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Article

Several Theorems on Single and Set-Valued Prešić Type Mappings

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Abstract: In this study, we introduce set-valued Prešić type almost contractive mapping, Prešić type almost F-contractive mapping and set-valued Prešić type almost F-contractive mapping in metric space and prove some fixed point results for these mappings. Additionally, we give examples to show that our main theorems are applicable. These examples show that the new class of set-valued Prešić type almost F-contractive operators is not included in Prešić type class of set-valued Prešić type almost contractive operators.

Keywords: fixed point; set-valued Prešić type almost contractive; Prešić type almost F-contractive; set-valued Prešić type almost F-contractive

1. Introduction

Banach [1] introduced a famous fundamental fixed point theorem, also known as the Banach contraction principle. There are various extensions and generalizations of the Banach contraction principle in the literature. Prešić [2] gave a contractive condition on the finite product of metric spaces and proved a fixed point theorem. Additionally, Ćirić and Prešić [3], Abbas et al. [4], Shehzad et al. [5], Pacular [6], and Yeşilkaya et al. [7] have extended and generalized these results. Some generalizations and applications of the Prešić theorem can be seen in [8–11].

Considering the convergence of certain sequences S. B. Prešić [2] generalized Banach contraction principle as follows:

Theorem 1. *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leqslant q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1}), \tag{1}$$

for every $x_1, x_2, ..., x_{k+1}$ in X, where $q_1, q_2, ..., q_k$ are non negative constants such that $q_1 + q_2 + ... + q_k < 1$.

Subsequently, there exist a unique point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}), \quad (n = 1, 2, ...)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

Remark that condition (1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. Accordingly, Theorem 1 is a generalization of the Banach fixed point theorem.

Ćirić and Prešić [3] generalized the above result as follows:

Theorem 2. *Let (X,d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leqslant \lambda \max_{1 \leqslant i \leqslant k} \{d(x_i, x_{i+1})\}, \tag{2}$$

where $\lambda \in (0,1)$ is constant and $x_1, x_2, ..., x_{k+1}$ are arbitrary elements in X. Subsequently, there exist a point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}), \quad (n = 1, 2, ...),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

If, in addition, we suppose that on a diagonal $\triangle \subset X^k$

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v) \tag{3}$$

holds for all $u, v \in X$, with $u \neq v$, then x is the unique point in X with $T(x, x, ..., x) = x$.

In 1969, using Pompeiu-Hausdorff metric, Nadler [12] introduced the notion of set-valued contraction mapping and proved a set-valued version of the well known Banach contraction principle. Since then the metric fixed point theory of single-valued mappings has been extended to set-valued mappings, see for examples [13–17]. Denote by $P(X)$ the family of all nonempty subsets of X, $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. It is well known that, $H : CB(X) \times CB(X) \to \mathbb{R}$ is defined by,

$$H(K, L) = \max\left\{\sup_{k \in K} d(k, L), \sup_{l \in L} d(l, K)\right\}$$

for all $K, L \in CB(X)$. Then H is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d. For $K, L \in CB(X)$, we defined $D(k, L) = \inf\{d(k, l) : l \in L\}$ and $D(K, L) = \sup\{D(k, L) : k \in K\}$. We will use the following lemma:

Lemma 1. *Let (X, d) metric spaces and K compact subsets of X. Afterwards, for $x \in X$, there exists $k \in K$, such that*

$$d(x, k) = D(x, K).$$

Lemma 2. *[12] Let K and L be nonempty closed and bounded subsets of a metric space. Therefore, for any $k \in K$,*

$$D(k, L) \leqslant H(K, L).$$

Lemma 3. *[12] Let K and L be nonempty closed and bounded subsets of a metric space and $h > 1$. Subsequently, for all $k \in K$, there exists $l \in L$ such that*

$$d(k, l) \leqslant hH(K, L).$$

Berinde [18–20] defined almost contraction (or (δ, L)-weak contraction) mappings in a metric space. In the same paper, Berinde [15] introduced the concepts of set-valued almost contraction (the original name was set-valued (δ, L)-weak contraction) and proved the following nice fixed point theorem:

Theorem 3. *[15] Let (X, d) be a complete metric spaces, $M : X \to CB(X)$ be a set-valued almost contraction, which is, there exist two constants $\delta \in (0, 1)$ and $L \geqslant 0$, such that*

$$H(Mx, My) \leqslant \delta d(x, y) + LD(y, Mx)$$

for all $x, y \in X$. Subsequently, M is a set-valued almost contraction operator.

One of the most interesting generalizations of it was given by Wardowski [21]. First, we recall the concept of F-contraction, which was introduced by Wardowski.

Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing. That is, $\beta < \gamma \Rightarrow F(\beta) < F(\gamma)$ for all $\beta, \gamma \in \mathbb{R}_+$
(F2) For every sequence $\{\beta_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} \beta_n = 0$ if and only if $\lim_{n\to\infty} F(\beta_n) = -\infty$
(F3) There exists a number $z \in (0, 1)$, such that $\lim_{\beta \to 0^+} \beta^z F(\beta) = 0$.

