}$$

is a unique solution in $L^1(I^2)$ *for equation (92).*

If furthermore $\frac{1}{t} < |\gamma|_{max} < 1$, *then* $f \in Lip^{H_{min}}(I^2)$ *with* $0 < H_{min} < 1$.

The Sierpinski selfsimilar function is written as the superposition of similar anisotropic structures at different scales, reminiscent of some possible modelization of turbulence or cascade models. In [18], we proved that some Sierpinski selfsimilar functions don't satisfy the thermodynamic formalism.

Clearly if $\omega_l = (a_l, b_l)$ then

$$\Gamma\left(S_{\omega_l}^{-1}\cdots S_{\omega_1}^{-1}(x)\right) = \Lambda(s^l x_1 - s^{l-1}a_1 - \cdots - sa_{l-1} - a_l)\ \Lambda(t^l x_2 - t^{l-1}b_1 - \cdots - tb_{l-1} - b_l)\ .$$

Consider the "separated open set condition"

$$\forall\ (\omega,\omega') \in A^2 \qquad \omega \neq \omega' \Rightarrow \mathfrak{R}_\omega \cap \mathfrak{R}_{\omega'} = \emptyset\ . \tag{94}$$

If $x \notin K$ then there exists a neighborhood $\vartheta(x)$ of x and $L \in \mathbb{N}$ such that

$$\forall\ y \in \vartheta(x) \qquad f(y) = \Gamma(y) + \sum_{l=1}^{L} \sum_{(\omega_1,\cdots,\omega_l)\in A^l} \gamma_{\omega_1}\cdots\gamma_{\omega_l}\, \Gamma\left(S_{\omega_l}^{-1}\cdots S_{\omega_1}^{-1}(y)\right). \tag{95}$$

It follows that $f \in Lip^{\bar{\alpha}}(x)$ for all $\bar{\alpha} < \mathbf{1}$.

On the other hand, from the "separated open set condition" (94), any $x \in K$ has a unique expansion

$$x = \left(\sum_{l=1}^{\infty} \frac{a_l}{s^l}, \sum_{l=1}^{\infty} \frac{b_l}{t^l}\right) \ \text{with}\ (a_l, b_l) = (a_l(x), b_l(x)) = \omega_l = \omega_l(x) \in A\ . \tag{96}$$

For $L \geq 1$, denote by

$$\omega(L,x) = (\omega_1,\cdots,\omega_L) \ \text{ and } \ \gamma_{\omega(L,x)} = \gamma_{\omega_1}\cdots\gamma_{\omega_L}\ .$$

Let S and T be two positive integers. Assume that $s = 2^S$ and $t = 2^T$. Set

$$r(x) = \liminf_{L\to\infty} \frac{\log|\gamma_{\omega(L,x)}|}{\log 2^{-L}}\ .$$

Theorem 7. *Let* $\frac{1}{t} < |\gamma|_{max} < 1$. *Assume that the "separated open set condition" (94) holds. Assume furthermore that each column and each row of the grid contains at most one box* $\mathfrak{R}_\omega$, *with* $\omega \in A$. *If* $|\gamma|_{min} < |\gamma|_{max}$, *then the obtained class of Sierpinski selfsimilar functions* f *are pointwise rectangular and level coordinate axes multifractal. More precisely, if* $x \in K$ *and* $r(x) < S$ *then*

$$f \in Lip^{\overline{\alpha}}(x) \qquad \forall\ S\alpha_1 + T\alpha_2 < r(x)\ ,$$

$$f \notin Lip^{\overline{\alpha}}(x) \qquad \forall\ S\alpha_1 + T\alpha_2 > r(x)$$

