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Local Sharp Vector Variational Type Inequality and Optimization Problems

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Abstract: In this paper, our goal was to establish the relationship between solutions of local sharp vector variational type inequality and sharp efficient solutions of vector optimization problems, also Minty local sharp vector variational type inequality and sharp efficient solutions of vector optimization problems, under generalized approximate η-convexity conditions for locally Lipschitzian functions.

Keywords: vector variational type inequality problems; vector optimization problems; efficient solutions; approximate η-convexity; Lipschitzian functions

1. Introduction

The research of variational inequality problems is a part of development in the theory of optimization since optimization problems can often be specialized to the solution of variational inequality problems. It is very important to point out that these theories pertain to more than just optimization problems and there in lies much of their attractiveness. Several authors have presented numerous fascinating results on variational inequality problems; see cited references here [1–12].

In 1984, Loridan [13] studied the concept of ϵ-efficient solutions for vector minimization problems where the function to be optimized has its values in the R^n space, which is a generalization of the classical problem for Pareto solution. Later in 1986, White [14] extended ϵ-optimality for scalar problems to vector maximization problems, or efficiency problems, with m objective functions defined on a subset of R^n. In 1993, Burke et al. [15] studied the concept of weak sharp minima for scalar optimization problem which was motivated by the application in convex and convex composite mathematical programming.

Recently, in 2016, Zhu [16] suggested the necessary optimal conditions for the weak local sharp efficient solution of a constrained multi-objective optimization problem by using the generalized Fermat formula, the Mordukhovich subdifferential for maximum functions, the fuzzy sum rule for Fréchet subdifferentials, and the sum rule for Mordukhovich subdifferentials, and also got the some sufficient optimal conditions respectively for the local and global weak sharp efficient solutions of such a multi-objective optimization problem, by applying the approximate projection method, and some appropriate convexity and affineness conditions.

Motivated by the ideas of local sharp and weak local sharp efficient solutions, we define the local sharp vector variational type inequalities and Minty local sharp vector variational type inequalities, and establish the relations between local (or Minty local) sharp vector variational type inequality and vector optimization problems involving generated by locally Lipschitzian mappings.

2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with a norm $\|\cdot\|$. Let X be a nonempty convex subset of $\mathbb{R}^n$. The distance function $d(\cdot, X) : X \to \mathbb{R}$ is defined by

$$d(x, X) = \inf_{x_0 \in X} \|x - x_0\|, \ \ \forall x \in X.$$

A vector valued function $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a number $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x - y\|, \ \forall x, y \in X.$$

Definition 1. *Let $\eta : X \times X \to X$ be a function. A lower semicontinuous function $\varphi : X \to \mathbb{R}$ is said to be approximate η-convex at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, for all $x, y \in B(x_0, \delta) \cap X$,*

$$\varphi(y) \geq \varphi(x) + \langle x^\star, \eta(y,x)\rangle - \tau\|y - x\|, \forall x^* \in \partial\varphi(x).$$

Definition 2. *Let $\eta : X \times X \to X$ be a function. A function $\varphi : X \to \mathbb{R}$ is said to be*

(i) approximate η-pseudoconvex type-I at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\langle x^*, \eta(y,x)\rangle \geq 0, \text{ for some } x^* \in \partial\varphi(x),$$

then

$$\varphi(y) - \varphi(x) \geq -\tau\|y - x\|;$$

(ii) approximate η-pseudoconvex type-II (strictly approximate η-pseudoconvex type-II) at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\langle x^*, \eta(y,x)\rangle + \tau\|y - x\| \geq 0, \text{ for some } x^* \in \partial\varphi(x),$$

then

$$\varphi(y) \geq (>)\varphi(x);$$

(iii) approximate η-quasiconvex type-I at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\varphi(y) \leq \varphi(x),$$

then

$$\langle x^*, \eta(y,x)\rangle - \tau\|y - x\| \leq 0, \ \forall x^* \in \partial\varphi(x);$$

(iv) approximate η-quasiconvex type-II (strictly approximate η-quasiconvex type-II) at $x_0 \in X$ if for any $\tau > 0$, there exists $\delta > 0$, such that, whenever $x, y \in B(x_0, \delta) \cap X$ and

$$\varphi(y) \leq (<)\varphi(x) + \tau\|y - x\|,$$

then

$$\langle x^*, \eta(y,x)\rangle \leq 0, \ \forall x^* \in \partial\varphi(x).$$

(VOP): A vector optimization problem (VOP) is formulated as follows:

$$\begin{cases} \text{Min } f(x), \\ \text{Subject to } \ x \in X \subset \mathbb{R}^n, \end{cases}$$

where, $f; X \subset R^n \to R^p$ with $f(x) = (f_1(x), \cdots, f_p(x))$, is a vector valued function.

Definition 3 ([16]).