Definition 1. *[21] Let (X, d) be a complete metric space and $F \in \mathcal{F}$. A mapping $T : X \to X$ is said to be an F-contraction on X if there exists $\tau > 0$, such that*

$$d(Tx, Ty) > 0 \ \textit{implies that} \ \ \tau + F(d(Tx, Ty)) \leqslant F(d(x, y))$$

for all $x, y \in X$.

For more study on F-contractions, one may refer to [16,22–24]. Additionally, Altun et al. [17] introduced set-valued F-contraction mappings and fixed point result for these type mappings on complete metric space was given as:

Definition 2. *[17] Let (X, d) be a complete metric space and $F \in \mathcal{F}$. A mapping $T : X \to CB(X)$ is said to be a set-valued F- contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$H(Tx, Ty) > 0 \ \textit{implies that} \ \ \tau + F(H(Tx, Ty)) \leqslant F(d(x, y))$$

for all $x, y \in X$.

Theorem 4. *[17] Let (X, d) be a complete metric spaces and $T : X \to K(X)$ be a set-valued F-contraction, then T has a fixed point in X.*

Altun et al. [17] showed that we can get $CB(X)$ instead of $K(X)$, by adding the condition $(F4)$ on F, as follows:

$$(F4) \ \ F(\inf M) = \inf F(M) \ \text{ for all } \ M \subset (0, \infty) \ \text{ with } \ \inf M > 0.$$

If F satisfies $(F1)$, then, it is satisfies $(F4)$ if and only if it is right continuous. Let $\mathcal{F}*$ be the family of all functions F satisfying $(F1) - (F4)$.

Altun et al. [16] using the concept of F-contractive mappings introduced the concept of set-valued almost F-contractive mappings in metric spaces and proved fixed point theorems for such mappings.

In Section 1, some basic definitions, lemmas, and theorems in the literature that will be used later in the paper are given. In Section 2, inspired and motivated by Nadler [12], Wardowski [21], Berinde [15,18–20] and Altun et al. [16,24], and Abbas et al. [4], we consider appropriate conditions for a class of mappings on Prešić type, set-valued almost contraction, single and set-valued almost

F-contraction and establish some new fixed point results. We also present examples to illustrate our main theorems. In the last section, we give conclusions.

2. Main Results

In this section, we give a fixed point theorem for set-valued Prešić type almost contractive mapping. Later, we introduce Prešić type almost F-contractive mapping and set-valued Prešić type almost F-contractive mapping in metric space and prove some fixed point results for these mappings. Firstly, let us start with the definition of set-valued Prešić type almost contractive mapping.

Definition 3. *Let (X, d) be a metric space. We say that $M : X^r \to CB(X)$ is a set-valued Prešić type almost contraction mapping, where r is a positive integer, if there exist $\delta \in (0, 1)$ and $\lambda \geqslant 0$ such that*

$$H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})) \leqslant \delta \max_{1\leqslant t\leqslant r}\{d(x_t, x_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{D(x_{t+1}, M(x_1, x_2, ..., x_r))\}, \quad (4)$$

for all $(x_1, x_2, \cdots, x_{r+1}) \in X^{r+1}$.

Theorem 5. *Let (X, d) be a complete metric spaces and $M : X^r \to CB(X)$ be a set-valued Prešić type almost contraction mapping, where r is a positive integer. If $x_1, x_2, ..., x_r$ are arbitrary points in X and*

$$x_{n+r} \in M(x_n, x_{n+1}, ..., x_{n+r-1}), \quad (n = 1, 2, ...) \quad (5)$$

then, the sequence (x_n) converges to some $a \in X$ and a is a fixed point of M, that is, $a \in M(a, a, ..., a)$.

Proof. Let $\beta > 1$, $x_r \in X$ and $x_{r+1} \in M(x_1, x_2, ..., x_r)$. If $H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})) = 0$ then $x_{r+1} \in M(x_{r+1}, x_{r+1}, \ldots, x_{r+1})$ that is, x_{r+1} is a fixed point of M. Let $H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})) \neq 0$. By Lemma 3 there exists $x_{r+2} \in M(x_2, x_3, ..., x_{r+1})$ such that

$$d(x_{r+1}, x_{r+2}) \leqslant \beta H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})).$$

By (4), we have

$$d(x_{r+1}, x_{r+2}) \leqslant \beta[\delta \max_{1\leqslant t\leqslant r}\{d(x_t, x_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{D(x_{t+1}, M(x_1, x_2, ..., x_r))\}].$$

Using the fact that $x_{r+1} \in M(x_1, x_2, ..., x_r)$ implies

$$\min_{1\leqslant t\leqslant r}\{D(x_{t+1}, M(x_1, x_2, ..., x_r))\} = 0.$$

Subsequently, we obtain

$$d(x_{r+1}, x_{r+2}) \leqslant \beta\delta \max_{1\leqslant t\leqslant r}\{d(x_t, x_{t+1})\}.$$

