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Baker–Campbell–Hausdorff–Dynkin Formula for the Lie Algebra of Rigid Body Displacements

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Abstract: The paper proposes, for the first time, a closed form of the Baker–Campbell–Hausdorff–Dynkin (BCHD) formula in the particular case of the Lie algebra of rigid body displacements. For this purpose, the structure of the Lie group of the rigid body displacements $SE(3)$ and the properties of its Lie algebra $\mathfrak{se}(3)$ are used. In addition, a new solution to this problem in dual Lie algebra of dual vectors is delivered using the isomorphism between the Lie group $SE(3)$ and the Lie group of the orthogonal dual tensors.

Keywords: BCHD formula; Lie group; Lie algebra

1. Introduction

The BCHD theorem—named after the British mathematician Henry Frederick Baker (1866–1956), the Irish mathematician John Edward Campbell (1862–1924), the German mathematician Felix Hausdorff (1868–1942) and the Soviet, than American mathematician Eugene Borisovich Dynkin (1924–2014)—is well known as one of the most interesting outcome of the theory of groups of transformations. Let $\mathcal{A}$ be an associative unital algebra, over a field to have characteristcs zero. If $\mathbf{A} \in \mathcal{A}$, we define $\exp(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$.

Starting with 1897, based on the group theory, Campbell studies the existence of an element $\mathbf{C}$ that satisfies the identity $\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{C})$ in the particular case of the composition of the finite transformations $\exp(\mathbf{A})$, $\exp(\mathbf{B})$ of a continuous transformation group and he finds the solution without a substantial referring to the group theory.

The complete symbolical solution to this problem has been given independently by Baker [1] and by Hausdorff [2] in 1905 and 1906, respectively.

In 1947, Dynkin gave an explicit form for the commutator series of $\log(\exp(\mathbf{A})\exp(\mathbf{B}))$ and delivered a more general direct estimate for the convergence domain. The most important two consequences ware that the result could be generalized to the infinite-dimensional case of the Banach–Lie algebras and the Lie's Third Theorem received a very simple solution [3,4]. Later, the proof given by Dynkin to the Lie-series nature of $\log(\exp(\mathbf{A})\exp(\mathbf{B}))$ and his description of the combinatorial aspects of the exponential formula made possible the study of other presentations of $\log(\exp(\mathbf{A})\exp(\mathbf{B}))$.

The first applications of the Exponential Theorem were in Physics (starting in the 1960s) and in Quantum and Statistical Mechanics (see e.g., [5–13]).

The Bourbakist refoundation of Mathematics created the framework for the further mathematical formalization of the BCHD Theorem. In this context, the subsequent demonstrations of the Theorem used very general algebraic tools and, consequently, the BCHD Theorem should be considered as the result of noncommutative algebra and not as an outcome of the Lie group theory. The method has significant applications in both mathematics and physics, such as the theory of the structure of Lie algebras and Lie groups, group theory, operator theory, linear PDE analysis, and ODE theory, in control theory, in numerical analysis and other fields.

Different partial results that led to what is named Baker–Campbell–Hausdorff–Dynkin formula have been published during the last 100 years. Still, a completely general closed formula for this theorem has never been provided. Various approaches have directly targeted the series expansion, including the calculation of higher-order commutators by combinatorial methods or by the recurrence relations based on a better understanding of their algebraic properties [14,15]. So, for the operators **A** and **B**, the operator **C** within exp(**C**) = exp(**A**) exp(**B**) had the form of an infinite series of progressively higher order nested commutators of **A** and **B**, where is denoted $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.

$$\mathbf{C}(\mathbf{A},\mathbf{B}) = \mathbf{A} + \mathbf{B} + P_2(\mathbf{A},\mathbf{B}) + P_3(\mathbf{A},\mathbf{B}) + P_4(\mathbf{A},\mathbf{B}) + ..., \tag{1}$$

where

$$P_2(\mathbf{A},\mathbf{B}) = (1/2)[\mathbf{A},\mathbf{B}] \tag{2}$$

$$P_3(\mathbf{A},\mathbf{B}) = \frac{1}{12}\left([\mathbf{A},[\mathbf{A},\mathbf{B}]] + [\mathbf{B},[\mathbf{B},\mathbf{A}]]\right) \tag{3}$$

$$P_4(\mathbf{A},\mathbf{B}) = \frac{1}{24}\,[\mathbf{B},[\mathbf{A},[\mathbf{B},\mathbf{A}]]]\,. \tag{4}$$

Each higher $P_n(\mathbf{A},\mathbf{B})$ is a n-order homogeneous Lie polynomial in **A** and **B**. Existed techniques can determine the terms, but they rapidly become hard to be computed for higher ranks. Dynkin (1947) furnished the closed form of the explicit presentation of log((exp**A**) (exp**B**)) in terms of iterated brackets for the first time, more than 40 years after Hausdorff's paper. His solutions became the so-called Dynkin's Formula and it is mentioned under this name when his representation is involved in the Exponential Theorem. So, there are other significant merits of Dynkin that inspired the acronym BCHD such as that he provided another proof of the Exponential Theorem that enlightened all the combinatorial aspects around the theorem, completely different from the preceding ones (see Dynkin [16–19]) and that he gave the proof of the convergence matter, far more natural and simpler than Hausdorff's one.

The formula given by Dynkin can be found in [20] and has the bellow form:

$$H(\mathbf{A},\mathbf{B}) = \log\left(\exp(\mathbf{A})\exp(\mathbf{B})\right) = \sum_{k=1}^{\infty} \sum_{m_1+n_1>0} \cdots \sum_{m_k+n_k>0} \frac{(-1)^{k-1}}{k\sum_{i=1}^{k}(m_i+n_i)} \frac{1}{m_1!n_1!\cdots{}_k!n_k!} \overbrace{[\mathbf{A},[...,[\mathbf{A},}^{m_1} \overbrace{[\mathbf{B},[...,[\mathbf{B}\,,}^{n_1} [\ldots \tag{5}$$

$$... \overbrace{[\mathbf{A},[...,[\mathbf{A}\,,}^{m_k} \overbrace{[\mathbf{B},[\ldots,[\mathbf{B}\,,[\ldots]\;...]}^{n_k}\,.$$

If we denote $\mathrm{ad}_{\mathbf{A}}(\mathbf{B}) = [\mathbf{A},\mathbf{B}]$ the previous equation becomes:

$$H(\mathbf{A},\mathbf{B}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{m_1+n_1>0} \cdots \sum_{m_{k-1}+n_{k-1}>0} \left(\frac{1}{\sum_{i=1}^{k-1}(m_i+n_i)+1} \frac{ad_{\mathbf{A}}^{m_1} ad_{\mathbf{B}}^{n_1} \ldots ad_{\mathbf{A}}^{m_{k-1}} ad_{\mathbf{B}}^{n_{k-1}}(\mathbf{A})}{m_1!n_1!\ldots m_{k-1}!n_{k-1}!} + \right. \tag{6}$$

$$\left. \sum_{m_k\geq 0} \frac{1}{\sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i + 1} \frac{ad_{\mathbf{A}}^{m_1} ad_{\mathbf{B}}^{n_1} \ldots ad_{\mathbf{A}}^{m_{k-1}} ad_{\mathbf{B}}^{n_{k-1}}(\mathbf{B})}{m_1!n_1!\cdots{}_{k-1}!n_{k-1}!} \right).$$

The Dynkin formula is a difficult implementation for the case of non-nilpotent operators. The closed formulas of the BCHD series have been intensively used in Physics (see the papers of Van-Brunt and Visser [21]) because they admit significant extensions based on simple algorithms (see [22]). Marco Matone, in [23], proved that there are thirteen kinds of commutator algebras that admit closed forms of BCHD formula. Later, an iterative algorithm was introduced and the closed BCHD formulas for the generators of complex Lie algebras were derived [24].

Closed BCHD formulas have been used by Alessandro Bravetti, Angel Garcia-Chung and Diego Tapias in Heisenberg algebra [25]. Recent results about BCHD theorem can be found in [26–32].

For the particular case of the Lie group of rotations of the rigid body $SO(3)$, a closed form of BCHD formula was first written in [33] and then, in an equivalent form, in [34]. For the Lie group of rigid body displacements $SE(3)$ a closed form BCHD formula was first given by the authors of this work in the conference paper [35].

This paper proves the existence of the closed form of the Baker–Campbell–Hausdorff–Dynkin formula for the Lie algebra of rigid body displacement. For this purpose, the structure of the Lie group of the rigid body displacements $SE(3)$ and the properties of its Lie algebra $\mathfrak{se}(3)$ are used. A new solution to this problem in dual Lie algebra is given based on either the isomorphism between the Lie group $SE(3)$ and the Lie group of the orthogonal dual tensors or the homomorphism between $SE(3)$ and the dual quaternions Lie group.

The structure of the paper is as follows: in chapter two a new locally closed form solution is given for the BCHD formula in the case of the rotation group $SO(3)$. In the third chapter, using the Lie group structure for rigid body displacements $SE(3)$, and a new form for the exponential map on the corresponding Lie algebra $\mathfrak{se}(3)$, a compact form for the BCHD problem is obtained. The result is more interesting as there was a widespread belief that such a solution did not exist [36]. The fourth chapter begins with a series of results on the representation of rigid body displacements through dual algebra (ring of dual numbers, dual vectors, dual tensors, and dual quaternions). Closed form of BCHD formula in dual algebra is obtained based on isomorphic structures (such as groups and Lie algebras) and is obviously equivalent to that obtained in $SE(3)$. The fifth chapter summarizes the entire paper in a section of conclusions. In Appendix A we will propose a computational solution for both the singularity-free extraction of a unit dual quaternion from an orthogonal dual tensor.

2. Closed form BCHD Formula in SO(3)

Let be $SO(3) = \left\{ \boldsymbol{R} \in \mathbb{L}(\boldsymbol{V}_3, \boldsymbol{V}_3) \,\middle|\, \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}, \det \boldsymbol{R} = 1 \right\}$ the Lie group of the rotations and $\mathfrak{so}(3) = \left\{ \widetilde{\boldsymbol{\omega}} \in \mathbb{L}(\boldsymbol{V}_3, \boldsymbol{V}_3) \,\middle|\, \widetilde{\boldsymbol{\omega}}^{\mathrm{T}} + \widetilde{\boldsymbol{\omega}} = \boldsymbol{O} \right\}$ the Lie algebra of this Lie group. To simplify writing, we denote $\boldsymbol{T}(\mathbf{v}) = \boldsymbol{T}\mathbf{v}$ for any tensor $\boldsymbol{T} \in \mathbb{L}(\boldsymbol{V}_3, \boldsymbol{V}_3)$ and $\mathbf{v} \in \boldsymbol{V}_3$. In these conditions, the following remark takes place:

Remark 1. *The exponential map:*

$$\begin{array}{c} \exp : \mathfrak{so}(3) \to SO(3) \\ \exp(\widetilde{\boldsymbol{\omega}}) = \sum_{k=0}^{\infty} \frac{\widetilde{\boldsymbol{\omega}}^k}{k!} \end{array} \tag{7}$$

is well-defined and surjective.

