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The Ideal of σ-Nuclear Operators and Its Associated Tensor Norm

Ju Myung Kim *,† and Keun Young Lee †

Department of Mathematics and Statistics, Sejong University, Seoul 05006, Korea; bst21@sejong.ac.kr
* Correspondence: kjm21@sejong.ac.kr; Tel.: +82-269-352-462
† These authors contributed equally to this work.

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Abstract: We introduce a new tensor norm (σ-tensor norm) and show that it is associated with the ideal of σ-nuclear operators. In this paper, we investigate the ideal of σ-nuclear operators and the σ-tensor norm.

Keywords: tensor norm; Banach operator ideal; nuclear operator

1. Introduction

Let $X \otimes Y$ be the algebraic tensor product of Banach spaces X and Y. One may refer to [1] (Section 1) for tensor products and their elementary properties. If α is a norm on the tensor product, then the normed space $(X \otimes Y, \alpha)$ is denoted by $X \otimes_\alpha Y$ and $X\hat{\otimes}_\alpha Y$ is the completion of $X \otimes_\alpha Y$. The most classical two norms ε and π on $X \otimes Y$ are the injective norm and projective norm, respectively. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup\left\{\left|\sum_{n=1}^{l} x^*(x_n)y^*(y_n)\right| : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\},$$

where $\sum_{n=1}^{l} x_n \otimes y_n$ is any representation of u and B_Z is the closed unit ball of a Banach space Z, and

$$\pi(u; X, Y) := \inf\left\{\sum_{n=1}^{l} \|x_n\|\|y_n\| : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N}\right\}.$$

We refer to [1,2] for ε and π. Our main notion is the following concept.

Definition 1. *For $\sum_{n=1}^{l} x_n \otimes y_n \in X \otimes Y$, let*

$$\left|\sum_{n=1}^{l} x_n \otimes y_n\right|_\sigma := \sup\left\{\sum_{n=1}^{l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}.$$

For $u \in X \otimes Y$, let

$$\alpha_\sigma(u; X, Y) := \inf\left\{\left|\sum_{n=1}^{l} x_n \otimes y_n\right|_\sigma : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N}\right\}.$$

We call α_σ the σ-tensor norm.

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ is said to be associated with a tensor norm α if the natural map from $\mathcal{A}(M, N)$ to $M^* \otimes_\alpha N$ is an isometry for both finite-dimensional normed spaces M and N. Let $\|\cdot\|$ be the operator norm on the ideal $\mathcal{L}$ of all operators and let $\mathcal{F}$ be the ideal of all finite rank operators. A linear map $T : X \to Y$ is called approximable if there exists a sequence $(T_n)_n$ in $\mathcal{F}(X, Y)$ such that $\lim_{n\to\infty} \|T_n - T\| = 0$. We denote by $\overline{\mathcal{F}}(X, Y)$ the space of all approximable operators from X to Y. Then the ideal $[\overline{\mathcal{F}}, \|\cdot\|]$ of approximable operators is a Banach operator ideal.

A linear map $T : X \to Y$ is nuclear if there exists sequences $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y with

$$\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$$

such that

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n,$$

where $x_n^* \underline{\otimes} y_n$ is an operator from X to Y defined by $(x_n^* \underline{\otimes} y_n)(x) = x_n^*(x) y_n$. The space of all nuclear operators from X to Y is denoted by $\mathcal{N}(X, Y)$ with the norm

$$\|T\|_{\mathcal{N}} := \inf\left\{ \sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| : T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \right\},$$

where the infimum is taken over all such representations. It is well known that $[\overline{\mathcal{F}}, \|\cdot\|]$ is associated with ε and $[\mathcal{N}, \|\cdot\|_{\mathcal{N}}]$ is associated with π (cf. [2] (Section 17.12)).

Pietsch [3] introduced a natural extended notion of the nuclear operator. A linear map $T : X \to Y$ is called σ-nuclear if there exists sequences $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y such that

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$$

unconditionally converges in the operator norm. We denote by $\mathcal{N}_\sigma(X, Y)$ the space of all σ-nuclear operators from X to Y and for $T \in \mathcal{N}_\sigma(X, Y)$, let

$$\|T\|_{\mathcal{N}_\sigma} := \inf\left\{ \Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma : T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \right\},$$

where $|\sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n|_\sigma := \sup\{\sum_{n=1}^{\infty} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\}$ and the infimum is taken over all σ-nuclear representations. Then $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is a Banach operator ideal [3] (Theorem 23.2.2).

In this paper, we study the Banach operator ideal $\mathcal{N}_\sigma$ of σ-nuclear operators and the corresponding σ-tensor norm α_σ. In Section 2, we obtain a factorization of operators belonging to $\mathcal{N}_\sigma$ and show that the surjective hull and the injective hull of $\mathcal{N}_\sigma$ coincide with the ideal of compact operators. It turns out that $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is associated with α_σ. In Section 3, we show that α_σ is a finitely generated tensor norm and the completion $X \hat{\otimes}_{\alpha_\sigma} Y$ is identified. An isometric representation of the dual space $(X \otimes_{\alpha_\sigma} Y)^*$ is established. In Section 4, we show that

$$X \otimes_\varepsilon Y = X \otimes_{\alpha_\sigma} Y$$

holds isometrically when X or Y has a hyperorthogonal basis. As a consequence, we show that α_σ is neither injective nor projective.

2. The Ideal of σ-Nuclear Operators

For Banach spaces X and Y, we denote by

$$\ell^\sigma(X^*, Y)$$

the collection of sequences $(x_n^*, y_n)_n$ in $X^* \times Y$ satisfying

$$\lim_{l\to\infty} \sup\left\{ \sum_{n\geq l} |x_n^*(x)y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \right\} = 0$$

and let

$$|(x_n^*, y_n)_n|_{\ell^\sigma} = \sup\left\{ \sum_{n=1}^{\infty} |x_n^*(x)y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \right\}$$

for $(x_n^*, y_n)_n \in \ell^\sigma(X^*, Y)$.

A basis $(e_n)_n$ for a Banach space X is called hyperorthogonal if for every $n \in \mathbb{N}$ $|\alpha_n| \leq |\beta_n|$ implies

$$\left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\| \leq \left\| \sum_{n=1}^{\infty} \beta_n e_n \right\|.$$

Using a standard argument, we have the following lemma.