(i) *A vector $x_0 \in X$ is said to be a local sharp efficient solution of (VOP), if for any $\tau > 0$ there exists a δ-neighborhood of x_0, such that for all $x \in B(x_0, \delta) \cap X$,*

$$\max_{1 \le i \le p} \{f_i(x) - f_i(x_0)\} \ge \tau \|x - x_0\|;$$

(ii) *A vector $x_0 \in X$ is said to be a weak local sharp efficient solution of (VOP), if for any $\tau > 0$, there exists a δ-neighborhood of x_0, such that for all $x \in B(x_0, \delta) \cap X$,*

$$\max_{1 \le i \le p} \{f_i(x) - f_i(x_0)\} \ge \tau d(x, \bar{X}),$$

where

$$\bar{X} := \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0)).$$

3. Local Sharp Vector Variational Type Inequalities

In this section, we consider local sharp and weak local sharp formulations of vector variational type inequality problems as follows:

(LSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0) \rangle \ge \tau \|x - x_0\|, \ \forall x^*_{0_i} \in \varphi f_i(x_0). \tag{1}$$

(WLSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0) \rangle \ge \tau d(x, \bar{X}), \forall x^*_{0_i} \in \partial f_i(x_0), \tag{2}$$

where

$$\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0)).$$

We note that, if x_0 is a solution of (LSVVTI), then x_0 is also a solution of (WLSVVTI).

Special Cases: Assume that, if $\eta(x, x_0) = x - x_0$. Then,

- (1) reduces to local sharp vector variational inequalities (*LSVVI*): for finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \ge \tau \|x - x_0\|, \forall x^*_{0_i} \in \varphi f_i(x_0). \tag{3}$$

- In addition, (2) reduces to weak local sharp vector variational inequalities (*WLSVVI*) for finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and for any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \ge \tau d(x, \bar{X}), \ \forall x^*_{0_i} \in \partial f_i(x_0), \tag{4}$$

  where

$$\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0)).$$

- Again, we note that if $\eta(x, x_0) = x - x_0$, then the solution of (LSVVI) is also a solution of $(\text{AVVI})_1$ (defined by [17]), but the converse need not be true:
  Example: consider the function

$$f(x) = (f_1(x), f_2(x)), x \in \mathbb{R},$$

where $f_1(x) = | x | - x^2$ and $f_2(x) = -x^2$.
If we take $x_0 = 0$, then for any $\tau > 0$, there does not exist any $\delta > 0$ such that

$$\langle x^*_{0_i}, x - x_0 \rangle \leq \tau \| x - x_0 \|, \forall i \in \{1, 2\}, x^*_{0_i} \in \partial f_i(x_0), x \in B(x_0, \delta) \cap \mathbb{R},$$

that is, x_0 is a solution of $(\text{AVVI})_1$. When $x < 0$, then for every $\delta > 0$ and $\tau > 0$, we do not have

$$\max_{1 \leq i \leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, x - x_0 \rangle \geq \tau \| x - x_0 \|,$$

that is, x_0 is not a solution of (LSVVI).

Unless otherwise stated, the following condition (C) is always assumed in this section.

(C) For the bi-function $\eta : X \times X \to X$ and the mappings $f_i : X \to \mathbb{R}$, $i = 1, \cdots, p$,

$$\langle f_i(x), \eta(x, x) \rangle = 0$$

for all $x \in X$.

First of all, in this section, we give the relationship between the solutions of local sharp vector variational type inequalities (LSVVTI) and local sharp (or weak local sharp) efficient solutions of vector optimization problem (VOP).

Now we are at the stage of introducing and proving the main theorems:

Theorem 4. *Let $\eta : X \times X \to X$ be a function and $f_i : X \to \mathbb{R}$, $i = 1, \cdots, p$ be locally Lipschitz and approximate η-convex at $x_0 \in X$, and satisfies the condition (C). If x_0 solves (LSVVTI), then it is a local sharp efficient solution of (VOP).*

Proof. Contrary, assume that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$ such that

$$\max_{1 \leq i \leq p} \{ f_i(x) - f_i(x_0) \} < \frac{\tau}{2} \| x - x_0 \|,$$

it implies,

$$f_i(x) - f_i(x_0) < \frac{\tau}{2} \| x - x_0 \|. \tag{5}$$

Since f_i is approximate η-convex at $x_0 \in X$, there exists $\bar{\delta}_i > 0$ such that for $\delta := \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$f_i(x) \geq f_i(x_0) + \langle x^*_{0_i}, \eta(x, x_0) \rangle - \frac{\tau}{2} \| x - x_0 \|, \ \forall x \in B(x_0, \delta) \cap X \ \text{ and } x^*_{0_i} \in \partial f_i(x_0). \tag{6}$$

Hence, it follows from (5) and (6) that

$$\frac{\tau}{2} \| x - x_0 \| > \langle x^*_{0_i}, \eta(x, x_0) \rangle - \frac{\tau}{2} \| x - x_0 \|.$$

Therefore, we have

$$\tau \| x - x_0 \| > \langle x^*_{0_i}, \eta(x, x_0) \rangle,$$

it implies that

$$\max_{1\leq i\leq p}\max_{x_{0_i}^*\in\partial f_i(x_0)}\langle x_{0_i}^*,\eta(x,x_0)\rangle<\tau\|x-x_0\|,\ \forall x\in B(x_0,\delta)\cap X \text{ and } x_{0_i}^*\in\partial f_i(x_0).$$

This is a contradiction to the fact that x_0 solves (LSVVTI). □

In following theorem, we obtain the converse result of Theorem 4 by assuming the approximate η-convexity of $-f_i$ instead of f_i.