It takes place the closed form formula:

$$\exp(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{I} + \mathrm{sinc}\,\omega\,\widetilde{\boldsymbol{\omega}} + \frac{1}{2}\mathrm{sinc}^2\frac{\omega}{2}\,\widetilde{\boldsymbol{\omega}}^2, \tag{8}$$

where ω *was denoted with* $\omega = \sqrt{-\frac{1}{2}\mathrm{Tr}\widetilde{\boldsymbol{\omega}}^2} = \|\mathrm{vect}\widetilde{\boldsymbol{\omega}}\|$ *and* sinc *is a cardinal sin function. The Equation (8) is obtained from the definition given in Equation (7), using the following identity:*

$$\widetilde{\boldsymbol{\omega}}^3 = -\omega^2\widetilde{\boldsymbol{\omega}}. \tag{9}$$

The linear invariants of tensor $\boldsymbol{R} = \exp(\widetilde{\boldsymbol{\omega}})$ *that results from:*

$$\mathrm{Tr}\boldsymbol{R} = 1 + 2\cos\omega \tag{10}$$

$$\mathrm{vect}\boldsymbol{R} = \sin\omega\,\boldsymbol{\omega}. \tag{11}$$

The inverse of the exponential map (7), denoted log, is a multiple valued function [35] :

$$\log : SO(3) \rightarrow \mathfrak{so}(3) \tag{12}$$

$$\log(\boldsymbol{R}) = \alpha\,\widetilde{\boldsymbol{u}}, \tag{13}$$

where

$$\widetilde{\boldsymbol{u}} = \pm \frac{\boldsymbol{R} - \boldsymbol{R}^{\boldsymbol{T}}}{\sqrt{(1 + \mathrm{Tr}\boldsymbol{R})\,(3 - \mathrm{Tr}\boldsymbol{R})}}, \tag{14}$$

$$\alpha = 2k\pi \pm \arccos\frac{\mathrm{Tr}\boldsymbol{R} - 1}{2}, k \in \mathbb{Z}. \tag{15}$$

The Equation (14) has singularities for $\mathrm{Tr}\boldsymbol{R} = 3$ and $\mathrm{Tr}\boldsymbol{R} = -1$. The following Lemma takes place:

Lemma 1.

1. *If* $\boldsymbol{R} \in SO(3)$ *is an proper orthogonal tensor, then* $\mathrm{Tr}\boldsymbol{R} \in [-1, 3]$*;*
2. *If* $\boldsymbol{R} \in SO(3)$ *and* $\mathrm{Tr}\boldsymbol{R} = 3$*, then* $\boldsymbol{R} = \boldsymbol{I}$*;*
3. *If* $\boldsymbol{R} \in SO(3)$ *and* $\mathrm{Tr}\boldsymbol{R} = -1 \Longleftrightarrow \boldsymbol{R} = \boldsymbol{R}^T, \boldsymbol{R} \neq \boldsymbol{I}$*.*

Proof of Lemma 1. The properties from Lemma 1 result from Equations (8)–(11):

If $\mathrm{Tr}\boldsymbol{R} = 3$, $\boldsymbol{R} = \boldsymbol{I}$ and $\log(\boldsymbol{R}) = \boldsymbol{0}$;

For $\mathrm{Tr}\boldsymbol{R} = -1$, $\boldsymbol{R} = \boldsymbol{R}^T$ and $\log(\boldsymbol{R}) \supset \{\pm\pi(2k+1)\widetilde{\boldsymbol{u}}\}$ with $k \in \mathbb{Z}$ where $\mathbf{u}$ is computed as $\mathbf{u} = \frac{\boldsymbol{R}\mathbf{v} + \mathbf{v}}{\|\boldsymbol{R}\mathbf{v} + \mathbf{v}\|}$ $\forall \mathbf{v} \in \mathbf{V_3}$ that have the property $\boldsymbol{R}\mathbf{v} \neq -\mathbf{v}$.

A new closed form BCHD formula for Lie algebra $\mathfrak{so}(3)$ will be given in the following theorem. □

Theorem 1. *Let be* $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathfrak{so}(3)$*;* $\omega_1, \omega_2 < 2\pi$ *.The bellow identity takes place:*

$$\log\left(\exp(\widetilde{\omega}_1)\exp(\widetilde{\omega}_2)\right) = \widetilde{\omega}, \tag{16}$$

where

$$\widetilde{\omega} = \alpha_1\widetilde{\omega}_1 + \alpha_2\widetilde{\omega}_2 + \alpha_{12}\left[\widetilde{\omega}_1, \widetilde{\omega}_2\right], \tag{17}$$

with:

$$\alpha_1 = \frac{\mathrm{sinc}\frac{\omega_1}{2}\cos\frac{\omega_2}{2}}{\mathrm{sinc}\frac{\omega}{2}}, \tag{18}$$

$$\alpha_2 = \frac{\cos\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}}{\mathrm{sinc}\frac{\omega}{2}}, \tag{19}$$

$$\alpha_{12} = \frac{\mathrm{sinc}\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}}{2\mathrm{sinc}\frac{\omega}{2}}, \tag{20}$$

$$\omega = 2\arccos\left(cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2} + \frac{1}{8}\mathrm{sinc}\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}\mathrm{Tr}\widetilde{\omega}_1\widetilde{\omega}_2\right). \tag{21}$$

Proof of Theorem 1. Let be $\widetilde{\omega} \in \mathfrak{so}(3)$ such that:

$$\exp(\widetilde{\omega}) = \exp(\widetilde{\omega}_1)\exp(\widetilde{\omega}_2). \tag{22}$$

Taking into account Equations (8), (10), (11) and (16), after some algebra, we obtain:

$$\cos\frac{\omega}{2} = \cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2} - \frac{1}{4}\mathrm{sinc}\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 \tag{23}$$

$$\mathrm{sinc}\frac{\omega}{2}\boldsymbol{\omega} = \mathrm{sinc}\frac{\omega_1}{2}\cos\frac{\omega_2}{2}\boldsymbol{\omega}_1 + \cos\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}\boldsymbol{\omega}_2 + \frac{1}{2}\mathrm{sinc}\frac{\omega_1}{2}\mathrm{sinc}\frac{\omega_2}{2}\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2. \tag{24}$$

From (23) results Equation (21). Using the identities:

$$\widetilde{\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2} = \widetilde{\omega}_1\widetilde{\omega}_2 - \widetilde{\omega}_2\widetilde{\omega}_1 \overset{\text{def}}{=} [\widetilde{\omega}_1, \widetilde{\omega}_2] \tag{25}$$

$$\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 = -\frac{1}{2}\mathrm{Tr}\widetilde{\omega}_1\widetilde{\omega}_2, \tag{26}$$

from Equation (24), results Equations (17)–(20). □

Remark 2. *From Remark 1, it results that* $\forall \boldsymbol{R} \in SO(3)$, *an skew-symmetric tensor* $\widetilde{\omega}$ *exists so that* $\boldsymbol{R} = \exp(\widetilde{\omega})$. *The vector* $\boldsymbol{\omega} = \mathrm{vect}(\widetilde{\omega})$ *is named rotation vector or Euler vector associated to tensor* $\boldsymbol{R}$. *This vector parameterization of the tensor* $\boldsymbol{R}$ *is minimal (in the theory of Lie groups, it is called the first kind exponential parameterization). The Equation (17), written in vector form:*

$$\boldsymbol{\omega} = \alpha_1\boldsymbol{\omega}_1 + \alpha_2\boldsymbol{\omega}_2 + \alpha_{12}(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2), \tag{27}$$

determines the rotation vector (the Euler vector) that corresponds to the rotation that results from the successive composition of the rotations given by the tensors $\boldsymbol{R}_2$ *and* $\boldsymbol{R}_1$*, respectively. The Equation (27) doesn't have singularities without other common vector parameterization such as Rodrigues vector or Wiener–Milenkovic vector* [37].

3. Closed Form BCHD Formula in SE(3)

Let be $SE(3)$ the Lie group of the rigid body displacements and $\mathfrak{se}(3)$ its Lie algebra. As it is known, the generic elements from $SE(3)$ and $\mathfrak{se}(3)$ can be written in the following matrix form [38,39]:

$$\begin{gathered} g = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3) \\ \widehat{\xi} = \begin{bmatrix} \widetilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathfrak{se}(3), \end{gathered} \tag{28}$$

where $\boldsymbol{R} \in SO(3)$, $\widetilde{\omega} \in \mathfrak{so}(3)$, $\mathbf{t}, \mathbf{v} \in \mathbf{V}_3$. The author's searches in the literature did not report the existence of other closed form of the BCHD formula for the Lie algebra of the rigid body displacements.

Lemma 2. *The exponential mapping:* $\exp : \mathfrak{se}(3) \to SE(3);\ \exp(\widehat{\xi}) = \sum_{k=0}^{\infty} \frac{\widehat{\xi}^k}{k!}$ *is well defined and surjective. The below relationship takes place:*

$$\exp(\widehat{\xi}) = \begin{bmatrix} \exp(\widetilde{\omega}) & \mathrm{dexp}_{\widetilde{\omega}}\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{29}$$

where $\exp(\widetilde{\omega})$ *is given by Equation (8) and tensor* $\mathrm{dexp}_{\widetilde{\omega}}$ *has the following shape:*

$$\mathrm{dexp}_{\widetilde{\omega}} = \boldsymbol{I} + \frac{1}{2}\mathrm{sinc}^2\frac{\omega}{2}\widetilde{\omega} + (1 - \mathrm{sinc}\omega)\,\frac{\widetilde{\omega}^2}{\omega^2}. \tag{30}$$

Proof of Lemma 2. By simple computations it results that:

$$\widehat{\xi} = \begin{bmatrix} \widetilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \Longrightarrow \exp(\widehat{\xi}) = \mathbf{I}_4 + \sum_{k=1}^{\infty} \frac{1}{k!}\widehat{\xi}^k = \begin{pmatrix} \exp(\widetilde{\omega}) & \mathrm{dexp}_{\widetilde{\omega}}\mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{31}$$

where $\text{dexp}_{\widetilde{\boldsymbol{\omega}}}$ denotes the tensor:

$$\text{dexp}_{\widetilde{\boldsymbol{\omega}}} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \widetilde{\boldsymbol{\omega}}^k. \quad (32)$$

It results immediately that $\exp : \mathfrak{se}(3) \rightarrow SE(3)$ is well-defined. As we proved that $\exp : \mathfrak{se}(3) \rightarrow SE(3)$ is a surjective mapping, it remains only to prove that tensor $\text{dexp}_{\widetilde{\boldsymbol{\omega}}}$ is invertible. A useful remark is that tensor $\text{dexp}_{\widetilde{\boldsymbol{\omega}}}$ may be written:

$$\text{dexp}_{\widetilde{\boldsymbol{\omega}}} = \int_0^1 \exp(\widetilde{\boldsymbol{\omega}} t) dt. \quad (33)$$

The tensor $\text{dexp}_{\widetilde{\boldsymbol{\omega}}}$ has the following closed form [40]:

$$\text{dexp}_{\widetilde{\omega}} = \boldsymbol{I} + \frac{1}{2}\text{sinc}^2 \frac{\omega}{2} \widetilde{\omega} + (1 - \text{sinc}\omega) \frac{\widetilde{\omega}^2}{\omega^2}. \quad (34)$$

For $\omega \neq 2k\pi,\ k \in \mathbb{N}$, this tensor is invertible and we have:

$$\text{dexp}_{\widetilde{\omega}}^{-1} = \boldsymbol{I} - \frac{1}{2}\widetilde{\omega} + \left(1 - \frac{\omega}{2}\cot\frac{\omega}{2}\right) \frac{\widetilde{\omega}^2}{\omega^2}. \quad (35)$$

□

Therefore, the inverse of the function exp is a multiple valued function given by:

$$\log : SE(3) \rightarrow \mathfrak{se}(3) \quad (36)$$

$$\log(\mathbf{g}) = \begin{bmatrix} \widetilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}, \quad (37)$$

where $\mathbf{g} = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ with $\boldsymbol{R} \in SO(3)$, $\mathbf{t} \in \boldsymbol{V}_3$ $\widetilde{\boldsymbol{\omega}} \in \log(\boldsymbol{R})$ and $\mathbf{v} = \text{dexp}_{\widetilde{\omega}}^{-1}\mathbf{t}$.