We take $\beta > 1$ such that $\gamma = \beta\delta < 1$ and so,

$$d(x_{r+1}, x_{r+2}) \leqslant \gamma \max_{1\leqslant t\leqslant r}\{d(x_t, x_{t+1})\}. \quad (6)$$

Similarly, there exists $x_{r+2} \in M(x_2, x_3, ..., x_{r+1})$ such that

$$d(x_{r+2}, x_{r+3}) \leqslant \beta[\delta \max_{2\leqslant t\leqslant r+1}\{d(x_t, x_{t+1})\} + \lambda \min_{2\leqslant t\leqslant r+1}\{D(x_{t+1}, M(x_2, x_3, ..., x_{r+1}))\}].$$

Using the fact that $x_{r+2} \in M(x_2, x_3, ..., x_{r+1})$ implies

$$\min_{2 \leqslant t \leqslant r+1} \{D(x_{t+1}, M(x_2, x_3, ..., x_{r+1}))\} = 0.$$

Continuous this condition, we have

$$d(x_{r+2}, x_{r+3}) \leqslant \gamma \max_{2 \leqslant t \leqslant r+1} \{d(x_t, x_{t+1})\}.$$

If we continue recursively, for $x_{n+r} \in M(x_n, x_{n+1}, ..., x_{n+r-1})$ we obtain

$$d(x_{n+r}, x_{n+r+1}) \leqslant \beta[\delta \max_{n \leqslant t \leqslant n+r-1} \{d(x_t, x_{t+1})\} + \lambda \min_{n \leqslant t \leqslant n+r-1} \{D(x_{t+1}, M(x_n, x_{n+1}, ..., x_{n+r-1}))\}].$$

Using the fact that $x_{n+r} \in M(x_n, x_{n+1}, ..., x_{n+r-1})$ implies

$$\min_{n \leqslant t \leqslant n+r-1} \{D(x_{t+1}, M(x_n, x_{n+1}, ..., x_{n+r-1}))\} = 0.$$

Accordingly, we obtain

$$d(x_{n+r}, x_{n+r+1}) \leqslant \gamma \max_{n \leqslant t \leqslant n+r-1} \{d(x_t, x_{t+1})\}. \tag{7}$$

For simplicity, let $w_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, $A = \max\{\frac{w_1}{s}, \frac{w_2}{s^2}, ..., \frac{w_r}{s^r}\}$, where $s = \gamma^{\frac{1}{r}}$. We shall prove by induction that for each $n \in \mathbb{N}$

$$w_n \leqslant As^n. \tag{8}$$

According to the definition of A it is clear that (8) is true for $n = 1, 2, ..., r$. Now, let the following r inequalities:

$$w_n \leqslant As^n, \quad w_{n+1} \leqslant As^{n+1}, ..., w_{n+r-1} \leqslant As^{n+r-1}$$

be the induction hypothesis. According to (7) we obtain

$$\begin{aligned} w_{n+r} &= d(x_{n+r}, x_{n+r+1}) \\ &\leqslant \gamma \max_{n \leqslant t \leqslant n+r-1} \{d(x_t, x_{t+1})\} \\ &= \gamma \max\{w_n, w_{n+1}, ..., w_{n+r-1}\} \\ &\leqslant \gamma \max\{As^n, As^{n+1}, ..., As^{n+r-1}\} \\ &= \gamma As^n \\ &= As^{n+r}, \end{aligned}$$

Let $n, m \in \mathbb{N}$ with $m > n$, Using the fact that (8) implies

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leqslant w_n + w_{n+1} + \cdots + w_{m-1} \\ &\leqslant As^n + As^{n+1} + ... + As^{m-1} \\ &\leqslant As^n(1 + s + s^2 + ...) = \frac{As^n}{1-s}. \end{aligned}$$

Since $s = \gamma^{\frac{1}{r}} < 1$, thus, $\frac{As^n}{1-s} \to 0$ as $n \to \infty$. Hence, (x_n) is a Cauchy sequence. Because (X, d) is complete, there exists $a \in X$ such that $\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n\to\infty} d(x_n, a) = 0$.

Now, we prove that a is a fixed point of M. Then, for any integer n, we have

$$\begin{aligned}D(a, M(a,a,...,a)) \leqslant& d(a, x_{n+r}) + D(x_{n+r}, M(a,a,...,a))\\ \leqslant& d(a, x_{n+r}) + H(M(x_n, x_{n+1},...,x_{n+r-1}), M(a,a,...,a))\\ \leqslant& d(a, x_{n+r}) + H(M(x_n, x_{n+1},...,x_{n+r-1}), M(x_{n+1}, x_{n+2},...,x_{n+r-1}, a))\\ &+ H(M(x_{n+1}, x_{n+2},...,x_{n+r-1}, a), M(x_{n+2}, x_{n+3},...,x_{n+r-1}, a, a))\\ &+ ... + H(M(x_{n+r-1}, a, ..., a), M(a,a,...,a))\end{aligned}$$