Lemma 1. *Let K be a collection of sequences of positive numbers.*

If $\sup_{(k_n)_n\in K} \sum_{n=1}^{\infty} k_n < \infty$ and $\lim_{l\to\infty} \sup_{(k_n)_n\in K} \sum_{n\geq l} k_n = 0$, then for every $\varepsilon > 0$, there exists an increasing sequence $(\beta_n)_n$ with $\beta_n > 1$ and $\lim_{n\to\infty} \beta_n = \infty$ such that

$$\lim_{l\to\infty} \sup_{(k_n)_n\in K} \sum_{n\geq l} k_n\beta_n = 0 \ \text{ and } \ \sup_{(k_n)_n\in K} \sum_{n=1}^{\infty} k_n\beta_n \leq (1+\varepsilon) \sup_{(k_n)_n\in K} \sum_{n=1}^{\infty} k_n.$$

It is well known that a nuclear operator $T : X \to Y$ has the following factorization.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ c_0 & \xrightarrow[D]{} & \ell_1, \end{array}$$

where R and S are compact operators, and D is a diagonal operator which is nuclear. From a modification of [3] (Theorem 23.2.5), we have a similar form for σ-nuclear operators.

Theorem 1. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear map. Then the following statements are equivalent.*

(a) $T \in \mathcal{N}_\sigma(X, Y)$.

(b) *There exists $(x_n^*, y_n)_n \in \ell^\sigma(X^*, Y)$ such that*

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n.$$

(c) *There exist Banach spaces Z and W having hyperorthogonal bases, $R \in \mathcal{N}_\sigma(X, Z)$, a diagonal operator $D \in \mathcal{N}_\sigma(Z, W)$ with $\|D\|_{\mathcal{N}_\sigma} \leq 1$, and $S \in \mathcal{N}_\sigma(W, Y)$ with $\|S\|_{\mathcal{N}_\sigma} \leq 1$ such that the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ Z & \xrightarrow[D]{} & W. \end{array}$$

In this case,

$$\|T\|_{\mathcal{N}_\sigma} = \inf |(x_n^*, y_n)_n|_{\ell^\sigma} = \inf \|R\|_{\mathcal{N}_\sigma},$$

where the first infimum is taken over all such representations of T in (b) and the second infimum is taken over all such factorizations of T in (c).

Proof. (c)⇒(a) is trivial and $\|T\|_{\mathcal{N}_\sigma}$ is less than or equal to the infimum for factorizations of T in (c).

(a)⇒(b): This part is a well known result. For the sake of the completeness of presentation, we provide an explicit proof. Let $T \in \mathcal{N}_\sigma(X,Y)$ and let $\varepsilon > 0$ be given. Then there exists a representation

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n,$$

which unconditionally converges in $(\mathcal{L}(X,Y), \|\cdot\|)$, such that

$$\Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma \leq (1+\varepsilon)\|T\|_{\mathcal{N}_\sigma}.$$

It is well known that a series $\sum_{n=1}^{\infty} z_n$ in a Banach space Z unconditionally converges if and only if

$$\lim_{l\to\infty} \sup_{z^*\in B_{Z^*}} \sum_{n\geq l} |z^*(z_n)| = 0.$$

Thus,

$$\begin{aligned}&\lim_{l\to\infty} \sup \Big\{ \sum_{n\geq l} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \Big\} \\ &\leq \lim_{l\to\infty} \sup \Big\{ \sum_{n\geq l} |\varphi(x_n^* \underline{\otimes} y_n)| : \varphi \in B_{(\mathcal{L}(X,Y),\|\cdot\|)^*} \Big\} = 0.\end{aligned}$$

Hence (b) follows and the first infimum

$$\inf |\cdot|_{\ell^\sigma} \leq |(x_n^*, y_n)_n|_{\ell^\sigma} = \Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma \leq (1+\varepsilon)\|T\|_{\mathcal{N}_\sigma}.$$

Since $\varepsilon > 0$ was arbitrary, $\inf |\cdot|_{\ell^\sigma} \leq \|T\|_{\mathcal{N}_\sigma}$.

(b)⇒(c): Let $\varepsilon > 0$ be given. By (b), there exists $(x_n^*, y_n)_n \in \ell^\sigma(X^*, Y)$ such that

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$$

and

$$|(x_n^*, y_n)_n|_{\ell^\sigma} \leq (1+\varepsilon) \inf |\cdot|_{\ell^\sigma}.$$

By Lemma 1, there exists a sequence $(\beta_n)_n$ with $\beta_n > 1$ and $\lim_{n\to\infty} \beta_n = \infty$ such that

$$(\beta_n^2 x_n^*, y_n)_n \in \ell^\sigma(X^*, Y)$$

and

$$|(\beta_n^2 x_n^*, y_n)_n|_{\ell^\sigma} \leq (1+\varepsilon)|(x_n^*, y_n)_n|_{\ell^\sigma}.$$

Let

$$Z := \Big\{ (\alpha_n)_n \text{ in } \mathbb{C} : \sum_{n=1}^{\infty} \alpha_n \beta_n^2 y_n \text{ unconditionally converges in } Y \Big\}$$

and

$$\|(\alpha_n)_n\|_Z := \sup_{y^*\in B_{Y^*}} \sum_{n=1}^{\infty} \beta_n^2 |\alpha_n y^*(y_n)|.$$

Then $(Z, \|\cdot\|_Z)$ is a Banach space and the sequence $(e_n)_n$ of standard unit vectors forms a hyperorthogonal basis in Z. Let

$$W := \left\{(\gamma_n)_n \text{ in } \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} \gamma_n y_n \text{ unconditionally converges in } Y\right\}$$

and

$$\|(\gamma_n)_n\|_W := \sup_{y^* \in B_{Y^*}} \sum_{n=1}^{\infty} |\gamma_n y^*(y_n)|.$$

Then $(W, \|\cdot\|_W)$ is a Banach space and the sequence $(f_n)_n$ of standard unit vectors forms a hyperorthogonal basis in W.