One of the most fundamental results in spatial kinematics is the Mozzi–Chasles theorem [41]: the most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line. Because this displacement is reminiscent of the displacement of a screw, it is called a screw displacement and the line or axis is called the screw axis. The vectors $\boldsymbol{\omega} = \text{vect}\widetilde{\boldsymbol{\omega}}$ and $\mathbf{v}$ completely characterize the screw parameters of a rigid body displacement. The screw axis is a directed line that has the following vector equation:

$$\mathbf{r} = \frac{\boldsymbol{\omega} \times \mathbf{v}}{\|\boldsymbol{\omega}\|^2} + \lambda \boldsymbol{\omega}, \lambda \in \mathbb{R}_+. \quad (38)$$

The rotation angle around this axis is:

$$\alpha = \|\boldsymbol{\omega}\| \quad (39)$$

and the translation vector is:

$$\mathbf{d} = \frac{\boldsymbol{\omega} \cdot \mathbf{v}}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\omega}. \quad (40)$$

Remark 3. *For $\boldsymbol{R} \in SO(3)$, $\boldsymbol{R} = \exp(\widetilde{\omega})$, the following two identities hold:*

$$\boldsymbol{R} = \text{dexp}_{\widetilde{\boldsymbol{\omega}}} \text{dexp}_{\widetilde{\boldsymbol{\omega}}}^{-T} \quad (41)$$

$$\dot{\boldsymbol{R}} = \widetilde{\text{dexp}_{\widetilde{\omega}} \dot{\boldsymbol{\omega}}} \boldsymbol{R} = \boldsymbol{R} \widetilde{\text{dexp}_{-\widetilde{\omega}} \dot{\boldsymbol{\omega}}}, \quad (42)$$

where $\dot{\boldsymbol{\omega}}$ *denotes the time derivative of* $\boldsymbol{\omega}$.

The closed form BCHD formula for Lie algebra $\mathfrak{se}(3)$ is given by the following theorem.

Theorem 2. *Let be* $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{se}(3)$ *with* $\hat{\xi}_1 = \begin{bmatrix} \tilde{\omega}_1 & \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix}$ *and* $\hat{\xi}_2 = \begin{bmatrix} \tilde{\omega}_2 & \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix}$. *The below identity holds:*

$$\log\left(\exp(\hat{\xi}_1)\exp(\hat{\xi}_2)\right) = \begin{bmatrix} \tilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}, \tag{43}$$

where

$$\tilde{\omega} = \alpha_1\tilde{\omega}_1 + \alpha_2\tilde{\omega}_2 + \alpha_{12}\left[\tilde{\omega}_1, \tilde{\omega}_2\right] \tag{44}$$

$$\mathbf{v} = \boldsymbol{T}_1\mathbf{v}_1 + \boldsymbol{T}_2\mathbf{v}_2. \tag{45}$$

The coefficients $\alpha_1, \alpha_2, \alpha_{12}$ *are given by the Equations (18)–(20) and the invertible tensors* $\boldsymbol{T}_1$ *and* $\boldsymbol{T}_2$ *depend solely on* $\tilde{\omega}_1$ *and on* $\tilde{\omega}_2$ *and are given by the following equations:*

$$\boldsymbol{T}_1 = \mathrm{dexp}_{\tilde{\omega}}^{-1}\mathrm{dexp}_{\tilde{\omega}_1} \tag{46}$$

$$\boldsymbol{T}_2 = \mathrm{dexp}_{\tilde{\omega}}^{-1}\exp(\tilde{\omega}_1)\mathrm{dexp}_{\tilde{\omega}_2}. \tag{47}$$

Proof of Theorem 2. Let be $\tilde{\omega} \in \mathfrak{so}(3)$ and $\mathbf{v} \in \boldsymbol{V}_3$ such that:

$$\exp(\xi_1)\exp(\xi_2) = \exp\begin{bmatrix} \tilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}. \tag{48}$$

Considering Equation (29), from Equation (48), it follows:

$$\exp(\tilde{\omega}_1)\exp(\tilde{\omega}_2) = \exp(\tilde{\omega}) \tag{49}$$

$$\mathrm{dexp}_{\tilde{\omega}}\mathbf{v} = \exp(\tilde{\omega}_1)\mathrm{dexp}_{\tilde{\omega}_2}\mathbf{v}_2 + \mathrm{dexp}_{\tilde{\omega}_1}\mathbf{v}_1. \tag{50}$$

Taking into consideration the Equation (49) and Theorem 1, it follows Equation (44). From Equation (50), it follows the Equation (45), where:

$$\begin{array}{c} \boldsymbol{T}_1 = \mathrm{dexp}_{\tilde{\omega}}^{-1}\mathrm{dexp}_{\tilde{\omega}_1} \\ \boldsymbol{T}_2 = \mathrm{dexp}_{\tilde{\omega}}^{-1}\exp(\tilde{\omega}_1)\mathrm{dexp}_{\tilde{\omega}_2} \end{array}. \tag{51}$$

□

Remark 4. *The Equations (44) and (45) from Theorem 2 allow the determination of screw vectors* $(\boldsymbol{\omega}, \mathbf{v})$ *for the rigid body displacement obtained from the successive application of two rigid body displacements of the screw vectors* $(\boldsymbol{\omega}_2, \mathbf{v}_2)$ *and* $(\boldsymbol{\omega}_1, \mathbf{v}_1)$*, respectively. This observation can be used for the analysis and the synthesis of the spatial mechanisms* [42–44]. *For exemple, by composing two rigid body displacements, a pure rotation is obtained if and only if* $\boldsymbol{\omega} \cdot \mathbf{v} = 0$. *The rotation axis passes through the origin of reference frame if and only if* $\boldsymbol{T}_1\mathbf{v}_1 + \boldsymbol{T}_2\mathbf{v}_2 = \mathbf{0}$.

4. Closed Form BCHD Formula for the Dual Lie Algebra of Rigid Body Displacements

Applying the isomorphism between the Lie algebra $\mathfrak{se}(3)$ and Lie algebra of the dual vectors [37,45,46], a closed form solution of the BCHD formula for Lie algebra of rigid body displacements can be given.

4.1. Dual Algebra: Mathematical Preliminaries

In this section we present properties of: dual numbers, dual vectors and dual tensors. More details can be found in [37,46–50].

4.1.1. Dual Numbers

Let the set of real dual numbers be denoted by

$$\underline{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} = \{\underline{a} = a + \varepsilon a_0 | a, a_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}, \tag{52}$$

where $a = Re(\underline{a})$ is the real part of $\underline{a}$ and $a_0 = Du\,(\underline{a})$ the dual part. The sum and product between dual numbers generate a ring with zero divisors structure for $\underline{\mathbb{R}}$. Of all the properties that dual numbers have, this work uses mainly their magnitude. The magnitude of a dual number fulfills the condition $|\underline{a}|^2 = \underline{a}^2$ and can be computed using the $|\underline{a}| = |a| + \varepsilon \text{sgn}\,(a)\,a_0$ formulas. The inverse of a dual number, denoted by $\underline{a}^{-1} \in \underline{\mathbb{R}}$, exists if and only if $Re\,(\underline{a}) \neq 0$ and can be computed using the $\underline{a}^{-1} = \frac{1}{\underline{a}} = \frac{1}{a} - \varepsilon\frac{a_0}{a^2}$ formulas. Another property is that $\underline{a} \in \underline{\mathbb{R}}$ is a zero divisor if and only if $Re\,(\underline{a}) = 0$. Based on these properties, it can be said that $(\underline{\mathbb{R}}, +, \cdot)$ is a commutative and unitary ring and any element $\underline{a} \in \underline{\mathbb{R}}$ is either invertible or zero divisor.

Any differentiable function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R},\ f = f\,(a)$ is completely defined on $\underline{\mathbb{I}} \subset \underline{\mathbb{R}}$ such that: $f : \underline{\mathbb{I}} \subset \underline{\mathbb{R}} \to \underline{\mathbb{R}}; f(\underline{a}) = f(a) + \varepsilon a_0 f'\,(a)$.

Based on the previous property, we can compute: $\cos(\underline{a}) = \cos(a) - \varepsilon a_0 \sin(a); \sin(\underline{a}) = \sin(a) + \varepsilon a_0 \cos(a)$; $\sqrt[n]{(\underline{a})} = \sqrt[n]{(a)} + \varepsilon\frac{a_0}{n\sqrt[n]{(a)^{n-1}}}$; $\tan(\underline{a}) = \tan(a) + \varepsilon\frac{a_0}{\cos^2(a)}$; $\arctan(\underline{a}) = \arctan(a) + \varepsilon\frac{a_0}{1+(a)^2}$; $\text{arcos}(\underline{a}) = \text{arcos}(a) - \varepsilon\frac{a_0}{\sqrt{1-(a)^2}}$; $atan2(\underline{b},\ \underline{a}) = atan2(b, a) + \varepsilon\frac{b_0 a - b a_0}{a^2+b^2}$.