and

$$(N_f(x,e_1), N_f(x,e_2)) = \left(\frac{r(x)}{S}, \frac{r(x)}{T}\right)\ .$$

Proof of Theorem 7. Clearly Schauder leaders of f satisfy $d_{(2,2)} = C_{(2,2)} = \frac{1}{4}$, and if $L \geq 1, \mathbf{j} = (LS, LT)$, $\mathbf{m} = (m_1, m_2) \in \tilde{N}_{\mathbf{j}}$ with $(m_1, m_2) = (n_1 + 2^{LS}, n_2 + 2^{LT})$ and $((n_1-1)2^{-LS}, (n_2-1)2^{-LT}) = (\sum_{l=1}^{L} \frac{a_l}{s^l}, \sum_{l=1}^{L} \frac{b_l}{t^l})$, then $d_{\mathbf{m}} = C_{\mathbf{m}} = \frac{1}{4}|\gamma_{\omega_1}| \cdots |\gamma_{\omega_L}|$ (because $|\gamma|_{\max} < 1$).

Let $x \in K$ and $\mathbf{j} = (j_1, j_2) \geq \mathbf{0}$ with $\mathbf{j} \neq \mathbf{0}$. Write $(L_1 - 1)S < j_1 \leq L_1 S$ and $(L_2 - 1)T < j_2 \leq L_2 T$. Put $L = \max(L_1, L_2)$. Then (47), (50), (94) and the assumption that each column and each row of the grid contains at most one box $\mathfrak{R}_\omega$, with $\omega \in A$, imply that

$$\frac{1}{4}|\gamma_{\omega(L,x)}| \leq d_{\mathbf{j}}(x) \leq \frac{1}{4}|\gamma_{\omega(L-1,x)}| \,. \tag{97}$$

Hence Theorem 2 and Theorem 5 yield Proposition 7. Indeed

$$\forall\, \varepsilon > 0 \;\; \exists\, L_\varepsilon \;\; \forall\, L \geq L_\varepsilon \;\; |\gamma_{\omega(L,x)}| \leq 2^{-L(r(x)-\varepsilon)} \tag{98}$$

and

$$\varepsilon > 0 \;\; \exists\, L_n \nearrow \infty \;\; \forall\, n \;\; |\gamma_{\omega(L_n,x)}| > 2^{-L_n(r(x)+\varepsilon)} \,. \tag{99}$$

If $S\alpha_1 + T\alpha_2 < r(x)$ then for $\varepsilon = r(x) - S\alpha_1 - T\alpha_2$ relations (97) and (98) imply that

$$\exists\, L_\varepsilon \;\; \forall\, L > L_\varepsilon \;\; d_{\mathbf{j}}(x) \leq C2^{-LS\alpha_1 - LT\alpha_2} \leq C2^{-j_1\alpha_1 - j_2\alpha_2} \,.$$

If $S\alpha_1 + T\alpha_2 > r(x)$ then for $2\varepsilon = S\alpha_1 + T\alpha_2 - r(x)$ relations (97) and (99) imply that

$$\forall\, n \quad d_{(L_n S, L_n T)}(x) > \frac{1}{4} 2^{-L_n S\alpha_1 - L_n T\alpha_2 + \varepsilon L_n} \,.$$

□

5. Conclusions

In [2,7], it was proved that for any cube $\mathcal{Q} \subset \mathbb{R}^2$, the restrictions $B^{\bar{H}}_{\mathcal{Q}}$ of realizations of fractional Brownian sheets $B^{\bar{H}}$ to $\mathcal{Q}$ are uniform rectangular Lipschitz with order $\bar{H}' < \bar{H}$. In this paper, we first improved that result pointwisely, namely $B^{\bar{H}}_{\mathcal{Q}}$ are pointwise rectangular (respectively level coordinate axes) monofractal with order $\bar{H}' < \bar{H}$ (respectively H_i). The proof is based on some criteria of the latest pointwise regularities in terms of the rectangular Schauder system, obtained in this paper. A second application of these criteria was done, namely we constructed a class of Sierpinski selfsimilar functions that are pointwise rectangular and level coordinate axes multifractal.