and by (4), we obtain

$$\begin{aligned}D(a, M(a,a,...,a)) \leqslant& d(a, x_{n+r}) + \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), ..., d(x_{n+r-1}, a)\}+\\ &\lambda \min\{D(x_{n+1}, M(x_n, x_{n+1},...,x_{n+r-1})), D(x_{n+2}, M(x_n, x_{n+1},...,x_{n+r-1})),\\ &..., D(a, M(x_n, x_{n+1},...,x_{n+r-1}))\}\\ &+ \max\{d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+3}), ..., d(x_{n+r-1}, a), d(a,a)\}+\\ &\lambda \min\{D(x_{n+2}, M(x_{n+1}, x_{n+2},...,x_{n+r-1}, a)),\\ &D(x_{n+3}, M(x_{n+1}, x_{n+2},...,x_{n+r-1}, a)), ..., D(a, M(x_{n+1}, x_{n+2},...,x_{n+r-1}, a))\}\\ &+ ... + \max\{d(x_{n+r-1}, a), d(a,a)\} + \lambda \min\{D(a, M(x_{n+1}, a, ..., a, a))\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain $D(a, M(a,a,...,a)) \leqslant 0$, which implies that $a \in M(a,a,...,a)$, whcih is, a is a fixed point of M. Therefore, this completes the proof. □

Definition 4. *Let (X, d) be a metric space. We say that $M : X^r \to X$ is a Prešić type almost F-contraction mapping, where r is a positive integer, if $F \in \mathcal{F}$, and there exist $\tau > 0$ and $\lambda \geqslant 0$, such that*

$$\begin{aligned}\tau + F(d(M(x_1, x_2,...,x_r), M(x_2, x_3,...,x_{r+1})))\\ \leqslant F(\max_{1\leqslant t\leqslant r}\{d(x_t, x_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{d(x_{t+1}, M(x_1, x_2,...,x_r))\}),\end{aligned} \tag{9}$$

for all $(x_1, x_2, \cdots, x_{r+1}) \in X^{r+1}$.

Theorem 6. *Let (X, d) be a complete metric spaces, $M : X^r \to X$ be a Prešić type almost F-contraction mapping, where r is a positive integer. If $x_1, x_2, ..., x_r$ are arbitrary points in X and*

$$x_{n+r} = M(x_n, x_{n+1},...,x_{n+r-1}), \quad (n = 1, 2, ...) \tag{10}$$

then, the sequence (x_n) converges to $a \in X$ and a is a fixed point of M, that is, $a = M(a,a,...,a)$.

Example 1. *Let $X = \{x_r = \frac{2r^2+r}{2},\ r \in \mathbb{N}\} \cup \{0\}$, (X, d) be a complete metric spaces and $d(a, b) =\mid a - b \mid$. Define the mapping $M : X^2 \to X$ by*

$$M(x_r, y_r) = \frac{x_r + y_r}{2} \quad for\ \ all\ \ x_r, y_r \in X.$$

We claim that M is a Prešić type almost F-contractive with respect to $F(\nu) = \nu + \ln(\nu)$ and $\tau = 1$. To see this, we shall prove that M satisfies the condition (9). Subsequently, we obtain

$$\begin{aligned}F(d(M(x_{r-1}, x_r), M(x_r, x_{r+1}))) \leqslant& F(\max\{d(x_{r-1}, x_r), d(x_r, x_{r+1})\}\\ &+ \lambda \min\{d(x_r, M(x_{r-1}, x_r)), d(x_{r+1}, M(x_{r-1}, x_r))\}) - \tau.\end{aligned} \tag{11}$$

Accordingly, we obtain

$$
\begin{aligned}
& d(M(x_{r-1},x_r),M(x_r,x_{r+1})) \\
& e^{d(M(x_{r-1},x_r),M(x_r,x_{r+1}))-\max\{d(x_{r-1},x_r),d(x_r,x_{r+1})\}-\lambda\min\{d(x_r,M(x_{r-1},x_r)),d(x_{r+1},M(x_{r-1},x_r))\}} \\
=& \frac{4r+1}{2}e^{\frac{4r+1}{2}-\frac{4r+3}{2}-\frac{4r-1}{4}} \\
\leqslant & e^{-1}(\max\{d(x_{r-1},x_r),d(x_r,x_{r+1})\}+\lambda\min\{d(x_r,M(x_{r-1},x_r)),d(x_{r+1},M(x_{r-1},x_r))\}).
\end{aligned}
$$

Thus, the inequality (11) is satisfied with $\lambda = 1$*. Therefore, Theorem 6 shows that* M *has a unique fixed point, which is,* $M(0,0) = 0$*.*

Now, we give a fixed point theorem for set-valued Prešić type almost F-contractive mapping in metric space. Let us start with the definition of the set-valued Prešić type almost F-contractive mapping.