4.1.2. Dual Vectors

As is known, in the Euclidean space, the linear space of free vectors with dimension 3 is denoted by $\boldsymbol{V}_3$. The set of dual vectors is defined as:

$$\underline{\boldsymbol{V}_3} = \boldsymbol{V}_3 + \varepsilon\boldsymbol{V}_3 = \left\{\underline{\mathbf{a}} = \mathbf{a} + \varepsilon\mathbf{a}_0; \mathbf{a}, \mathbf{a}_0 \in \boldsymbol{V}_3, \varepsilon^2 = 0,\ \varepsilon \neq 0\right\}, \tag{53}$$

where $\mathbf{a} = Re\,(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = Du\,(\underline{\mathbf{a}})$ the dual part. It is In the particular case of dual vectors, three products are considered: scalar product (denoted by $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product (denoted by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$) and triple scalar product (denoted by $\langle\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}\rangle = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$. Regarding algebraic structure, $(\underline{\boldsymbol{V}_3}, +, \cdot_{\underline{\mathbb{R}}})$ is a free $\underline{\mathbb{R}}$-module [37].

For any dual vector $\underline{\mathbf{a}} \in \underline{\boldsymbol{V}_3}$, the magnitude of $\underline{\mathbf{a}}$, denoted by $|\underline{\mathbf{a}}|$, is the dual number which fulfills $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be computed using

$$|\underline{\mathbf{a}}| = \begin{cases} \|\mathbf{a}\| + \varepsilon\frac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}, Re\,(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \varepsilon\,\|\mathbf{a}_0\|\,,\ Re\,(\underline{\mathbf{a}}) = \mathbf{0}, \end{cases} \tag{54}$$

where $\|\cdot\|$ is the Euclidean norm. If $|\underline{\mathbf{a}}| = 1$ then $\underline{\mathbf{a}}$ is called unit dual vector.

Theorem 3. *For any $\underline{\mathbf{a}} \in \underline{\boldsymbol{V}_3}$, a dual number $\underline{\alpha} \in \underline{\mathbb{R}}$, and a unit dual vector $\underline{\mathbf{u}}_a \in \underline{\boldsymbol{V}_3}$ exist in order to have*

$$\underline{\mathbf{a}} = \underline{\alpha}\underline{\mathbf{u}}_a. \tag{55}$$

The computational formulas for $\underline{\alpha}$ and $\underline{\mathbf{u}}_a$, are:

$$\pm\,\underline{\alpha} = |\underline{\mathbf{a}}| \tag{56}$$

$$\pm\underline{\mathbf{u}}_a=\begin{cases}\frac{\mathbf{a}}{\|\mathbf{a}\|}+\varepsilon\frac{\mathbf{a}\times(\mathbf{a}_0\times\mathbf{a})}{\|\mathbf{a}\|^3} & Re\left(\underline{\mathbf{a}}\right)\neq\mathbf{0}\\ \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}+\varepsilon\mathbf{v}\times\frac{\mathbf{a}_0}{\|\mathbf{a}_0\|},\forall\mathbf{v}\in V_3 & Re\left(\underline{\mathbf{a}}\right)=\mathbf{0}.\end{cases} \tag{57}$$

For $Re\left(\underline{\mathbf{a}}\right)\neq\mathbf{0}$, $\underline{\alpha}$, *and* $\underline{\mathbf{u}}_a$ *are unique up to a sign change.*

The proof of this Theorem was presented by the author in [37]. The previous result emphasizes that any dual vector $\underline{\mathbf{a}}\in\underline{V}_3$, with $Re\left(\underline{\mathbf{a}}\right)\neq\mathbf{0}$ can be associated with a labeled directed line in the Euclidean three-dimensional space. This directed line has the following parametric equation: $\mathbf{r}=\frac{\mathbf{a}\times\mathbf{a}_0}{\|\mathbf{a}\|^2}+\lambda\frac{\mathbf{a}}{\|\mathbf{a}\|},\forall\lambda\in\mathbb{R}$. If $Re\left(\underline{\mathbf{a}}\right)=\mathbf{0}$, the parametric equation is $\mathbf{r}=\mathbf{v}+\lambda\frac{\mathbf{a}_0}{\|\mathbf{a}_0\|},\forall\mathbf{v}\in V_3,\forall\lambda\in\mathbb{R}$.

4.1.3. Dual Tensors

An $\underline{\mathbb{R}}$-linear mapping of $\underline{V}_3$ into $\underline{V}_3$ is called an Euclidean dual tensor:

$$\underline{\boldsymbol{T}}\left(\underline{\lambda}_1\underline{\mathbf{v}}_1+\underline{\lambda}_2\underline{\mathbf{v}}_2\right)=\underline{\lambda}_1\underline{\boldsymbol{T}}\left(\underline{\mathbf{v}}_1\right)+\underline{\lambda}_2\underline{\boldsymbol{T}}\left(\underline{\mathbf{v}}_2\right),\forall\underline{\lambda}_1,\underline{\lambda}_2\in\underline{\mathbb{R}},\forall\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2\in\underline{V}_3. \tag{58}$$

A Euclidean dual tensor will be called dual tensor and $\mathbf{L}\left(\underline{V}_3,\underline{V}_3\right)$ will denote the free $\underline{\mathbb{R}}$-module of dual tensors. To simplify writing, we denote $\underline{\boldsymbol{T}}(\underline{\mathbf{v}})=\underline{\boldsymbol{T}}\,\underline{\mathbf{v}}$ for any tensor $\underline{\boldsymbol{T}}\in\mathbf{L}\left(\underline{V}_3,\underline{V}_3\right)$ and $\underline{\mathbf{v}}\in\underline{V}_3$. A dual tensor $\underline{\boldsymbol{T}}\in\mathbf{L}\left(\underline{V}_3,\underline{V}_3\right)$ can be decomposed in $\underline{\boldsymbol{T}}=\boldsymbol{T}+\varepsilon\boldsymbol{T}_0$, with $\boldsymbol{T},\boldsymbol{T}_0\in\mathbf{L}\left(V_3,V_3\right)$ are real tensors. The transposed dual tensor is denoted by $\underline{\boldsymbol{T}}^{\mathrm{T}}$ and is defined by

$$\underline{\mathbf{v}}_1\cdot\left(\underline{\boldsymbol{T}}\underline{\mathbf{v}}_2\right)=\underline{\mathbf{v}}_2\cdot\left(\underline{\boldsymbol{T}}^{\mathrm{T}}\underline{\mathbf{v}}_1\right),\forall\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2\in\underline{V}_3, \tag{59}$$

while , $\forall\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\in\underline{V}_3$, $Re\left(\langle\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle\right)\neq 0$, the determinant is:

$$\langle\underline{\boldsymbol{T}}\underline{\mathbf{v}}_1,\underline{\boldsymbol{T}}\underline{\mathbf{v}}_2,\underline{\boldsymbol{T}}\underline{\mathbf{v}}_3\rangle=\det\underline{\boldsymbol{T}}\,\langle\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle\,. \tag{60}$$

For any dual vector $\underline{\mathbf{a}}\in\underline{V}_3$ the associated skew-symmetric dual tensor will be denoted by $\underline{\widetilde{\mathbf{a}}}$ and will be defined by:

$$\underline{\widetilde{\mathbf{a}}}\,\underline{\mathbf{b}}=\underline{\mathbf{a}}\times\underline{\mathbf{b}},\forall\underline{\mathbf{b}}\in\underline{V}_3. \tag{61}$$

The previous definition produces the following result: for any skew-symmetric dual tensor $\underline{\boldsymbol{A}}\in\mathbf{L}\left(\underline{V}_3,\underline{V}_3\right),\underline{\boldsymbol{A}}=-\underline{\boldsymbol{A}}^{\mathrm{T}}$, a uniquely defined dual vector $\underline{\mathbf{a}}=vect\underline{\boldsymbol{A}}$, $\underline{\mathbf{a}}\in\underline{V}_3$ exists in order to have $\underline{\boldsymbol{A}}\underline{\mathbf{b}}=\underline{\mathbf{a}}\times\underline{\mathbf{b}},\forall\underline{\mathbf{b}}\in\underline{V}_3$. The set of skew-symmetric dual tensors is structured as a free $\underline{\mathbb{R}}$-module of rank 3, and is isomorph with $\underline{V}_3$.

An important class of invariants that will be used to describe the dual tensor are called linear invariants and are denoted by $\mathrm{vect}\underline{\boldsymbol{T}}=\mathrm{vect}\frac{1}{2}\left[\underline{\boldsymbol{T}}-\underline{\boldsymbol{T}}^{\mathrm{T}}\right]$, $\mathrm{Tr}\underline{\boldsymbol{T}}$ [37,41], where

$$\mathrm{Tr}\underline{\boldsymbol{T}}=\frac{\langle\underline{\boldsymbol{T}}\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle+\langle\underline{\mathbf{v}}_1,\underline{\boldsymbol{T}}\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle+\langle\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\boldsymbol{T}}\underline{\mathbf{v}}_3\rangle}{\langle\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle}, \tag{62}$$

for any $\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\in\underline{V}_3$ with $Re\left(\langle\underline{\mathbf{v}}_1,\underline{\mathbf{v}}_2,\underline{\mathbf{v}}_3\rangle\right)\neq 0$.

Given two dual vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}\in\underline{V}_3$, $\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}$ denotes a dual tensor called tensor (dyadic) product and is defined by:

$$\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}:\underline{V}_3\to\underline{V}_3,\ \left(\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}\right)\underline{\mathbf{v}}=\left(\underline{\mathbf{v}}\cdot\underline{\mathbf{b}}\right)\underline{\mathbf{a}},\forall\underline{\mathbf{v}}\in\underline{V}_3. \tag{63}$$

An important property of (63) is: $\left(\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}\right)\left(\underline{\mathbf{c}}\otimes\underline{\mathbf{d}}\right)=\left(\underline{\mathbf{b}}\cdot\underline{\mathbf{c}}\right)\underline{\mathbf{a}}\otimes\underline{\mathbf{d}}$.

Remark 5. *If* $\mathbf{B}=\{\underline{\mathbf{e}}_1,\underline{\mathbf{e}}_2,\underline{\mathbf{e}}_3\}$ *is an orthonormal basis of dual vectors and* $\underline{\mathbf{a}}=\sum_{i=1}^{3}\underline{a}^i\underline{\mathbf{e}}_i$, $\underline{\mathbf{b}}=\sum_{i=1}^{3}\underline{b}^i\underline{\mathbf{e}}_i$ *the dyadic product* $\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}$ *is linked to a matrix of dual numbers computed as* $\left[\underline{\mathbf{a}}\otimes\underline{\mathbf{b}}\right]=\left.\underline{\mathbf{a}}\right]\left.\underline{\mathbf{b}}\right]^{\mathrm{T}}$, *where* $\left.\underline{\mathbf{a}}\right]=\left[\underline{a}^1,\underline{a}^2,\underline{a}^3\right]^{\mathrm{T}}$ *and* $\left.\underline{\mathbf{b}}\right]=\left[\underline{b}^1,\underline{b}^2,\underline{b}^3\right]^{\mathrm{T}}$. *The skew-symmetric tensor is linked to a matrix of* *dual numbers* $\widetilde{\underline{\mathbf{a}}}\rceil = \begin{bmatrix} \underline{0} & -\underline{a}^3 & \underline{a}^2 \\ \underline{a}^3 & \underline{0} & -\underline{a}^1 \\ -\underline{a}^2 & \underline{a}^1 & \underline{0} \end{bmatrix}$.