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References

1. Ayache, A.; Wu, D.; Xiao, Y. Joint continuity of the local times of fractional Brownian sheets. *Annales de l'IHP Probabilités et Statistiques* **2008**, *44*, 727–748. [CrossRef]
2. Ayache, A.; Léger, S.; Pontier, M. Drap brownien fractionnaire. *Potential Anal.* **2002**, *17*, 31–43. [CrossRef]
3. Ayache, A.; Xiao, Y. Asymptotic properties and Hausdorff dimension of fractional Brownian sheets. *J. Fourier Anal. Appl.* **2005**, *11*, 407–439. [CrossRef]
4. Bonami, A.; Estrade, A. Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.* **2003**, *9*, 215–236. [CrossRef]

5. Clausel, M.; Vedel, B. Explicit constructions of operator scaling Gaussian fields. *Fractals* **2011**, *19*, 101–111. [CrossRef]
6. Dunker, T. Estimates for the small ball probabilities of the fractional Brownian sheet. *J. Theor. Probab.* **2000**, *13*, 357–382. [CrossRef]
7. Kamont, A. On the fractional anisotropic Wiener field. *Probab. Math. Statist.* **1996**, *16*, 85–98.
8. Mason, D.M.; Shi, Z. Small deviations for some multi-parameter Gaussian processes. *J. Theor. Probab.* **2001**, *14*, 213–239. [CrossRef]
9. Øksendal, B.; Zhang, T. Multiparameter Fractional Brownian Motion and Quasi-Linear Stochastic Partial Differential Equations. 2000. Available online: https://www.duo.uio.no/handle/10852/42605 (accessed on 6 July 2020).
10. Richard, F.; Bierme, H. Statistical tests of anisotropy for fractional brownian textures. Application to full-field digital mammography. *J. Math. Imaging Vis.* **2010**, *36*, 227–240. [CrossRef]
11. Roux, S.; Clausel, M.; Vedel, B.; Jaffard, S.; Abry, P. Self-Similar Anisotropic Texture Analysis: The Hyperbolic Wavelet Transform Contribution. *IEEE Trans. Image Proc.* **2010**, *22*, 4353–4363. [CrossRef] [PubMed]
12. Wu, D.; Xiao, Y. Geometric properties of fractional Brownian sheets. *J. Fourier Anal. Appl.* **2007**, *13*, 1–37. [CrossRef]
13. Xiao, Y.; Zhang, T. Local times of fractional Brownian sheets. *Probab. Theory Related Fields* **2002**, *124*, 204–226. [CrossRef]
14. Abry, P.; Jaffard, S.; Wendt, H. Irregularities and scaling in signal and image processing: Multifractal analysis. In *Benoit Mandelbrot: A Life in Many Dimensions*; Frame, M., Cohen, N., Eds.; World Scientific Publishing: Singapore, 2015; pp. 31–116.
15. Ben Slimane, M.; Ben Abid, M.; Ben Omrane, I.; Halouani, B. Criteria of pointwise and uniform directional Lipschitz regularities on tensor products of Schauder functions. *J. Fourier Anal. Appl.* **2018**, *460*, 496–515. [CrossRef]
16. Jaffard, S.; Mandelbrot, B. Local regularity of nonsmooth wavelet expansions and applications to the Polya function. *Adv. Math.* **1996**, *120*, 265–282. [CrossRef]
17. Jaffard, S. Wavelet techniques in multifractal analysis. In *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*; AMS: Providence, RI, USA, 2004.
18. Ben Slimane, M. Multifractal formalism and anisotropic selfsimilar functions. *Math. Proc. Camb. Philos. Soc.* **1998**, *124*, 329–363. [CrossRef]
19. King, J. The singularity spectrum for general Sierpinski carpets. *Adv. Math.* **1995**, *116*, 1–11. [CrossRef]
20. Olsen, L. Self-affine multifractal Sierpinski sponges in R^d. *Pac. J. Math.* **1998**, *183*, 143–199. [CrossRef]
21. Falconer, K.J. *Fractal Geometry: Mathematical Foundations and Applications*; John Wiley and Sons: Toronto, ON, USA, 1990.
22. Hutchinson, J. Fractals and self-similarity. *Indiana Univ. Math. J.* **1981**, *30*, 713–747. [CrossRef]