Let

$$R : X \to Z, Rx = (x_n^*(x))_n,$$
$$D : Z \to W, D(\alpha_n)_n = (\beta_n \alpha_n)_n,$$
$$S : W \to Y, S(\gamma_n)_n = \sum_{n=1}^{\infty} \frac{\gamma_n}{\beta_n} y_n.$$

To show that $R = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} e_n$ unconditionally converges in $\mathcal{L}(X, Z)$, let $\delta > 0$. Choose an $l_\delta \in \mathbb{N}$ such that

$$\sup\left\{\sum_{n \geq l_\delta} \beta_n^2 |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\right\} \leq \delta.$$

Then for every finite subset F of $\mathbb{N}$ with $\min F > l_\delta$,

$$\begin{aligned}\Big\|\sum_{n \in F} x_n^* \underline{\otimes} e_n\Big\| &= \sup_{x \in B_X} \Big\|\sum_{n \in F} x_n^*(x) e_n\Big\|_Z \\ &= \sup\left\{\sum_{n \in F} \beta_n^2 |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\right\} \leq \delta.\end{aligned}$$

Hence $R \in \mathcal{N}_\sigma(X, Z)$. Since for every $x \in B_X$ and $z^* \in B_{Z^*}$,

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n^*(x) z^*(e_n)| &= \sum_{n=1}^{\infty} \lambda_n x_n^*(x) z^*(e_n) \ (|\lambda_n| = 1) \\ &\leq \Big\|\sum_{n=1}^{\infty} \lambda_n x_n^*(x) e_n\Big\|_Z \\ &= \sup_{y^* \in B_{Y^*}} \sum_{n=1}^{\infty} \beta_n^2 |x_n^*(x) y^*(y_n)| \leq |(\beta_n^2 x_n^*, y_n)_n|_{\ell^\sigma},\end{aligned}$$

$\|R\|_{\mathcal{N}_\sigma} \leq |(\beta_n^2 x_n^*, y_n)_n|_{\ell^\sigma} \leq (1+\varepsilon)|(x_n^*, y_n)_n|_{\ell^\sigma} \leq (1+\varepsilon)^2 \inf|\cdot|_{\ell^\sigma}$.

To show that $D = \sum_{n=1}^{\infty} \beta_n e_n^* \underline{\otimes} f_n$ unconditionally converges in $\mathcal{L}(Z, W)$, where each $e_n^* \in Z^*$ is the n-th coordinate functional, let $\delta > 0$. Choose an $N_\delta \in \mathbb{N}$ such that $1/\beta_n \leq \delta$ for every $n \geq N_\delta$. Then for every finite subset F of $\mathbb{N}$ with $\min F > N_\delta$,

$$\begin{aligned}\Big\|\sum_{n \in F} \beta_n e_n^* \underline{\otimes} f_n\Big\| &= \sup_{(\alpha_n)_n \in B_Z} \Big\|\sum_{n \in F} \beta_n \alpha_n f_n\Big\|_W \\ &= \sup\left\{\sum_{n \in F} \beta_n |\alpha_n y^*(y_n)| : (\alpha_n)_n \in B_Z, y^* \in B_{Y^*}\right\} \\ &\leq \delta \sup\left\{\sum_{n \in F} \beta_n^2 |\alpha_n y^*(y_n)| : (\alpha_n)_n \in B_Z, y^* \in B_{Y^*}\right\} \\ &\leq \delta \sup_{(\alpha_n)_n \in B_Z} \|(\alpha_n)_n\|_Z \leq \delta.\end{aligned}$$

Hence $D \in \mathcal{N}_\sigma(Z, W)$ and $\|D\|_{\mathcal{N}_\sigma} \leq 1$, indeed, for every $(\alpha_k)_k \in B_Z$ and $w^* \in B_{W^*}$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |\beta_n e_n^*((\alpha_k)_k) w^*(f_n)| &= \sum_{n=1}^{\infty} |\beta_n \alpha_n w^*(f_n)| \\
&= \sum_{n=1}^{\infty} \delta_n \beta_n \alpha_n w^*(f_n) \ (|\delta_n| = 1) \\
&\leq \Big\| \sum_{n=1}^{\infty} \delta_n \beta_n \alpha_n f_n \Big\|_W \\
&= \sup_{y^* \in B_{Y^*}} \sum_{n=1}^{\infty} \beta_n |\alpha_n y^*(y_n)| \\
&\leq \sup_{y^* \in B_{Y^*}} \sum_{n=1}^{\infty} \beta_n^2 |\alpha_n y^*(y_n)| = \|(\alpha_k)_k\|_Z \leq 1.
\end{aligned}$$

Let $f_n^* \in W^*$ be the n-th coordinate functional. In order to show that $S = \sum_{n=1}^{\infty}(1/\beta_n) f_n^* \underline{\otimes} y_n$ converges unconditionally in $\mathcal{L}(W, Y)$, we take $\delta > 0$. Choose an $N_\delta \in \mathbb{N}$ such that $1/\beta_n \leq \delta$ for every $n \geq N_\delta$. Then for every finite subset F of $\mathbb{N}$ with $\min F > N_\delta$,

$$\begin{aligned}
\Big\| \sum_{n \in F} (1/\beta_n) f_n^* \underline{\otimes} y_n \Big\| &= \sup_{(\gamma_n)_n \in B_W} \Big\| \sum_{n \in F} \frac{\gamma_n}{\beta_n} y_n \Big\|_Y \\
&= \sup \Big\{ \Big| \sum_{n \in F} \frac{\gamma_n}{\beta_n} y^*(y_n) \Big| : (\gamma_n)_n \in B_W, y^* \in B_{Y^*} \Big\} \\
&\leq \delta \sup \Big\{ \Big| \sum_{n \in F} \gamma_n y^*(y_n) \Big| : (\gamma_n)_n \in B_W, y^* \in B_{Y^*} \Big\} \\
&\leq \delta \sup_{(\gamma_n)_n \in B_W} \|(\gamma_n)_n\|_W \leq \delta.
\end{aligned}$$