More details on relations between dual numbers, dual vectors and dual matrices can be found in [37,47].

4.2. Rigid Body Displacement Parameterization through Orthogonal Dual Tensors

Let be the orthogonal dual tensor set denoted by:

$$\underline{SO}(3) = \left\{ \underline{\boldsymbol{R}} \in \mathbf{L}\left(\underline{V}_3, \underline{V}_3\right) \middle| \underline{\boldsymbol{R}}\underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}}, \det\underline{\boldsymbol{R}} = 1 \right\}, \tag{64}$$

with $\underline{\boldsymbol{I}}$ being the unit orthogonal dual tensor. The internal structure of any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{SO}(3)$ is presented in a series of results that were detailed in our previous work [37].

Theorem 4. *(Structure theorem): For any* $\underline{\boldsymbol{R}} \in \underline{SO}(3)$*, an unique decomposition is viable*

$$\underline{\boldsymbol{R}} = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}}\right)\boldsymbol{R}, \tag{65}$$

where $\boldsymbol{R} \in SO(3)$ *and* $\boldsymbol{\rho} \in \boldsymbol{V}_3$ *are called structural invariants.*

Theorem 5. *(Representation theorem): For any orthogonal dual tensor* $\underline{\mathbf{R}}$ *defined as in Equation (64), a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *can be computed in order to have* [37]*:*

$$\underline{\boldsymbol{R}}\left(\underline{\alpha}, \underline{\mathbf{u}}\right) = \underline{\boldsymbol{I}} + \sin\underline{\alpha}\widetilde{\underline{\mathbf{u}}} + \left(1 - \cos\underline{\alpha}\right)\widetilde{\underline{\mathbf{u}}}^2 = \exp\left(\underline{\alpha}\widetilde{\underline{\mathbf{u}}}\right). \tag{66}$$

The computational formulas for α, $\mathbf{u}$, d, $\mathbf{u}_0$*, are:*

$$\alpha = \operatorname{atan2}\left(\pm\frac{1}{2}\sqrt{\left(1 + \operatorname{Tr}\boldsymbol{R}\right)\left(3 - \operatorname{Tr}\boldsymbol{R}\right)}; \frac{\operatorname{Tr}\boldsymbol{R} - 1}{2}\right) \tag{67}$$

$$\mathbf{u} = \begin{cases} \pm\operatorname{vect}\left(\frac{1}{\sqrt{(1+\operatorname{Tr}\boldsymbol{R})(3-\operatorname{Tr}\boldsymbol{R})}}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right)\right), \text{ when } \operatorname{Tr}\boldsymbol{R} \in (-1, 3) \\ \frac{\boldsymbol{R}\mathbf{v}+\mathbf{v}}{\|\boldsymbol{R}\mathbf{v}+\mathbf{v}\|}, \ \forall\mathbf{v} \in \boldsymbol{V}_3, \text{ when } \operatorname{Tr}\boldsymbol{R} = -1 \ (\boldsymbol{R} \text{ is symmetric}) \\ \frac{\boldsymbol{\rho}}{\|\boldsymbol{\rho}\|}, \text{ when } \operatorname{Tr}\boldsymbol{R} = 3 \ (\boldsymbol{R} = \mathbf{I}) \end{cases} \tag{68}$$

$$d = \boldsymbol{\rho} \cdot \mathbf{u} \tag{69}$$

$$\mathbf{u}_0 = \begin{cases} \frac{1}{2}\boldsymbol{\rho} \times \mathbf{u} + \frac{1}{2}\cot\frac{\alpha}{2}\mathbf{u} \times \left(\boldsymbol{\rho} \times \mathbf{u}\right), \alpha \neq 0 \\ \frac{1}{2}\boldsymbol{\rho} \times \mathbf{u}, \alpha = 0 \end{cases}. \tag{70}$$

Both parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the natural invariants of $\underline{\boldsymbol{R}}$. The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi–Chalses axis [48], while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translation distance d. If $\underline{\alpha} \in \mathbb{R}$, there is the case of a rotation parameterization, while for $\underline{\alpha} \in \varepsilon\mathbb{R}$, the parameterization describes a translation.

The Lie algebra of the Lie group $\underline{SO}(3)$ is the skew-symmetric dual tensor set denoted by $\underline{\mathfrak{so}}(3) = \left\{\widetilde{\underline{\omega}} \in \mathbf{L}\left(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3\right) \middle| \widetilde{\underline{\omega}} = -\widetilde{\underline{\omega}}^{\mathrm{T}}\right\}$, where the internal mapping is $\langle\widetilde{\underline{\omega}}_1, \widetilde{\underline{\omega}}_2\rangle = \widetilde{\underline{\omega}_1\underline{\omega}_2}$. The Lie algebra $\underline{\mathfrak{so}}(3)$ is isomorphic to the Lie algebra of dual vectors $\underline{\mathbf{V}}_3$, having as internal operation the cross product of dual vectors.

The link between the Lie algebra $\underline{\mathfrak{so}}(3)$, the Lie group $\underline{SO}(3)$, and the exponential map is given below.

Theorem 6. *The mapping:*

$$\begin{array}{c} \exp : \underline{\mathfrak{so}}(3) \to \underline{SO}(3), \\ \exp(\underline{\widetilde{\omega}}) = e^{\underline{\widetilde{\omega}}} = \sum_{k=0}^{\infty} \frac{\underline{\widetilde{\omega}}^{k}}{k!}. \end{array} \tag{71}$$

is well-defined and surjective. It takes place the following closed form formula [37]*:*

$$\exp(\underline{\widetilde{\omega}}) = \underline{\boldsymbol{I}} + \mathrm{sinc}\underline{\omega}\underline{\widetilde{\omega}} + \frac{1}{2}\mathrm{sinc}^2\frac{\underline{\omega}}{2}\underline{\widetilde{\omega}}^2, \tag{72}$$

where

$$\underline{\omega} = \sqrt{-\frac{1}{2}\mathrm{Tr}\underline{\widetilde{\omega}}^2} = |\mathrm{vect}\underline{\widetilde{\omega}}| \tag{73}$$

$$sinc\underline{\omega} = \begin{cases} \frac{sin\underline{\omega}}{\underline{\omega}}, & \text{if } Re\underline{\omega} \neq 0 \\ 1, & \text{if } Re\underline{\omega} = 0 \end{cases}. \tag{74}$$

Proof of Theorem 6. Equation (72) is a new shape of Equation (66). □

A dual vector $\underline{\boldsymbol{\omega}}$ parameterizes any screw axis of a rigid body displacement, whereas the screw parameters (angle of rotation about the screw axis and the translation along the screw axis). The computation of the screw axis is related to the finding the logarithm of an orthogonal dual tensor $\underline{\boldsymbol{R}}$, which is a multiple valued function defined by:

$$\begin{array}{c} \log : \underline{SO}(3) \to \underline{\mathfrak{so}}(3), \\ \log(\underline{\boldsymbol{R}}) = \{\underline{\widetilde{\omega}} \in \underline{\mathfrak{so}}(3) \,|\exp(\underline{\widetilde{\omega}}) = \underline{\boldsymbol{R}}\} \end{array} \tag{75}$$

and is the inverse of Equation (71).

Based on Theorem 5 and Theorem 6, for any orthogonal dual tensor $\underline{\boldsymbol{R}}$, a dual vector $\underline{\boldsymbol{\omega}} = \underline{\alpha}\underline{\mathbf{u}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$ can be computed and it represents the Euler dual vector or screw dual vector, which embeds the screw axis and screw parameters. The form of $\underline{\boldsymbol{\omega}}$ implies that $\underline{\widetilde{\omega}} \in \log(\underline{\boldsymbol{R}})$.

If $\|\boldsymbol{\omega}\| < 2\pi$, Theorem 4 and Theorem 5 can be used to uniquely recover the Euler dual vector $\underline{\boldsymbol{\omega}}$, which is equivalent with computing $\log(\underline{\boldsymbol{R}})$.

Next, we will introduce the isomorphism between the Lie group $SE(3)$ and the Lie group $\underline{SO}(3)$.

Theorem 7. *(Isomorphism theorem)* [46]*: The special Euclidean group* $(SE(3),\cdot)$ *and* $(\underline{SO}(3),\cdot)$ *are connected via the isomorphism of the Lie groups*

$$\begin{array}{c} \Phi : SE(3) \to \underline{SO}(3), \\ \Phi(g) = (\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}})\,\boldsymbol{R}, \end{array} \tag{76}$$

where $g = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\boldsymbol{R} \in SO(3)$, $\boldsymbol{\rho} \in \boldsymbol{V}_3$.

The Lie algebra $(\underline{\mathfrak{se}}(3), [\cdot])$ *and* $(\underline{\boldsymbol{V}}_3, \times)$ *are connected via the isomorphism*

$$\begin{array}{c} \varphi : \mathfrak{se}(3) \to \underline{\boldsymbol{V}}_3 \\ \varphi\left(\widehat{\xi}\right) = \boldsymbol{\omega} + \varepsilon\mathbf{v} \end{array}, \tag{77}$$

where $\widehat{\xi} = \begin{bmatrix} \widetilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$, $\widetilde{\omega} \in \mathfrak{so}(3)$, $\mathbf{v} \in \boldsymbol{V}_3$.

Proof of Theorem 7. For any $g_1, g_2 \in SE_3$, the map defined in (76) yields:

$$\Phi(g_1 g_2) = \Phi(g_1)\,\Phi(g_2)\,. \tag{78}$$

Let $\underline{\boldsymbol{R}} \in \underline{SO}(3)$. Based on Theorem 2 , which ensures an unique decomposition, we can conclude that the only choice for $\boldsymbol{g}$, such that $\Phi(g) = \underline{\boldsymbol{R}}$, is $\boldsymbol{g} = \begin{bmatrix} \boldsymbol{R} & \rho \\ \mathbf{0} & 1 \end{bmatrix}$.

This underlines that Φ is a bijection and keeps all the internal operations, where $\boldsymbol{R}$ and $\boldsymbol{\rho}$ are denoted as structural invariant of orthogonal dual tensor $\underline{\boldsymbol{R}}$.

For any $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{se}(3)$, the mapping defined by Equation (77) verifies the identity

$$\varphi\left(\left[\hat{\xi}_1, \hat{\xi}_2\right]\right) = \varphi\left(\hat{\xi}_1\right) \times \varphi\left(\hat{\xi}_2\right). \tag{79}$$

For any $\underline{\boldsymbol{\omega}} \in \underline{\boldsymbol{V}}_3$, $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$, there is only determined $\hat{\xi} = \begin{bmatrix} \tilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$ such that $\varphi\left(\hat{\xi}\right) = \underline{\boldsymbol{\omega}}$.