Definition 5. *Let* (X,d) *be a metric space. We say that* $M : X^r \to CB(X)$ *is a set-valued Prešić type almost* F*-contraction mapping, where* r *is a positive integer, if* $F \in \mathcal{F}$*, and there exist* $\tau > 0$ *and* $\lambda \geqslant 0$*, such that*

$$
\begin{aligned}
&\tau + F(H(M(x_1,x_2,...,x_r),M(x_2,x_3,...,x_{r+1}))) \\
&\qquad \leqslant F(\max_{1\leqslant t\leqslant r}\{d(x_t,x_{t+1})\}+\lambda\min_{1\leqslant t\leqslant r}\{D(x_{t+1},M(x_1,x_2,...,x_r))\}),
\end{aligned}
\tag{12}
$$

for all $(x_1,x_2,\cdots,x_{r+1}) \in X^{r+1}$.

Theorem 7. *Let* (X,d) *be a complete metric spaces,* $M : X^r \to K(X)$ *be a set-valued Prešić type almost* F*-contraction mapping, where* r *is a positive integer. If* $x_1,x_2,...,x_r$ *are any arbitrary points in* X *and*

$$
x_{n+r} \in M(x_n,x_{n+1},...,x_{n+r-1}), \quad (n = 1,2,...) \tag{13}
$$

then, the sequence (x_n) *converges to some* $a \in X$ *and* a *is a fixed point of* M*, that is,* $a \in M(a,a,...,a)$*.*

Proof. Firstly, we shows that M has a fixed point. Let $x_1,x_2,...,x_r$, be arbitrary r elements in X. Define the sequence (x_n) in X by

$$
x_{n+r} \in M(x_n,x_{n+1},\ldots,x_{n+r-1}), \qquad (n = 1,2,\ldots).
$$

If for some $t \in \{1,2,\ldots r\}$, we have $x_{t+1} = x_{t+2}$, then, we have $x_{t+1} \in M(x_{t+1},x_{t+1},\ldots,x_{t+1})$ that is, x_{t+1} is a fixed point of M and the proof is finished. We assume that $x_{n+r} \neq x_{n+r+1}$ for all $n \in \mathbb{N}$. Subsequently, as $M(x_2,x_3,...,x_{r+1})$ is closed, we have $D(x_{r+1},M(x_2,x_3,...,x_{r+1})) > 0$ for any arbitrary points $x_2,x_3,...,x_{r+1} \in X$. From Lemma 2, we obtain

$$
0 < D(x_{r+1},M(x_2,x_3,...,x_{r+1})) \leqslant H(M(x_1,x_2,...,x_r),M(x_2,x_3,...,x_{r+1})).
$$

From $(F1)$, we have

$$
F(D(x_{r+1},M(x_2,x_3,...,x_{r+1}))) \leqslant F(H(M(x_1,x_2,...,x_r),M(x_2,x_3,...,x_{r+1}))),
$$

and from (12), we can write that

$$
\begin{aligned}
&F(D(x_{r+1},M(x_2,x_3,...,x_{r+1}))) \\
&\qquad \leqslant F(H(M(x_1,x_2,...,x_r),M(x_2,x_3,...,x_{r+1}))) \\
&\qquad \leqslant F(\max_{1\leqslant t\leqslant r}\{d(x_t,x_{t+1})\}+\lambda\min_{1\leqslant t\leqslant r}\{D(x_{t+1},M(x_1,x_2,...,x_r))\})-\tau.
\end{aligned}
\tag{14}
$$

Denote

$$W = \min_{1 \leqslant t \leqslant r} \{D(x_{t+1}, M(x_1, x_2, ..., x_r))\},$$

so, we have $x_{r+1} \in M(x_1, x_2, ..., x_r)$. So, we obtain $W = 0$. Moreover, since $M(x_2, x_3, ..., x_{r+1})$ is compact, then from Lemma 1 there exists $x_{r+2} \in M(x_2, x_3, ..., x_{r+1})$ such that
$d(x_{r+1}, x_{r+2}) = D(x_{r+1}, M(x_2, x_3, ..., x_{r+1}))$. Additionally, denote,

$$P = \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_r, x_{r+1})\}$$

then we have $P > 0$. From (14), we obtain

$$\begin{aligned} F(d(x_{r+1}, x_{r+2})) &\leqslant F(H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1}))) \\ &\leqslant F(\max_{1 \leqslant t \leqslant r} \{d(x_t, x_{t+1})\} + 0) - \tau \\ &= F(P) - \tau. \end{aligned}$$

If we continue recursively, we obtain a sequence $x_{n+r} \in X$, such that $x_{n+r} \in M(x_n, x_{n+1}, ..., x_{n+r-1})$, and

$$\begin{aligned} F(d(x_{n+r}, x_{n+r+1})) &\leqslant F(H(M(x_n, x_{n+1}, ..., x_{n+r-1}), M(x_{n+1}, x_{n+2}, ..., x_{n+r}))) \\ &\leqslant F(P) - n\tau \end{aligned} \tag{15}$$

for $n \in \mathbb{N}$. On taking limit as $n \to \infty$, we obtain

$$\lim_{n \to \infty} F(d(x_{n+r}, x_{n+r+1})) = -\infty.$$

Thus, from $(F2)$, we have

$$\lim_{n \to \infty} d(x_{n+r}, x_{n+r+1}) = 0.$$

From $(F3)$, there exists $h \in (0, 1)$, such that

$$\lim_{n \to \infty} (d(x_{n+r}, x_{n+r+1}))^h F(d(x_{n+r}, x_{n+r+1})) = 0. \tag{16}$$