Hence $S \in \mathcal{N}_\sigma(W, Y)$ and $\|S\|_{\mathcal{N}_\sigma} \leq 1$, indeed, for every $(\gamma_k)_k \in B_W$ and $y^* \in B_{Y^*}$,

$$\sum_{n=1}^{\infty} |(1/\beta_n) f_n^*((\gamma_k)_k) y^*(y_n)| \leq \sum_{n=1}^{\infty} |\gamma_n y^*(y_n)| \leq \|(\gamma_n)_n\|_W \leq 1.$$

Clearly, $T = SDR$ and the second infimum $\inf \|\cdot\|_{\mathcal{N}_\sigma} \leq \|R\|_{\mathcal{N}_\sigma} \leq (1+\varepsilon)^2 \inf |\cdot|_{\ell^\sigma}$. Since $\varepsilon > 0$ was arbitrary, $\inf \|\cdot\|_{\mathcal{N}_\sigma} \leq \inf |\cdot|_{\ell^\sigma}$. $\square$

The surjective hull $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur}$ of an operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{sur}(X, Y) := \{T \in \mathcal{L}(X, Y) : Tq_X \in \mathcal{A}(\ell_1(B_X), Y)\},$$

where $q_X : \ell_1(B_X) \to X$ is the natural quotient operator, and $\|T\|_{\mathcal{A}^{sur}} := \|Tq_X\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{sur}(X, Y)$ (see [2] (p. 113) and [3] (Section 8.5)).

Lemma 2. *(see Proposition 8.5.4 in [3]) Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let X and Y be Banach spaces. A linear map $T \in \mathcal{A}^{sur}(X, Y)$ if and only if there exists a Banach space Z and an $S \in \mathcal{A}(Z, Y)$ such that $T(B_X) \subset S(B_Z)$. In this case,*

$$\|T\|_{\mathcal{A}^{sur}} = \inf \|S\|_{\mathcal{A}},$$

where the infimum is taken over all the above inclusions.

Lemma 3. *[4] A subset K of a Banach space X is relatively compact if and only if for every $\varepsilon > 0$, there exists a null sequence $(x_n)_n$ in X with $\sup_{n \in \mathbb{N}} \|x_n\| \leq (1+\varepsilon) \sup_{x \in K} \|x\|$ such that*

$$K \subset \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \Big\}.$$

The surjective hull of the ideal of nuclear operators is identified in [3] (Proposition 8.5.5).

Theorem 2. *The surjective hull $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{sur}$ of the ideal of σ-nuclear operators can be identified with the ideal $[\mathcal{K}, \|\cdot\|]$ of compact operators.*

Proof. Since $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}] \subset [\overline{\mathcal{F}}, \|\cdot\|]$ and $[\overline{\mathcal{F}}, \|\cdot\|]^{sur} = [\mathcal{K}, \|\cdot\|]$,

$$[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{sur} \subset [\mathcal{K}, \|\cdot\|].$$

To show the opposite inclusion, let X and Y be Banach spaces. Let $T \in \mathcal{K}(Y, X)$ and let $\varepsilon > 0$. Then by Lemma 3, there exists a null sequence $(x_n)_n$ in X with $\sup_{n\in\mathbb{N}} \|x_n\| \leq (1+\varepsilon)\|T\|$ such that

$$T(B_Y) \subset \Big\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \Big\}.$$

Let us consider the map

$$E : \ell_1 \to X, E = \sum_{n=1}^{\infty} e_n \underline{\otimes} x_n,$$

where each e_n is the standard unit vector in c_0. Since

$$\lim_{l\to\infty} \sup \Big\{ \sum_{n\geq l} |\alpha_n x^*(x_n)| : (\alpha_n)_n \in B_{\ell_1}, x^* \in B_{X^*} \Big\} \leq \lim_{l\to\infty} \sup_{(\alpha_n)_n \in B_{\ell_1}} \sum_{n\geq l} |\alpha_n| \|x_n\| = 0,$$

in view of Theorem 1, $E \in \mathcal{N}_\sigma(\ell_1, X)$ and

$$\begin{aligned} \|E\|_{\mathcal{N}_\sigma} &\leq |(e_n, x_n)_n|_{\ell^\sigma} \\ &= \sup \Big\{ \sum_{n=1}^{\infty} |\alpha_n x^*(x_n)| : (\alpha_n)_n \in B_{\ell_1}, x^* \in B_{X^*} \Big\} \\ &\leq \sup_{n\in\mathbb{N}} \|x_n\| \leq (1+\varepsilon)\|T\|. \end{aligned}$$

Since $T(B_Y) \subset E(B_{\ell_1})$, by Lemma 2, $T \in \mathcal{N}_\sigma^{sur}(Y, X)$ and

$$\|T\|_{\mathcal{N}_\sigma^{sur}} \leq \|E\|_{\mathcal{N}_\sigma} \leq (1+\varepsilon)\|T\|.$$

□

The injective hull $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj}$ of an operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{inj}(X, Y) := \{T \in \mathcal{L}(X, Y) : I_Y T \in \mathcal{A}(X, \ell_\infty(B_{Y^*}))\},$$

where $I_Y : Y \to \ell_\infty(B_{Y^*})$ is the natural isometry, and $\|T\|_{\mathcal{A}^{inj}} := \|I_Y T\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{inj}(X, Y)$ (see [2] (p. 112) and [3] (Section 8.4)).

Lemma 4. *If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is a symmetric Banach operator ideal, then*

$$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj} = ([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur})^{dual}.$$

Proof. The symmetric operator ideal means that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] \subset [\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{dual}$. Then by [3] (Theorem 8.5.9),

$$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj} \subset ([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{dual})^{inj} \subset ([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur})^{dual}.$$

Additionally, since $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur} \subset ([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{dual})^{sur}$, by [3] (Theorem 8.5.9),

$$([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur})^{dual} \subset (([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{dual})^{sur})^{dual} = (([\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj})^{dual})^{dual}.$$

Note that $(([\mathcal{B}, \|\cdot\|_{\mathcal{B}}]^{inj})^{dual})^{dual} \subset [\mathcal{B}, \|\cdot\|_{\mathcal{B}}]^{inj}$ for every Banach operator ideal $\mathcal{B}$. Hence the assertion follows. □

The injective hull of the ideal of nuclear operators is identified in [3] (Proposition 8.4.5). The following theorem is a consequence of the fact that the ideal $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is symmetric (cf. [3] (Theorem 23.2.7)).