Thus, φ is a bijective mapping. □

Remark 6. *The inverse of* Φ *is:*

$$\Phi^{-1}: \underline{SO}(3) \leftrightarrow SE(3);\ \Phi^{-1}\left(\underline{\boldsymbol{R}}\right) = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{80}$$

where $\boldsymbol{R} = Re\left(\underline{\boldsymbol{R}}\right), \boldsymbol{\rho} = vect\left(Du\left(\underline{\boldsymbol{R}}\right)\boldsymbol{R}^{\mathrm{T}}\right)$.

Theorem 8. *The recovery of* $\underline{\alpha}, \underline{\mathbf{u}}$ *can also be done if we use the linear and structural invariants. This leads to:*

$$\underline{\mathbf{u}} = \begin{cases} \pm\frac{\mathrm{vect}\underline{\boldsymbol{R}}}{\left|\mathrm{vect}\underline{\boldsymbol{R}}\right|}, \text{ when } Re\left(\mathrm{vect}\underline{\boldsymbol{R}}\right) \neq \mathbf{0} \\ \frac{\boldsymbol{R}\mathbf{v}+\mathbf{v}}{\|\boldsymbol{R}\mathbf{v}+\mathbf{v}\|} + \varepsilon\rho \times \frac{\boldsymbol{R}\mathbf{v}+\mathbf{v}}{\|\boldsymbol{R}\mathbf{v}+\mathbf{v}\|}, \forall\mathbf{v}\in V_3, \text{ when } Re\left(\mathrm{vect}\underline{\boldsymbol{R}}\right) = \mathbf{0} \text{ and } \mathrm{Tr}\boldsymbol{R} = -1 \\ \frac{\rho}{\|\rho\|}, \text{ when } Re\left(\mathrm{vect}\underline{\boldsymbol{R}}\right) = \mathbf{0} \text{ and } \mathrm{Tr}\boldsymbol{R} = 3 \end{cases} \tag{81}$$

$$\underline{\alpha} = atan2\left(\underline{\mathbf{u}} \cdot \mathrm{vect}\underline{\boldsymbol{R}}, \frac{1}{2}\left[\mathrm{Tr}\underline{\boldsymbol{R}} - 1\right]\right). \tag{82}$$

Proof of Theorem 8. The following identities are used:

$$\begin{array}{c} \underline{\mathbf{u}}\sin\underline{\alpha} = \mathrm{vect}\underline{\boldsymbol{R}} \\ \cos\underline{\alpha} = \frac{1}{2}\left[\mathrm{Tr}\underline{\boldsymbol{R}} - 1\right] \end{array}, \tag{83}$$

equations that emerge from Equation (66). For more details see [37]. □

Theorem 9. *The natural invariants* $\underline{\alpha} = \alpha + \varepsilon d$, $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *are very useful to directly recover the structural invariants* $\boldsymbol{R}$ *and* $\boldsymbol{\rho}$ *from Equation (76):*

$$\begin{array}{c} \boldsymbol{R} = \boldsymbol{I} + \sin\alpha\tilde{\mathbf{u}} + (1 - \cos\alpha)\,\tilde{\mathbf{u}}^2 \\ \boldsymbol{\rho} = d\mathbf{u} + \sin\alpha\mathbf{u}_0 + (1 - \cos\alpha)\,\mathbf{u} \times \mathbf{u}_0 \end{array}. \tag{84}$$

Proof of Theorem 9. Using Equations (76) and (66), the Equation (84) is proven. If these equations are equal, then the structure of their dual parts lead to the result presented in Equation (84). □

4.3. Parameterization of Orthogonal Dual Tensor through Dual Quaternions

It is well-known that a dual quaternion is defined as an associated pair of a dual scalar quantity and a free dual vector [51,52]:

$$\underline{\hat{\mathbf{q}}} = \left(\underline{q}, \underline{\mathbf{q}}\right), \underline{q} \in \underline{\mathbb{R}}, \underline{\mathbf{q}} \in \underline{\boldsymbol{V}}_3. \tag{85}$$

A set of dual quaternions will be denoted $\underline{\mathbf{R}}$ and will be a $\underline{\mathbb{R}}$-module of rank 4, if dual quaternion addition and multiplication with dual numbers are considered.

The product of two dual quaternions $\underline{\widehat{\mathbf{q}}}_1 = \left(\underline{q}_1, \underline{\mathbf{q}}_1\right)$ and $\underline{\widehat{\mathbf{q}}}_2 = \left(\underline{q}_2, \underline{\mathbf{q}}_2\right)$ is defined by

$$\underline{\widehat{\mathbf{q}}}_1\underline{\widehat{\mathbf{q}}}_2 = \left(\underline{q}_1\underline{q}_2 - \underline{\mathbf{q}}_1 \cdot \underline{\mathbf{q}}_2, \underline{q}_1\underline{\mathbf{q}}_2 + \underline{q}_2\underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_1 \times \underline{\mathbf{q}}_2\right). \tag{86}$$

Using the above properties, results that the $\underline{\mathbb{R}}$-module $\underline{\mathbf{R}}$ becomes an associative, non-commutative linear dual algebra of order 4 over the ring of dual numbers. For any dual quaternion defined by Equation (85), the following can be computed: the conjugate denoted by $\underline{\widehat{\mathbf{q}}}^* = \left(\underline{q}, -\underline{\mathbf{q}}\right)$ and the norm denoted by $\left|\underline{\widehat{\mathbf{q}}}\right| = \sqrt{\underline{\widehat{\mathbf{q}}}\underline{\widehat{\mathbf{q}}}^*}$. Regarded solely as a free $\underline{\mathbb{R}}$-module, $\underline{\mathbf{R}}$ contains two remarkable sub-modules: $\underline{\mathbf{R}}_{\underline{R}}$ and $\underline{\mathbf{R}}_{\underline{V}_3}$. The first one composed from pairs $\left(\underline{q}, \underline{\mathbf{0}}\right), \underline{q} \in \underline{\mathbb{R}}$, isomorphic with $\underline{\mathbb{R}}$, and the second one, containing the pairs $\left(\underline{0}, \underline{\mathbf{q}}\right), \underline{\mathbf{q}} \in \underline{V}_3$, isomorphic with $\underline{V}_3$. Any dual quaternion can be written as $\underline{\widehat{\mathbf{q}}} = \underline{q} + \underline{\mathbf{q}}$, where $\underline{q} = \left(\underline{q}, \underline{\mathbf{0}}\right)$ and $\underline{\mathbf{q}} = \left(\underline{0}, \underline{\mathbf{q}}\right)$, or $\underline{\widehat{\mathbf{q}}} = \widehat{\mathbf{q}} + \varepsilon\widehat{\mathbf{q}}_0$, where $\widehat{\mathbf{q}}$, $\widehat{\mathbf{q}}_0$ are real quaternions.

Let $\mathbb{U}$ denote the set of real unit quaternions ($|\widehat{\mathbf{q}}| = 1$) and $\underline{\mathbb{U}}$ denote the set of dual unit quaternions $\left(\left|\underline{\widehat{\mathbf{q}}}\right| = 1\right)$. The scalar part and the vector part of a unit dual quaternion are also called dual Euler parameters [53].

Theorem 10. *For any $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}$, the following representation is valid:*

$$\underline{\widehat{\mathbf{q}}} = \left(1 + \varepsilon\frac{1}{2}\widehat{\rho}\right)\widehat{\mathbf{q}}, \tag{87}$$

where $\boldsymbol{\rho} \in \underline{V}_3$ and $\widehat{\mathbf{q}} \in \mathbb{U}$.

This representation is the quaternionic counterpart to Equation (76). Based on Theorem 5, a dual number $\underline{\alpha}$ and a unit dual vector $\underline{\mathbf{u}}$ exist so that

$$\underline{\widehat{\mathbf{q}}} = \cos\frac{\underline{\alpha}}{2} + \underline{\mathbf{u}}\sin\frac{\underline{\alpha}}{2} = \exp(\frac{\underline{\alpha}}{2}\underline{\mathbf{u}}), \forall\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}. \tag{88}$$

If we denote $\underline{\boldsymbol{\omega}} = \underline{\alpha}\underline{\mathbf{u}}$, the Euler dual vector, will be written as:

$$\underline{\widehat{\mathbf{q}}} = cos\frac{\underline{\omega}}{2} + \frac{1}{2}sinc\frac{\underline{\omega}}{2}\underline{\boldsymbol{\omega}} = \exp(\frac{\underline{\boldsymbol{\omega}}}{2}). \tag{89}$$

Remark 7. *The mapping* $\exp : \underline{V}_3 \to \underline{\mathbb{U}}$, $\underline{\widehat{\mathbf{q}}} = \exp(\frac{\underline{\boldsymbol{\omega}}}{2})$ *is well defined and surjective.*

Remark 8. *The set of unit dual quaternions $\underline{\mathbb{U}}$ and the internal operation from Equation (86) is a Lie group. The corresponding Lie algebra is $\underline{V}_3$ with the cross product between dual vectors as its' internal operation. Lie group $\underline{\mathbb{U}}$ can be used to global parameterize all rigid body displacements.*

Using the internal structure of any element from $\underline{SO}(3)$ the following theorem is valid.

Theorem 11. *The Lie groups $\underline{\mathbb{U}}$ and $\underline{SO}(3)$ are linked by a surjective homomorphism*

$$\Delta : \underline{\mathbb{U}} \to \underline{SO}(3),\ \Delta\left(\underline{q} + \underline{\mathbf{q}}\right) = \underline{\boldsymbol{I}} + 2\underline{q}\underline{\widetilde{\mathbf{q}}} + 2\underline{\widetilde{\mathbf{q}}}^2. \tag{90}$$

Proof of Theorem 11. Considering that any $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}$ can be decomposed as in Equation (88), results that $\Delta\left(\underline{\widehat{\mathbf{q}}}\right) = exp(\underline{\alpha}\underline{\widetilde{\mathbf{u}}}) \in \underline{SO}(3)$ and this proves that the Equation (90) is well defined. Using direct calculus, it can be also acknowledged that $\Delta\left(\underline{\widehat{\mathbf{q}}}_2\underline{\widehat{\mathbf{q}}}_1\right) = \Delta\left(\underline{\widehat{\mathbf{q}}}_2\right)\Delta\left(\underline{\widehat{\mathbf{q}}}_1\right)$.

Regarding surjectivity, any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{SO}(3)$ can be represented as in Theorem 5, $\underline{\boldsymbol{R}} = exp(\underline{\alpha}\underline{\widetilde{\mathbf{u}}})$. Thus, we can find a dual quaternion $\underline{\widehat{\mathbf{q}}} = exp(\frac{\underline{\alpha}}{2}\underline{\mathbf{u}})$ in order to have $\Delta\left(\underline{\widehat{\mathbf{q}}}\right) = \underline{\boldsymbol{R}}$, which proves that Δ is a surjective homomorphism. □

A significant property of the above homomorphism is that, for $\underline{\widehat{\mathbf{q}}}$ and $-\underline{\widehat{\mathbf{q}}}$, the same orthogonal dual tensor can be associated and this proves that Equation (90) is not injective and that $\underline{\mathbb{U}}$ is a double cover of $\underline{SO}(3)$.