By (15), we have

$$(d(x_{n+r}, x_{n+r+1}))^h F(d(x_{n+r}, x_{n+r+1})) - (d(x_{n+r}, x_{n+r+1}))^h F(P) \leqslant -(d(x_{n+r}, x_{n+r+1}))^h n\tau \leqslant 0. \tag{17}$$

On taking limit as $n \to \infty$, we obtain

$$\lim_{n \to \infty} n(d(x_{n+r}, x_{n+r+1}))^h = 0. \tag{18}$$

Thus, from (18), there exists $n_0 \in \mathbb{N}$ such that $n(d(x_{n+r}, x_{n+r+1}))^h \leqslant 1$ for all $n \geqslant n_0$. Accordingly, we have

$$d(x_{n+r}, x_{n+r+1}) \leqslant \frac{1}{n^{\frac{1}{h}}} \tag{19}$$

for all $n \geqslant n_0$. In order to show that (x_n) is a Cauchy sequence, consider $n, m \in \mathbb{N}$, such that $m > n \geqslant n_0$. Using the triangular inequality for the metric and from (19), we have

$$\begin{aligned} d(x_{n+r}, x_{m+r}) \leqslant & d(x_{n+r}, x_{n+r+1}) + d(x_{n+r+1}, x_{n+r+2}) + \ldots + d(x_{m+r-1}, x_{m+r}) \\ & \leqslant \sum_{t=n}^{m-1} d(x_{t+r}, x_{t+r+1}) \leqslant \sum_{t=n}^{m-1} \frac{1}{t^{\frac{1}{h}}} \to 0. \end{aligned}$$

This shows that (x_n) is a Cauchy sequence in (X, d). Since (X, d) is complete metric spaces, there exists $a \in X$, such that

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n\to\infty} d(x_n, a) = 0.$$

Now, we prove that a is a fixed point of M. From (12) for any arbitrary points $x_1, x_2, ..., x_r \in X$ with $H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})) > 0$, we get

$$\begin{aligned} & H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})) \\ & \qquad \leqslant \max_{1\leqslant t\leqslant r} \{d(x_t, x_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r} \{D(x_{t+1}, M(x_1, x_2, ..., x_r))\}. \end{aligned}$$

Subsequently, we obtain

$$\begin{aligned} D(x_{n+r}, M(a, a, ..., a)) & \leqslant H(M(x_n, x_{n+1}, ..., x_{n+r-1}), M(a, a, ..., a)) \\ & \leqslant \max_{n\leqslant t\leqslant n+r-1} \{d(x_t, a)\} + \lambda D(a, M(x_n, x_{n+1}, ..., x_{n+r-1})) \\ & \leqslant \max_{n\leqslant t\leqslant n+r-1} \{d(x_t, a)\} + \lambda D(a, x_{n+r}) \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get $D(a, M(a, a, ..., a)) = 0$, that is, a is a fixed point of M. Therefore, this completes the proof. □

Remark 1. *Note that, in Theorem 7, $M(x_1, x_2, ..., x_r)$ is compact for any arbitrary points $x_1, x_2, ..., x_r \in X$. Thus, we can present the following problem: let (X, d) be a complete metric space and $M : X^r \to CB(X)$ be a set-valued Prešić type almost F-contraction mapping. Does M have a fixed point? By adding the condition (F4) on F, we can give a answer to this problem, as follows:*

Corollary 1. *Let (X, d) be a complete metric space, r a positive integer and $M : X^r \to CB(X)$ be a given mapping. Suppose that $F \in \mathcal{F}*$ and there exist $\tau > 0$ and $\lambda \geqslant 0$ such that*

$$\begin{aligned} & \tau + F(H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1}))) \\ & \qquad \leqslant F(\max_{1\leqslant t\leqslant r} \{d(x_t, x_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r} \{D(x_{t+1}, M(x_1, x_2, ..., x_r))\}), \end{aligned} \tag{20}$$

for all $(x_1, x_2, \cdots, x_{r+1}) \in X^{r+1}$. Then, for arbitrary points $x_1, x_2, ..., x_r \in X$ the sequence (x_n) defined by

$$x_{n+r} \in M(x_n, x_{n+1}, ..., x_{n+r-1}), \quad (n = 1, 2, ...) \tag{21}$$

converges to $a \in X$ and a is a fixed point of M, that is, $a \in M(a, a, ..., a)$.