Theorem 3. *For the ideal of σ-nuclear operators, the following equality is valid:*

$$[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{inj} = [\mathcal{K}, \|\cdot\|].$$

Proof. Since $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is symmetric, by Theorem 2 and Lemma 4,

$$[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{inj} = ([\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{sur})^{dual} = [\mathcal{K}, \|\cdot\|]^{dual} = [\mathcal{K}, \|\cdot\|].$$

□

For $T \in \mathcal{F}(X, Y)$, let

$$\|T\|_{\mathcal{N}_\sigma^0} := \inf\left\{ \Big| \sum_{n=1}^{l} x_n^* \underline{\otimes} y_n \Big|_\sigma : T = \sum_{n=1}^{l} x_n^* \underline{\otimes} y_n, l \in \mathbb{N} \right\}.$$

Then $\|\cdot\|_{\mathcal{N}_\sigma^0}$ is a norm on $\mathcal{F}$ [3] (Proposition 23.2.10).

Proposition 1. *Suppose that X or Y is a finite-dimensional normed space. Then*

$$\|T\|_{\mathcal{N}_\sigma^0} = \|T\|_{\mathcal{N}_\sigma}$$

for every $T \in \mathcal{L}(X, Y)$.

Proof. Let $T \in \mathcal{L}(X, Y)$ and let $\delta > 0$ be given. Let

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n,$$

be a σ-nuclear representation in Theorem 1(b) such that

$$|(x_n^*, y_n)_n|_{\ell^\sigma} \leq (1+\delta)\|T\|_{\mathcal{N}_\sigma}.$$

If X is finite-dimensional, then there exists an $l \in \mathbb{N}$ such that

$$\sup\left\{ \sum_{n \geq l+1} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \right\} \leq \delta \|T\|_{\mathcal{N}_\sigma} / \|id_X\|_{\mathcal{N}_\sigma^0},$$

where id_X is the identity operator on X. We have

$$\begin{aligned}
\|T\|_{\mathcal{N}_\sigma^0} &\leq \Big\| \sum_{n=1}^{l} x_n^* \underline{\otimes} y_n \Big\|_{\mathcal{N}_\sigma^0} + \Big\| \sum_{n \geq l+1} x_n^* \underline{\otimes} y_n \Big\|_{\mathcal{N}_\sigma^0} \\
&\leq |(x_n^*, y_n)_n|_{\ell^\sigma} + \Big\| \sum_{n \geq l+1} x_n^* \underline{\otimes} y_n \Big\|_{\mathcal{N}_\sigma} \|id_X\|_{\mathcal{N}_\sigma^0} \\
&\leq (1+2\delta)\|T\|_{\mathcal{N}_\sigma}.
\end{aligned}$$

If Y is finite-dimensional, then id_X can be replaced by id_Y in the above proof. □

Corollary 1. *$[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is associated with α_σ.*

Proof. Let X and Y be Banach spaces. Let $\sum_{n=1}^{l} x_n^* \otimes y_n \in X^* \otimes Y$. The by an application of Helly's lemma,

$$\Big|\sum_{n=1}^{l} x_n^* \otimes y_n\Big|_\sigma = \sup\Big\{\sum_{n=1}^{l} |x_n^*(x)y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\Big\} = \Big|\sum_{n=1}^{l} x_n^* \underline{\otimes} y_n\Big|_\sigma.$$

Consequently, for every $u \in X^* \otimes Y$, we have

$$\alpha_\sigma(u; X^*, Y) = \inf\Big\{\Big|\sum_{n=1}^{l} x_n^* \underline{\otimes} y_n\Big|_\sigma : u = \sum_{n=1}^{l} x_n^* \otimes y_n, l \in \mathbb{N}\Big\}.$$

Hence the assertion follows from Proposition 1. □

3. The σ-Tensor Norm

Let us recall that a tensor norm α is a norm on $X \otimes Y$ for each pair of Banach spaces X and Y such that

(TN1) $\varepsilon \le \alpha \le \pi$.
(TN2) for operators $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$,

$$\|T_1 \otimes T_2 : X_1 \otimes_\alpha X_2 \to Y_1 \otimes_\alpha Y_2\| \le \|T_1\|\|T_2\|.$$

A tensor norm α is said to be finitely generated if

$$\alpha(u; X, Y) = \inf\{\alpha(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\}$$

for every $u \in X \otimes Y$. The transposed tensor norm α^t of α is defined by

$$\alpha^t(u; X, Y) := \alpha(u^t; Y, X)$$

for $u \in X \otimes Y$.

Proposition 2. *α_σ is a finitely generated tensor norm and $\alpha^t = \alpha$.*

Proof. We see that α_σ is a norm and satisfies (TN1) on $X \otimes Y$ for each pair of Banach spaces X and Y.

To check (TN2), let $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ be operators. Let $u \in X_1 \otimes X_2$ and let $u = \sum_{n=1}^{l} x_n^1 \otimes x_n^2$ be an arbitrary representation. Then

$$\begin{aligned}
\alpha_\sigma((T_1 \otimes T_2)(u); Y_1, Y_2) &= \alpha_\sigma\Big(\sum_{n=1}^{l} T_1 x_n^1 \otimes T_2 x_n^2; Y_1, Y_2\Big) \\
&\le \Big|\sum_{n=1}^{l} T_1 x_n^1 \otimes T_2 x_n^2\Big|_\sigma \\
&= \|T_1\|\|T_2\|\Big|\sum_{n=1}^{l} (T_1/\|T_1\|)(x_n^1) \otimes (T_2/\|T_2\|)(x_n^2)\Big|_\sigma \\
&\le \|T_1\|\|T_2\|\Big|\sum_{n=1}^{l} x_n^1 \otimes x_n^2\Big|_\sigma.
\end{aligned}$$

Hence

$$\alpha_\sigma((T_1 \otimes T_2)(u); Y_1, Y_2) \leq \|T_1\|\|T_2\|\alpha_\sigma(u; X_1, X_2).$$