The inverse of the mapping given by the Equation (90) is a multiple valued function with two branches and presumes the determination of both unit dual quaternions that correspond to an orthogonal dual tensor. A general solution to this problem is given in the Appendix A of this work.

The next theorem presents the closed form BCHD formula in $\underline{SO}(3)$.

Theorem 12. *Let be $\underline{\boldsymbol{\omega}}_\mathbf{1}=\boldsymbol{\omega}_\mathbf{1}+\varepsilon\mathbf{v}_\mathbf{1}$ and $\underline{\boldsymbol{\omega}}_\mathbf{2}=\boldsymbol{\omega}_\mathbf{2}+\varepsilon\mathbf{v}_\mathbf{2}$ the dual vectors that corresponds to $\widehat{\underline{\boldsymbol{\xi}}}_\mathbf{1}$ and, respectively, $\widehat{\underline{\boldsymbol{\xi}}}_\mathbf{2}$ from $\mathfrak{se}(3)$. The below identities take place:*

$$\begin{gathered}\log(\exp(\underline{\widetilde{\boldsymbol{\omega}}}_1)\log(\underline{\widetilde{\boldsymbol{\omega}}}_2))=\underline{\widetilde{\boldsymbol{\omega}}}\\ \underline{\widetilde{\boldsymbol{\omega}}}=\underline{\alpha}_1\underline{\widetilde{\boldsymbol{\omega}}}_1+\underline{\alpha}_2\underline{\widetilde{\boldsymbol{\omega}}}_2+\underline{\alpha}_{12}\left[\underline{\widetilde{\boldsymbol{\omega}}}_1,\underline{\widetilde{\boldsymbol{\omega}}}_2\right]\end{gathered} \tag{91}$$

$$\underline{\boldsymbol{\omega}}=\underline{\alpha}_1\underline{\boldsymbol{\omega}}_1+\underline{\alpha}_2\underline{\boldsymbol{\omega}}_2+\underline{\alpha}_{12}(\underline{\boldsymbol{\omega}}_1\times\underline{\boldsymbol{\omega}}_2), \tag{92}$$

where $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$ and the dual numbers $\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_{12}$ are written in closed form:

$$\underline{\alpha}_1 = \frac{\operatorname{sinc}\frac{\underline{\omega}_1}{2}\cos\frac{\underline{\omega}_2}{2}}{\operatorname{sinc}\frac{\underline{\omega}}{2}} \tag{93}$$

$$\underline{\alpha}_2 = \frac{\cos\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}}{\operatorname{sinc}\frac{\underline{\omega}}{2}} \tag{94}$$

$$\underline{\alpha}_{12} = \frac{\operatorname{sinc}\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}}{2\operatorname{sinc}\frac{\underline{\omega}}{2}} \tag{95}$$

$$\underline{\omega}= 2\arccos(\cos\frac{\underline{\omega}_1}{2}\cos\frac{\underline{\omega}_2}{2} - \frac{1}{4}\operatorname{sinc}\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}\underline{\boldsymbol{\omega}}_1 \cdot \underline{\boldsymbol{\omega}}_2). \tag{96}$$

Proof of Theorem 12. Let be $\underline{\boldsymbol{\omega}}_1, \underline{\boldsymbol{\omega}}_2 \in \underline{\boldsymbol{V}}_3$ so that:

$$\exp(\frac{\underline{\boldsymbol{\omega}}}{2}) = \exp(\frac{\underline{\boldsymbol{\omega}}_1}{2})\exp(\frac{\underline{\boldsymbol{\omega}}_2}{2}). \tag{97}$$

Using Equations (86) and (89), the following equations are obtained:

$$\cos\frac{\underline{\omega}}{2} = \cos\frac{\underline{\omega}_1}{2}\cos\frac{\underline{\omega}_2}{2} - \frac{1}{4}\operatorname{sinc}\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}\underline{\boldsymbol{\omega}}_1 \cdot \underline{\boldsymbol{\omega}}_2 \tag{98}$$

$$\operatorname{sinc}\frac{\underline{\omega}}{2}\underline{\boldsymbol{\omega}} = \operatorname{sinc}\frac{\underline{\omega}_1}{2}\cos\frac{\underline{\omega}_2}{2}\underline{\boldsymbol{\omega}}_1 + \cos\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}\underline{\boldsymbol{\omega}}_2 + \frac{1}{2}\operatorname{sinc}\frac{\underline{\omega}_1}{2}\operatorname{sinc}\frac{\underline{\omega}_2}{2}\underline{\boldsymbol{\omega}}_1 \times \underline{\boldsymbol{\omega}}_2. \tag{99}$$

Considering the Equations (98) and (99), the Equations (91) and (92) are the dual coefficients of Equations (93)–(96). In this proof, the identity $\widetilde{\underline{\boldsymbol{\omega}}_1 \times \underline{\boldsymbol{\omega}}_2} = [\underline{\widetilde{\omega}}_1, \underline{\widetilde{\omega}}_2], \forall\underline{\widetilde{\omega}}_1, \underline{\widetilde{\omega}}_2 \in \mathfrak{se}(3)$ was used. □

Remark 9. *The results of Theorem 1 can be obtained from Equation (91) if $\underline{\boldsymbol{\omega}}_1 = \boldsymbol{\omega}_1 \in \boldsymbol{V}_3$ and $\underline{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_2 \in \boldsymbol{V}_3$. If the real part and the dual part of Equation (91) are separated, the Equations (44) and (45) of the Theorem 2 are obtained.*

The Equation (92) represents the Euler dual vector that corresponds to the rigid body displacement resulted from a successive composition of two rigid body displacements that correspond to the Euler dual vectors $\underline{\boldsymbol{\omega}}_2$ *and, respectively* $\underline{\boldsymbol{\omega}}_1$*. The resulting rigid body displacement is a pure rotation if and only if* $|\underline{\boldsymbol{\omega}}| \in \mathbb{R}$*. The axis of rotation passes through the origin of the reference frame if and only if* $Du(\underline{\boldsymbol{\omega}})=\boldsymbol{0}$*. Rigid body displacement is a pure translation if and only if* $|\underline{\boldsymbol{\omega}}| \in \varepsilon\mathbb{R}$*. In the general case, from the dual screw vector* $\underline{\boldsymbol{\omega}}$*, the screw axis, the rotation angle and the translation vector of the rigid body displacement can be recovered. In fact, Theorem 12 is the fundamental result of this work.*

5. Conclusions

In this paper, we give a new purely algebraic proof of the Baker–Campbell–Hausdorff–Dynkin theorem for Lie algebra of rigid body displacements in three-dimensional Euclidian space. Although Dynkin indicated in 1947 a general procedure to determine the expansion, this formula is of difficult implementation for the case of non-nilpotent operators. The results are especially useful in the context of applications (robotics, computer vision, image analysis and tomography, pose determination and sensor calibration, estimation and control of spacecraft, etc.) where explicit formulae and the possibility of measuring the error are crucial. In order to obtain a closed form coordinate-free formula, the structure of the Lie group of the rigid body displacements and the properties of its Lie algebra are used. A new solution to the problem in dual Lie algebra of dual vectors is given based on the isomorphism between the Lie group of rigid body displacement and the Lie group of the orthogonal dual tensors.

Further, for the first time are presented the applications that result regarding the composition of the rigid body displacements (Euler dual vector determination). All presented theoretical solutions are useful tools for the development of future applications because they are suitable for direct implementation into numerical methods.

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Appendix A. Singularity-Free Extraction of a Unit Dual Quaternion from Orthogonal Dual Tensor

In this section we will propose a computational solution both the singularity-free extraction of a unit dual quaternions from an orthogonal dual tensor. Thus, we will demonstrate that the inverse of the mapping given by the Equation (90) is a multiple valued function with two branches.

Let be $\boldsymbol{R} \in \underline{SO}(3)$, the orthogonal dual tensor that models a rigid body motion and $\pm\hat{\underline{\mathbf{q}}}$ corresponding dual quaternions. We build the symmetric tensor:

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{R}} + \underline{\boldsymbol{R}}^{\mathrm{T}} + (1 - \mathrm{Tr}\underline{\boldsymbol{R}})\,\underline{\boldsymbol{I}} \tag{A1}$$

and the following theorem takes place:

Theorem A1. *For any* $\underline{\boldsymbol{R}} \in \underline{SO}(3)$*, the following identities take place:*

$$\underline{\boldsymbol{R}}\underline{\boldsymbol{S}} = \underline{\boldsymbol{S}}\underline{\boldsymbol{R}} = \underline{\boldsymbol{S}} \tag{A2}$$

$$\underline{\boldsymbol{S}} = 4\underline{\mathbf{q}} \otimes \underline{\mathbf{q}}. \tag{A3}$$

Proof of Theorem A1. From Theorem 3 it follows: $\underline{\boldsymbol{S}} = \underline{\boldsymbol{I}} + \sin\underline{\alpha}\widetilde{\underline{\mathbf{u}}} + (1-\cos\underline{\alpha})\,\widetilde{\underline{\mathbf{u}}}^2 + \underline{\boldsymbol{I}} - \sin\underline{\alpha}\,\widetilde{\underline{\mathbf{u}}} + (1-\cos\underline{\alpha})\,\widetilde{\underline{\mathbf{u}}}^2 = 2\underline{\boldsymbol{I}} + 2\,(1-\cos\underline{\alpha})\,\widetilde{\underline{\mathbf{u}}}^2$. Taking into account Equation (83), it follows:

$$\underline{\boldsymbol{S}} = (3 - \mathrm{Tr}\underline{\boldsymbol{R}})\,\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}. \tag{A4}$$

Because $\underline{\boldsymbol{R}}\underline{\mathbf{u}} = \underline{\mathbf{u}}\underline{\boldsymbol{R}} = \underline{\mathbf{u}}, \forall \underline{\boldsymbol{R}} \in \underline{SO}(3)_3$, from Equation (A1) it follows Equation (88).