Proof. Firstly, we show that M has a fixed point. Let $x_1, x_2, ..., x_r$, be arbitrary r elements in X. Define the sequence (x_n) in X by

$$x_{n+r} \in M(x_n, x_{n+1}, \ldots, x_{n+r-1}), \qquad (n = 1, 2, \ldots).$$

If for some $t \in \{1, 2, \ldots r\}$, we have $x_{t+1} = x_{t+2}$ then, we have $x_{t+1} \in M(x_{t+1}, x_{t+1}, \ldots, x_{t+1})$ that is, x_{t+1} is a fixed point of M. We assume that $x_{n+r} \neq x_{n+r+1}$ for all $n \in \mathbb{N}$. Subsequently, as $M(x_2, x_3, ..., x_{r+1})$ is closed, we obtain $D(x_{r+1}, M(x_2, x_3, ..., x_{r+1})) > 0$ for any arbitrary points $x_2, x_3, ..., x_{r+1} \in X$. From Lemma 2, we obtain

$$0 < D(x_{r+1}, M(x_2, x_3, ..., x_{r+1})) \leqslant H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1})).$$

From $(F1)$, we have

$$F(D(x_{r+1}, M(x_2, x_3, ..., x_{r+1}))) \leqslant F(H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1}))),$$

and from (20), we can write that

$$\begin{aligned} F(D(x_{r+1}, &M(x_2, x_3, ..., x_{r+1}))) \\ &\leqslant F(H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1}))) \\ &\leqslant F(\max_{1 \leqslant t \leqslant r} \{d(x_t, x_{t+1})\} + \lambda \min_{1 \leqslant t \leqslant r} \{D(x_{t+1}, M(x_1, x_2, ..., x_r))\}) - \tau. \end{aligned} \tag{22}$$

Denote

$$W = \min_{1 \leqslant t \leqslant r} \{D(x_{t+1}, M(x_1, x_2, ..., x_r))\},$$

then we have $x_{r+1} \in M(x_1, x_2, ..., x_r)$. Accordingly, we obtain $W = 0$. Subsequently, we can write

$$\begin{aligned} F(D(x_{r+1}, &M(x_2, x_3, ..., x_{r+1}))) \\ &\leqslant F(H(M(x_1, x_2, ..., x_r), M(x_2, x_3, ..., x_{r+1}))) \\ &\leqslant F(\max_{1 \leqslant t \leqslant r} \{d(x_t, x_{t+1})\} + 0) - \tau. \end{aligned} \tag{23}$$

From $(F4)$, we can obtain

$$F(D(x_{r+1}, M(x_2, x_3, ..., x_{r+1}))) = \inf_{u \in M(x_2, x_3, ..., x_{r+1})} F(d(x_{r+1}, u))$$

and, thus, from (23), we obtain

$$\inf_{u \in M(x_2, x_3, ..., x_{r+1})} F(d(x_{r+1}, u)) \leqslant F(\max_{1 \leqslant t \leqslant r} \{d(x_t, x_{t+1})\}) - \tau. \tag{24}$$

Therefore, from (24) there exists $x_{r+2} \in M(x_2, x_3, ..., x_{r+1})$ such that

$$F(d(x_{r+1}, x_{r+2})) \leqslant F(\max_{1 \leqslant t \leqslant r} \{d(x_t, x_{t+1})\}) - \tau. \tag{25}$$

The rest of the proof can be completed as in the proof of Theorem 7. □

Example 2. *Let $X = \{x_r = \frac{r^2 + r}{2}, \ r \in \mathbb{N}\} \cup \{0\}$, (X, d) be a complete metric spaces and $d(a, b) = \mid a - b \mid$. Define the mapping $M : X^2 \to CB(X)$ by*

$$M(x, y) = \begin{cases} \{\frac{x+y}{2}\}, & \text{if } x = x_r, \ y = x_{r+1} \\ \{0\}, & \text{otherwise} \end{cases}$$

We claim that M is a set-valued Prešić type almost F-contractive mapping with respect to $F(\nu) = \nu + \ln(\nu)$, $\tau = 1$ and $\lambda = \frac{2}{r+1}$, where $F \in \mathcal{F}*$. To see this, we shall prove that M satisfies the condition (12). Subsequently, we obtain

$$\tau + F(H(M(x_r, x_{r+1}), M(x_{r+1}, x_{r+2}))) \leqslant F(\max\{d(x_r, x_{r+1}), d(x_{r+1}, x_{r+2})\} + \lambda \min\{D(x_{r+1}, M(x_r, x_{r+1})), D(x_{r+2}, M(x_r, x_{r+1}))\}). \quad (26)$$

Afterwards, for $r = 1$, we obtain

$$\tau + F(H(M(x_1, x_2), M(x_2, x_3))) \leqslant F(\max\{d(x_1, x_2), d(x_2, x_3)\} + \lambda \min\{D(x_2, M(x_1, x_2)), D(x_3, M(x_1, x_2))\}). \quad (27)$$