To show that α_σ is finitely generated, let $u \in X \otimes Y$ and let $u = \sum_{n=1}^{l} x_n \otimes y_n$ be an arbitrary representation. Let $M_0 = \text{span}\{x_n\}_{n=1}^{l}$ and $N_0 = \text{span}\{y_n\}_{n=1}^{l}$. Using the Hahn–Banach extension theorem, we have

$$\begin{aligned}
&\inf\{\alpha_\sigma(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\} \\
&\leq \alpha_\sigma(u; M_0, N_0) \\
&\leq \sup\Big\{\sum_{n=1}^{l} |m^*(x_n)n^*(y_n)| : m^* \in B_{M_0^*}, n^* \in B_{N_0^*}\Big\} \\
&= \sup\Big\{\sum_{n=1}^{l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\}.
\end{aligned}$$

Hence

$$\inf\{\alpha_\sigma(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\} \leq \alpha_\sigma(u; X, Y).$$

The other part of the assertion follows from the definition of the σ-tensor norm. □

We now consider the completion $X\hat{\otimes}_{\alpha_\sigma}Y$ of $X \otimes_{\alpha_\sigma} Y$. When $\sum_{n=1}^{\infty} x_n \otimes y_n$ converges in $X\hat{\otimes}_{\alpha_\sigma}Y$ and $\sup\{\sum_{n=1}^{\infty} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} < \infty$, we let

$$\Big|\sum_{n=1}^{\infty} x_n \otimes y_n\Big|_\sigma := \sup\Big\{\sum_{n=1}^{\infty} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\}.$$

Lemma 5. *Let X and Y be Banach spaces and let $(x_n)_n$ and $(y_n)_n$ be sequences in X and Y, respectively. Then*

$$\lim_{l\to\infty} \sup\Big\{\sum_{n\geq l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} = 0$$

if and only if the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ unconditionally converges in $X\hat{\otimes}_{\alpha_\sigma}Y$.

Proof. Suppose that $\lim_{l\to\infty} \sup\{\sum_{n\geq l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} = 0$. Let $\delta > 0$ be given. Choose an $l_\delta \in \mathbb{N}$ such that

$$\sup\Big\{\sum_{n\geq l_\delta} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \leq \delta.$$

Then for every finite subset F of $\mathbb{N}$ with $\min F > l_\delta$,

$$\begin{aligned}
\alpha_\sigma\Big(\sum_{n\in F} x_n \otimes y_n; X, Y\Big) &\leq \Big|\sum_{n\in F} x_n \otimes y_n\Big|_\sigma \\
&\leq \sup\Big\{\sum_{n\geq l_\delta} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \leq \delta.
\end{aligned}$$

Suppose that $\sum_{n=1}^{\infty} x_n \otimes y_n$ unconditionally converges in $X\hat{\otimes}_{\alpha_\sigma}Y$. Then

$$\begin{aligned}
&\lim_{l\to\infty} \sup\Big\{\sum_{n\geq l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \\
&\leq \lim_{l\to\infty} \sup\Big\{\sum_{n\geq l} |\varphi(x_n \otimes y_n)| : \varphi \in B_{(X\hat{\otimes}_{\alpha_\sigma}Y)^*}\Big\} = 0.
\end{aligned}$$

□

The following lemma is well known.

Lemma 6. *Let $(Z, \|\cdot\|)$ be a normed space and let $(\hat{Z}, \|\cdot\|)$ be its completion. If $z \in \hat{Z}$, then for every $\delta > 0$, there exists a sequence $(z_n)_n$ in Z such that*

$$\sum_{n=1}^{\infty} \|z_n\| \leq (1+\delta)\|z\|$$

and $z = \sum_{n=1}^{\infty} z_n$ converges in $\hat{Z}$.

Proposition 3. *Let X and Y be Banach spaces. If $u \in X\hat{\otimes}_{\alpha_\sigma} Y$, then there exists sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that*

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converges in $X\hat{\otimes}_{\alpha_\sigma} Y$ and

$$\alpha_\sigma(u; X, Y) = \inf\Big\{\Big|\sum_{n=1}^{\infty} x_n \otimes y_n\Big|_\sigma : u = \sum_{n=1}^{\infty} x_n \otimes y_n \text{ unconditionally converges in } X\hat{\otimes}_{\alpha_\sigma} Y\Big\}.$$

Proof. We use Lemma 5. Let $u \in X\hat{\otimes}_{\alpha_\sigma} Y$ and let $\delta > 0$ be given. Then by Lemma 6, there exists a sequence $(u_n)_n$ in $X \otimes Y$ such that

$$\sum_{n=1}^{\infty} \alpha_\sigma(u_n; X, Y) \leq (1+\delta)\alpha_\sigma(u; X, Y)$$

and $u = \sum_{n=1}^{\infty} u_n$ converges in $X\hat{\otimes}_{\alpha_\sigma} Y$.

For every $n \in \mathbb{N}$, let

$$u_n = \sum_{k=1}^{m_n} x_k^n \otimes y_k^n$$

be such that

$$\Big|\sum_{k=1}^{m_n} x_k^n \otimes y_k^n\Big|_\sigma \leq (1+\delta)\alpha_\sigma(u_n; X, Y).$$

Then for every $\gamma > 0$, there exists an $N_\gamma \in \mathbb{N}$ such that

$$\sup\Big\{\sum_{n \geq N_\gamma} \sum_{k=1}^{m_n} |x^*(x_k^n) y^*(y_k^n)|; x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \leq \sum_{n \geq N_\gamma} \Big|\sum_{k=1}^{m_n} x_k^n \otimes y_k^n\Big|_\sigma \leq \gamma.$$

This shows that

$$u = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} x_k^n \otimes y_k^n$$

unconditionally converges in $X\hat{\otimes}_{\alpha_\sigma} Y$. In addition, the infimum

$$\inf\{\cdot\} \leq \Big|\sum_{n=1}^{\infty} \sum_{k=1}^{m_n} x_k^n \otimes y_k^n\Big|_\sigma \leq \sum_{n=1}^{\infty} \Big|\sum_{k=1}^{m_n} x_k^n \otimes y_k^n\Big|_\sigma \leq (1+\delta)^2 \alpha_\sigma(u; X, Y).$$

Since $\delta > 0$ was arbitrary, $\inf\{\cdot\} \leq \alpha_\sigma(u; X, Y)$.