Knowing $3 - \mathrm{Tr}\underline{\boldsymbol{R}} = 4\cos^{\mathbf{2}}\frac{\underline{\alpha}}{2}$ and $\underline{\mathbf{q}} = \cos\frac{\underline{\alpha}}{2}\underline{\mathbf{u}}$, and taking into account Equation (A4), it follows Equation (A3). □

Remark A1. *From Equation (A4) it follows* $\underline{\boldsymbol{S}} \neq \underline{\boldsymbol{O}}$ *if and only if* $Re\,[\mathrm{Tr}\underline{\boldsymbol{R}} - 3] \neq 0$, *namely* $\underline{\boldsymbol{S}} = \underline{\boldsymbol{O}} \Longleftrightarrow \underline{\boldsymbol{R}} = \boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}}$, *so* $\underline{\boldsymbol{R}}$ *models a pure translation. Except this case, it follows from Equation (A4), the screw unit dual vector* $\underline{\mathbf{u}}$ *corresponding to the orthogonal dual tensor* $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}\,(\underline{\alpha}, \underline{\mathbf{u}}) \in SO(3)$*:*

$$\underline{\mathbf{u}} = \pm\frac{\underline{\boldsymbol{S}}\underline{\mathbf{v}}}{|\underline{\boldsymbol{S}}\underline{\mathbf{v}}|}, \forall \underline{\mathbf{v}} \in \underline{V}_3,\; Re\,(\underline{\boldsymbol{S}}\underline{\mathbf{v}}) \neq \mathbf{0}. \tag{A5}$$

This results shows a singularity-free method for Theorem 5, dual angles $\underline{\alpha}$ being uniquely determined by:

$$\underline{\alpha} = \mathrm{atan2}\left(\underline{\mathbf{u}} \cdot \mathrm{vect}\underline{\boldsymbol{R}}, \frac{1}{2}\,[\mathrm{Tr}\underline{\boldsymbol{R}} - 1]\right). \tag{A6}$$

In the following, we will note by $\widehat{\underline{\mathbf{q}}}_1 \otimes \widehat{\underline{\mathbf{q}}}_2$ the dyadic product of two dual quaternions, defined by:

$$\left(\widehat{\underline{\mathbf{q}}}_1 \otimes \widehat{\underline{\mathbf{q}}}_2\right)\widehat{\underline{\mathbf{q}}} = \left(\widehat{\underline{\mathbf{q}}}_2 \cdot \widehat{\underline{\mathbf{q}}}\right)\widehat{\underline{\mathbf{q}}}_1, \forall \widehat{\underline{\mathbf{q}}} \in \underline{Q}. \tag{A7}$$

The following theorem takes place:

Theorem A2. *For* $\forall \underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$*, the identity takes place:*

$$\frac{1}{4}\begin{bmatrix} 1 + \mathrm{Tr}\underline{\boldsymbol{R}} & 2(\mathrm{vect}\underline{\boldsymbol{R}})^{\mathrm{T}} \\ 2\mathrm{vect}\underline{\boldsymbol{R}} & \underline{\boldsymbol{S}} \end{bmatrix} = \widehat{\underline{\mathbf{q}}} \otimes \widehat{\underline{\mathbf{q}}}. \tag{A8}$$

Proof of Theorem A2. Let be a unit dual quaternion $\widehat{\underline{\mathrm{q}}} = \underline{\mathrm{q}} + \underline{\mathbf{q}} = \cos\frac{\underline{\alpha}}{2} + \underline{\mathbf{u}}\sin\frac{\underline{\alpha}}{2}$, then $1 + \mathrm{Tr}\underline{\boldsymbol{R}} = 2\,(1 + \cos\underline{\alpha}) = 4\cos^2\frac{\underline{\alpha}}{2} = 4\underline{\mathrm{q}}^2$. From Equations (66) and (90) it follows $\mathrm{vect}\underline{\boldsymbol{R}} = 2\underline{\mathrm{q}}\underline{\mathbf{q}}$. The left member of the Equation (A8) will be denoted by $\widehat{\underline{\mathrm{S}}}$ and taking into account the previous equations and Equation (A3) it follows:

$$\widehat{\underline{\mathrm{S}}} = \frac{1}{4}\begin{bmatrix} 4\underline{\mathrm{q}}^2 & 4\underline{\mathrm{q}}\underline{\mathbf{q}}^{\mathrm{T}} \\ 4\underline{\mathrm{q}}\underline{\mathbf{q}} & 4\underline{\mathbf{q}} \otimes \underline{\mathbf{q}} \end{bmatrix} = \widehat{\underline{\mathbf{q}}} \otimes \widehat{\underline{\mathbf{q}}}. \tag{A9}$$

□

The previous Theorem allows singularity-free extraction of a dual quaternion from a orthogonal dual tensor. Thus, let be $\widehat{\underline{\mathbf{v}}}$ a dual quaternion such that $Re\widehat{\underline{\mathbf{v}}} \neq \widehat{\underline{\mathbf{0}}}$. From Equation (A6) and Equation (A5) it follows:

$$\pm\,\underline{\mathbf{q}} = \frac{\widehat{\underline{\mathrm{S}}}\widehat{\underline{\mathbf{v}}}}{\left|\widehat{\underline{\mathrm{S}}}\widehat{\underline{\mathbf{v}}}\right|}; \forall \underline{\mathbf{v}} \in \underline{Q},\; Re\widehat{\underline{\mathrm{S}}}\widehat{\underline{\mathbf{v}}} \neq \widehat{\underline{\mathbf{0}}}. \tag{A10}$$

Remark A2. *Let be* $\{\underline{\widehat{\mathbf{e}}}_0, \underline{\widehat{\mathbf{e}}}_1, \underline{\widehat{\mathbf{e}}}_2, \underline{\widehat{\mathbf{e}}}_3\}$ *an orthonormal basis in the dual quaternions set:*

$$\begin{array}{l} \underline{\widehat{\mathbf{e}}}_0 = [1,\ 0,\ 0,\ 0]^{\mathrm{T}} \\ \underline{\widehat{\mathbf{e}}}_1 = [0,\ 1,\ 0,\ 0]^{\mathrm{T}} \\ \underline{\widehat{\mathbf{e}}}_2 = [0,\ 0,\ 1,\ 0]^{\mathrm{T}} \\ \underline{\widehat{\mathbf{e}}}_3 = [0,\ 0,\ 0,\ 1]^{\mathrm{T}} \end{array}. \tag{A11}$$

In this case, the dual matrix corresponding with the dual orthogonal tensor $\underline{\boldsymbol{R}}$ *is:*

$$\underline{\boldsymbol{R}} = \left(\underline{\mathrm{r}}_{ij}\right)_{i,j=\overline{1,3}},\ \underline{\mathrm{r}}_{ij} \in \underline{\mathbb{R}}. \tag{A12}$$

The matrix $\underline{\widehat{\mathbf{S}}} = \frac{1}{4}\begin{bmatrix} 1 + \mathrm{Tr}\underline{\boldsymbol{R}} & 2(\mathrm{vect}\underline{\boldsymbol{R}})^{\mathrm{T}} \\ 2\mathrm{vect}\underline{\boldsymbol{R}} & \underline{\boldsymbol{S}} \end{bmatrix} = [\underline{\mathbf{s}}_0, \underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \underline{\mathbf{s}}_3]$, *where the column matrix* $\underline{\mathbf{s}}_k$, $k = \overline{0,3}$, *has the expressions:*

$$\begin{array}{l} \underline{\mathbf{s}}_0 = \frac{1}{4}[1 + \underline{\mathrm{r}}_{11} + \underline{\mathrm{r}}_{22} + \underline{\mathrm{r}}_{33},\ \underline{\mathrm{r}}_{32} - \underline{\mathrm{r}}_{23}, \underline{\mathrm{r}}_{13} - \underline{\mathrm{r}}_{31}, \underline{\mathrm{r}}_{21} - \underline{\mathrm{r}}_{12}]^{\mathrm{T}} \\ \underline{\mathbf{s}}_1 = \frac{1}{4}[\underline{\mathrm{r}}_{32} - \underline{\mathrm{r}}_{23}, 1 + \underline{\mathrm{r}}_{11} - \underline{\mathrm{r}}_{22} - \underline{\mathrm{r}}_{33}, \underline{\mathrm{r}}_{21} + \underline{\mathrm{r}}_{12}, \underline{\mathrm{r}}_{31} + \underline{\mathrm{r}}_{13}]^{\mathrm{T}} \\ \underline{\mathbf{s}}_2 = \frac{1}{4}[\underline{\mathrm{r}}_{13} - \underline{\mathrm{r}}_{31}, \underline{\mathrm{r}}_{21} + \underline{\mathrm{r}}_{12}, 1 - \underline{\mathrm{r}}_{11} + \underline{\mathrm{r}}_{22} - \underline{\mathrm{r}}_{33}, \underline{\mathrm{r}}_{31} + \underline{\mathrm{r}}_{23}]^{\mathrm{T}} \\ \underline{\mathbf{s}}_3 = \frac{1}{4}[\underline{\mathrm{r}}_{21} - \underline{\mathrm{r}}_{12}, \underline{\mathrm{r}}_{31} + \underline{\mathrm{r}}_{13}, \underline{\mathrm{r}}_{32} + \underline{\mathrm{r}}_{23}, 1 - \underline{\mathrm{r}}_{11} - \underline{\mathrm{r}}_{22} + \underline{\mathrm{r}}_{33}]^{\mathrm{T}} \end{array}. \tag{A13}$$

Knowing $\underline{\widehat{\mathbf{S}}}\underline{\widehat{\mathbf{e}}}_k = \underline{\widehat{\mathbf{s}}}_k, k = \overline{0,3}$, and using Equation (A7) it follows:

$$\pm\underline{\widehat{\mathbf{q}}} = \frac{\underline{\widehat{\mathbf{s}}}_k}{\|\underline{\mathbf{s}}_k\|}, k = \overline{0,3}\ if\ Re\|\underline{\widehat{\mathbf{s}}}_k\|^2 \neq \underline{0}. \tag{A14}$$

Noting $\|\underline{\widehat{\mathrm{s}}}\| = \max_{k=\overline{0,3}} \|\underline{\widehat{\mathbf{s}}}_k\|$, it follows:

$$\pm\underline{\widehat{\mathbf{q}}} = \frac{\underline{\widehat{\mathbf{s}}}}{\|\underline{\widehat{\mathrm{s}}}\|}. \tag{A15}$$

The Equation (A15), is the final solution of the singularity-free extraction of unit dual quaternion from orthogonal dual tensor problem.

Remark A3. *The dual matrix corresponding to unit dual quaternion* $\underline{\widehat{\mathbf{q}}}$ *will be noted with* $\underline{\widehat{\mathbf{q}}} = \left[\underline{\mathrm{q}}_0, \underline{\mathrm{q}}_1, \underline{\mathrm{q}}_2, \underline{\mathrm{q}}_3\right]^{\mathrm{T}}; \underline{\mathrm{q}}_k \in \underline{\mathbb{R}}, k = \overline{0,3}$ *and taking into account the Equations (A9) and (A13) it follows:*

$$\underline{\mathrm{q}}_k^2 = \|\underline{\widehat{\mathbf{s}}}_k\|^2, k = \overline{0,3}. \tag{A16}$$

Knowing that $\underline{\widehat{\mathbf{q}}}$ *is a unit dual quaternion, from Equation (A16) results that* $\sum_{k=0}^{3} \|\underline{\widehat{\mathbf{s}}}_k\|^2 = 1$. *Therefore,* $Re(\|\underline{\widehat{\mathrm{s}}}\|) \neq 0$, *so that Equation (A15) can be written for any orthogonal dual tensor. Equation (A15) generalizes to the case of dual quaternions, the result was obtained in the case of the real quaternions* [54]. *In* [54] *the results were obtained using Cayley's factorization in Lie group* $SO(4)$.

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