Accordingly, we have

$$\begin{aligned} &H(M(x_1, x_2), M(x_2, x_3)) \\ &= H\left(\left\{\frac{x_1 + x_2}{2}, 0\right\}, \left\{\frac{x_2 + x_3}{2}, 0\right\}\right) \\ &= \max\left\{\inf\left\{\frac{x_3 - x_1}{2}, \frac{x_1 + x_2}{2}\right\}, \inf\left\{\frac{x_3 - x_1}{2}, \frac{x_2 + x_3}{2}\right\}\right\} \\ &= \frac{x_3 - x_1}{2} = \frac{5}{2} \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &H(M(x_1, x_2), M(x_2, x_3)) \\ &e^{H(M(x_1,x_2),M(x_2,x_3)) - \max\{d(x_1,x_2),d(x_2,x_3)\} - \lambda \min\{D(x_2,M(x_1,x_2)),D(x_3,M(x_1,x_2))\}} \\ &= \frac{5}{2} e^{\frac{5}{2} - 4} \\ &\leqslant e^{-1}(\max\{d(x_1, x_2), d(x_2, x_3)\} + \lambda \min\{D(x_2, M(x_1, x_2)), d(x_3, M(x_1, x_2))\}). \end{aligned}$$

Thus, the inequality (27) is satisfied with $\lambda = 1$. Moreover, for $r > 3$, we obtain

$$\begin{aligned} &H(M(x_r, x_{r+1}), M(x_{r+1}, x_{r+2})) \\ &= H\left(\left\{\frac{x_r + x_{r+1}}{2}, 0\right\}, \left\{\frac{x_{r+1} + x_{r+2}}{2}, 0\right\}\right) \\ &= \max\left\{\inf\left\{\frac{x_{r+2} - x_r}{2}, \frac{x_r + x_{r+1}}{2}\right\}, \inf\left\{\frac{x_{r+2} - x_r}{2}, \frac{x_{r+1} + x_{r+2}}{2}\right\}\right\} \\ &= \frac{x_{r+2} - x_r}{2} \end{aligned}$$

$$\begin{aligned} &H(M(x_r, x_{r+1}), M(x_{r+1}, x_{r+2})) \\ &e^{H(M(x_r,x_{r+1}),M(x_{r+1},x_{r+2})) - \max\{d(x_r,x_{r+1}),d(x_{r+1},x_{r+2})\} - \lambda \min\{D(x_{r+1},M(x_r,x_{r+1})),D(x_{r+2},M(x_r,x_{r+1}))\}} \\ &= \frac{2r + 3}{2} e^{\frac{2r+3}{2} - (r+3)} \\ &\leqslant e^{-1}(\max\{d(x_r, x_{r+1}), d(x_{r+1}, x_{r+2})\} + \lambda \min\{D(x_{r+1}, M(x_r, x_{r+1})), d(x_{r+2}, M(x_r, x_{r+1}))\}). \end{aligned}$$

Thus, the inequality (26) is satisfied with $\lambda = \frac{2}{r+1}$. Therefore, Theorem 7 implies that M has two fixed point, which is, $M(0, 0) = 0$, $M(1, 1) = 1$.

On the other hand, it is not set-valued Prešić type almost contraction in metric spaces, to see this, we obtain

$$\begin{aligned} &\lim_{r \to \infty} \frac{H(M(x_r, x_{r+1}), M(x_{r+1}, x_{r+2})) - \lambda \min\{D(x_{r+1}, M(x_r, x_{r+1})), D(x_{r+2}, M(x_r, x_{r+1}))\}}{\max\{d(x_r, x_{r+1}), d(x_{r+1}, x_{r+2})\}} \\ &= \lim_{r \to \infty} \frac{2r + 1}{2r + 4} = 1. \end{aligned}$$

Subsequently,

$$H(M(x_r, x_{r+1}), M(x_{r+1}, x_{r+2})) \leqslant \delta \max\{d(x_r, x_{r+1}), d(x_{r+1}, x_{r+2})\} + \lambda \min\{D(x_{r+1}, M(x_r, x_{r+1})), D(x_{r+2}, M(x_r, x_{r+1}))\}$$

does not hold for $\delta \in (0, 1)$. *Hence, the condition of Theorem 5 is not satisfied. This example shows the new class of set-valued Prešić type almost F-contractive operators is not included in Prešić type class of set-valued Prešić type almost contractive operators.*

3. Conclusions

Berinde [15,18–20] defined almost contraction (or (δ, L)-weak contraction) and set-valued almost contraction mappings in metric space. Altun et al. [16,24], handling the concept of F-contractive, introduced the concept of almost F-contractive mappings and set-valued almost F-contractive mappings in metric spaces. Abbas et al. [4] introduced a certain fixed point theorem for the Prešić type F-contractive mapping. In this article is introduced new some fixed point theorems, by combining the ideas of Berinde, Altun et al. and Abbas et al. We prove a fixed point theorem for set-valued Prešić type almost contractive mapping. After we give Prešić type almost F-contractive mapping and set-valued Prešić type almost F-contractive mapping in metric space and prove several fixed point results for these mappings. Additionally, we introduce examples showing that our main results are applicable. The second of these examples show the new class of set-valued Prešić type almost F-contractive operators is not included in Prešić type class of set-valued Prešić type almost contractive mappings. These results extend the main results of many comparable results from the current literature.

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