Since for every representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converging in $X\hat{\otimes}_{\alpha_\sigma}Y$,

$$\alpha_\sigma(u;X,Y) = \lim_{l\to\infty}\alpha_\sigma\Big(\sum_{n=1}^{l}x_n\otimes y_n\Big) \le \lim_{l\to\infty}\Big|\sum_{n=1}^{l}x_n\otimes y_n\Big|_\sigma = \Big|\sum_{n=1}^{\infty}x_n\otimes y_n\Big|_\sigma,$$

$\alpha_\sigma(u;X,Y) \le \inf\{\cdot\}$. □

Let α be a tensor norm and let M and N be finite-dimensional normed spaces. let

$$\alpha_0'(u;M,N) := \sup\{|\langle v,u\rangle| : \alpha(v;M^*,N^*) \le 1\}$$

for $u \in M\otimes N$. Then the dual tensor norm is defined by

$$\alpha'(u;X,Y) := \inf\{\alpha_0'(u;M,N) : u \in M\otimes N, \dim M, \dim N < \infty\}$$

for $u \in X\otimes Y$. The adjoint tensor norm is

$$\alpha^* := (\alpha')^t = (\alpha^t)'.$$

If α is finitely generated, then α', α^t and α^* are all finitely generated and $(\alpha')' = \alpha$

The adjoint ideal $[\mathcal{A}^{adj}, \|\cdot\|_{\mathcal{A}^{adj}}]$ is the maximal Banach operator ideal associated with the adjoint tensor norm α^*.

Lemma 7. *(see Theorem 17.5 in [2]) Let $\mathcal{A}$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α. Then for all Banach spaces X and Y,*

$$(X\otimes_{\alpha'}Y)^* = \mathcal{A}(X,Y^*)$$

holds isometrically.

Pietsch [3] introduced a stronger notion of the absolutely p-summing operator. A linear map $T : X \to Y$ is called absolutely τ-summing if there exists a $C > 0$ such that

$$\sum_{n=1}^{l}|y_n^*(Tx_n)| \le C\sup\Big\{\sum_{n=1}^{l}|x^*(x_n)y_n^*(y)| : x^* \in B_{X^*}, y \in B_Y\Big\}$$

for every $x_1,...,x_l \in X$ and $y_1^*,...,y_l^* \in Y^*$. We denote by $\mathcal{P}_\tau(X,Y)$ the space of all absolutely τ-summing operators from X to Y and for $T \in \mathcal{P}_\tau(X,Y)$, let

$$\|T\|_{\mathcal{P}_\tau} := \inf C,$$

where the infimum is taken over all such inequalities. Then it was shown in [3] (Theorems 23.1.2 and 23.1.3) that $[\mathcal{P}_\tau, \|\cdot\|_{\mathcal{P}_\tau}]$ is a maximal Banach operator ideal.

Pietsch [3] also introduced the ideal of σ-integral operators as follows;

$$[\mathcal{I}_\sigma, \|\cdot\|_{\mathcal{I}_\sigma}] := [\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]^{max}.$$

It was shown that

$$\mathcal{I}_\sigma^{adj} = \mathcal{P}_\tau \text{ and } \mathcal{P}_\tau^{adj} = \mathcal{I}_\sigma$$

hold isometrically [3] (Theorem 23.3.6).

We now have:

Corollary 2. *For all Banach spaces X and Y,*

$$(X \otimes_{\alpha_\sigma} Y)^* = \mathcal{P}_\tau(X, Y^*)$$

holds isometrically.

Proof. Since

$$\alpha'_\sigma = (\alpha^t_\sigma)' = \alpha^*_\sigma$$

is associated with $\mathcal{I}^{adj}_\sigma = \mathcal{P}_\tau$, by Lemma 7,

$$(X \otimes_{\alpha_\sigma} Y)^* = (X \otimes_{(\alpha'_\sigma)'} Y)^* = \mathcal{P}_\tau(X, Y^*)$$

holds isometrically. □

4. Non-Injectiveness and Non-Projectiveness of the σ-Tensor Norm

Proposition 4. *Let X and Y be Banach spaces. If X or Y has a hyperorthogonal basis, then*

$$X \otimes_\varepsilon Y = X \otimes_{\alpha_\sigma} Y$$

holds isometrically.

Proof. Suppose that Y has a hyperorthogonal basis $(e_i)_i$. Let $(e^*_i)_i$ be the sequence of coordinate functionals for $(e_i)_i$. Let $u = \sum_{n=1}^{l} x_n \otimes y_n \in X \otimes Y$ and let U be the corresponding *weak** to *weak* continuous finite rank operator for u, namely, $U = \sum_{n=1}^{l} x_n \underline{\otimes} y_n : X^* \to Y$. Then for every $x^* \in X^*$,

$$Ux^* = \sum_{i=1}^{\infty} (e^*_i U x^*) e_i$$

and $e^*_i U \in X \hookrightarrow X^{**}$ for every $i \in \mathbb{N}$. Moreover, since $U(B_{X^*})$ is relatively compact,

$$\begin{aligned} \lim_{l \to \infty} \varepsilon\Big(\sum_{i=1}^{l} e^*_i U \otimes e_i - u; X, Y \Big) &= \lim_{l \to \infty} \Big\| \sum_{i=1}^{l} e^*_i U \underline{\otimes} e_i - U \Big\| \\ &= \lim_{l \to \infty} \sup_{x^* \in B_{X^*}} \Big\| \sum_{i=1}^{l} (e^*_i U x^*) e_i - Ux^* \Big\| = 0. \end{aligned}$$

Consequently,

$$u = \sum_{i=1}^{\infty} e^*_i U \otimes e_i$$

converges in $X \hat{\otimes}_\varepsilon Y$. We will use Lemma 5 to show that the series unconditionally converges in $X \hat{\otimes}_{\alpha_\sigma} Y$.

Let $\eta > 0$ be given. Let $\{Ux^*_k\}_{k=1}^{m}$ be an $\eta/2$-net for $U(B_{X^*})$. Choose an $l \in \mathbb{N}$ so that

$$\Big\| \sum_{i \geq l} (e^*_i U x^*_k) e_i \Big\| \leq \frac{\eta}{2}$$

for every $k = 1, ..., m$. Let $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$.

Let $k_0 \in \{1, ..., m\}$ be such that

$$\|Ux^* - Ux^*_{k_0}\| \leq \frac{\eta}{2}.$$

Then we have

$$\begin{aligned}
&\sum_{i\geq l}|(e_i^*Ux^*)y^*(e_i)| \\
&\leq \sum_{i\geq l}|(e_i^*U(x^*-x_{k_0}^*))y^*(e_i)| + \sum_{i\geq l}|(e_i^*Ux_{k_0}^*)y^*(e_i)| \\
&\leq \sum_{i\geq l}\gamma_i(e_i^*U(x^*-x_{k_0}^*))y^*(e_i) + \sum_{i\geq l}\delta_i(e_i^*Ux_k^*)y^*(e_i) \;\; (|\gamma_i| = 1 = |\delta_i|) \\
&\leq \Big\|\sum_{i\geq l}\gamma_i(e_i^*U(x^*-x_{k_0}^*))e_i\Big\| + \Big\|\sum_{i\geq l}\delta_i(e_i^*Ux_k^*)e_i\Big\| \\
&\leq \Big\|\sum_{i=1}^{\infty}(e_i^*U(x^*-x_{k_0}^*))e_i\Big\| + \Big\|\sum_{i\geq l}(e_i^*Ux_k^*)e_i\Big\| \\
&\leq \|Ux^*-Ux_{k_0}^*\| + \frac{\eta}{2} \leq \eta.
\end{aligned}$$

By Proposition 3 and the above argument,

$$\alpha_\sigma(u;X,Y) \leq \Big|\sum_{i=1}^{\infty}e_i^*U\otimes e_i\Big|_\sigma = \|U\| = \varepsilon(u;X,Y).$$

The other part of the assertion follows from $\alpha_\sigma^t = \alpha_\sigma$ and $\varepsilon^t = \varepsilon$. □

A tensor norm α is called right-injective (respectively, right-projective) if for every isometry $I : Y \to Z$ (respectively, quotient operator $q : Y \to Z$), the operator

$$id_X \otimes I \text{ (respectively, } id_X \otimes q) : X \otimes_\alpha Y \to X \otimes_\alpha Z$$

is an isometry (respectively, a quotient operator) for all Banach spaces X, Y and Z. If α^t is right-injective (respectively, right-projective), then α is called left-injective (respectively, left-projective).

An operator ideal is said to be surjective if $[\mathcal{A}, \|\cdot\|_\mathcal{A}]^{sur} = [\mathcal{A}, \|\cdot\|_\mathcal{A}]$. According to [2] (Theorem 20.11), a maximal operator ideal is surjective if and only if its associated tensor norm is left-injective.

Example 1 (Non-injectiveness of α_σ)**.** *We show that*

$$\mathcal{I}_\sigma^{sur} \neq \mathcal{I}_\sigma.$$

For every separable Banach space X and every Banach space Y,

$$\mathcal{I}_\sigma^{sur}(X,Y) = \mathcal{L}(X,Y).$$

Indeed, according to [3] (Theorem 23.3.4), an operator $T : X \to Y$ is σ-integral if and only if $i_Y T$ is factored through some Banach lattice, where $i_Y : Y \to Y^{**}$ is the canonical isometry. Consequently, $Tq_X \in \mathcal{I}_\sigma(\ell_1, Y)$ for every $T \in \mathcal{L}(X,Y)$.

On the other hand, there exists a separable Banach space Z such that $id_Z \notin \mathcal{I}_\sigma(Z,Z)$ (cf. [5] (p. 364)). Hence

$$\mathcal{I}_\sigma^{sur}(Z,Z) = \mathcal{L}(Z,Z) \neq \mathcal{I}_\sigma(Z,Z).$$

Example 2. *(Non-projectiveness of α_σ) The following argument is due to the proof of [2] (Proposition 4.3). Let $q_{\ell_2} : \ell_1 \to \ell_2$ be the canonical quotient operator. Consider the map*

$$id_{\ell_2} \otimes q_{\ell_2} : \ell_2 \hat{\otimes}_{\alpha_\sigma} \ell_1 \to \ell_2 \hat{\otimes}_{\alpha_\sigma} \ell_2.$$

By Proposition 4,

$$\ell_2\hat{\otimes}_{\alpha_\sigma}\ell_1 = \ell_2\hat{\otimes}_\varepsilon\ell_1 = \mathcal{K}(\ell_2,\ell_1) \text{ and } \ell_2\hat{\otimes}_{\alpha_\sigma}\ell_2 = \ell_2\hat{\otimes}_\varepsilon\ell_2 = \mathcal{K}(\ell_2,\ell_2)$$

hold isometrically. Consequently, the map $id_{\ell_2} \otimes q_{\ell_2}$ *can be viewed from* $\mathcal{K}(\ell_2,\ell_1)$ *from* $\mathcal{K}(\ell_2,\ell_2)$*.*

Now, let $T : \ell_2 \to \ell_2$ *be a compact operator failed to be Hilbert–Schmidt. If* $id_{\ell_2} \otimes q_{\ell_2}$ *would be surjective, then there exists an* $R \in \mathcal{K}(\ell_2,\ell_1)$ *such that*

$$q_{\ell_2}R = id_{\ell_2} \otimes q_{\ell_2}(R) = T.$$

This is a contradiction because $q_{\ell_2}R$ *is Hilbert–Schmidt.*

5. Discussion

We introduce a new tensor norm and associate it with an operator ideal. This work continues the study of theory of tensor norms and we expect that several more results on tensor norms and operator ideals can be developed. We introduce one of the important subjects. For a finitely generated tensor norm α, a Banach space X is said to have the α-approximation property (α-AP) if for every Banach space Y, the natural map

$$J_\alpha : Y\hat{\otimes}_\alpha X \longrightarrow Y\hat{\otimes}_\varepsilon X$$

is injective (cf. [3] (Section 21.7)). We can consider the α_σ-AP and the following problems.

Problem 1. Does every Banach space have the α_σ-AP?

Problem 2. For every Banach space X, if X^* has the α_σ-AP, then does X have the α_σ-AP?

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