Theorem 5. *For each $i=1,\cdots,p$, let η and f_i be same as in Theorem 4, $-f_i$ be approximate η-convex at $x_0\in X$, and satisfies the condition (C). Then the converse statement of Theorem 4 is true.*

Proof. Suppose that $x_0\in X$ is not a solution of the (LSVVTI). Then, for any $\delta_0>0$ and $\frac{\tau}{2}>0$, there exists $x\in B(x_0,\delta_0)\cap X$ and $x_{0_i}^*\in\partial f_i(x_0)$, such that

$$\max_{1\leq i\leq p}\max_{x_{0_i}^*\in\partial f_i(x_0)}\langle x_{0_i}^*,\eta(x,x_0)\rangle<\frac{\tau}{2}\|x-x_0\|,$$

it implies

$$\langle x_{0_i}^*,\eta(x,x_0)\rangle<\frac{\tau}{2}\|x-x_0\|. \tag{7}$$

Since $-f_i$ is approximate η-convex at $x_0\in X$, for any $\frac{\tau}{2}>0$, there exists $\bar{\delta}_i>0$, such that for $\delta:=\min\left\{\delta_0,\bar{\delta}_i : i=1,\cdots p\right\}$, we have

$$-f_i(x)\geq -f_i(x_0)+\langle x_{0_i}^*,\eta(x,x_0)\rangle-\frac{\tau}{2}\|x-x_0\|,\forall x\in B(x_0,\delta)\cap X \text{ and } x_{0_i}^*\in-\partial f_i(x_0),$$

we can write it as

$$\langle x_{0_i}^*,\eta(x,x_0)\rangle\geq f_i(x)-f_i(x_0)-\frac{\tau}{2}\|x-x_0\|. \tag{8}$$

From (7) and (8), we have

$$\frac{\tau}{2}\|x-x_0\|>f_i(x)-f_i(x_0)-\frac{\tau}{2}\|x-x_0\|,\ \forall x\in B(x_0,\delta)\cap X.$$

Hence, we have

$$f_i(x)-f_i(x_0)<\tau\|x-x_0\|,$$

this implies that

$$\max_{1\leq i\leq p}\left\{f_i(x)-f_i(x_0)\right\}<\tau\|x-x_0\|,\ \forall x\in B(x_0,\delta)\cap X,$$

which is a contradiction to the fact that x_0 is a local sharp efficient solution of (VOP). □

In next theorem, the same result of Theorem 4 is obtained by substituting the strictly approximate η-quasiconvex type-II condition instead of approximate η-convexity condition on f_i.

Theorem 6. *Let η and f_i be the same as in Theorem 4, $f_i: X\to\mathbb{R}$ be a strictly approximate η-quasiconvex type-II at $x_0\in X$, for each $i=1,\cdots,p$, and satisfies the condition (C). If x_0 solves (LSVVTI), then it is a local sharp efficient solution of (VOP).*

Proof. Assume that $x_0\in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0>0$ and $\tau>0$, there exists $x\in B(x_0,\delta_0)\cap X$, such that

$$\max_{1\leq i\leq p}\left\{f_i(x)-f_i(x_0)\right\}<\tau\|x-x_0\|,$$

it implies,

$$f_i(x) - f_i(x_0) < \tau\|x - x_0\|.$$

Since f_i is a strictly approximate η-quasiconvex type-II at $x_0 \in X$, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$, such that by setting $\delta = \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$\langle x^*_{0_i}, \eta(x, x_0)\rangle \leq 0 < \tau\|x - x_0\|, \ \forall x \in B(x_0, \delta) \cap X \ \text{ and } x^*_{0_i} \in \partial f_i(x_0),$$

implies that

$$\max_{1\leq i\leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0)\rangle < \tau\|x - x_0\|, \forall x \in B(x_0, \delta) \cap X \text{ and } x^*_{0_i} \in \partial f_i(x_0). \tag{9}$$

This means that x_0 is not a solution of (LSVVTI). □

In the following theorem, we can get the the generalization of Theorem 5 by assuming the strictly approximate η-pseudoconvex type-II condition on $-f_i$.

Theorem 7. *For each $i = 1, \cdots, p$, let η and f_i be same as in Theorem 4, $-f_i$ be a strictly approximate η-pseudoconvex type-II at $x_0 \in X$, and satisfies the condition (C). If x_0 is a weak local sharp efficient solution of (VOP), then it is also a solution of (LSVVTI).*

Proof. Suppose that $x_0 \in X$ is not a solution of (LSVVTI). Then, for any $\delta_0 > 0$ and $\tau > 0$, there exists $x \in B(x_0, \delta_0) \cap X$ and $x^*_{0_i} \in \partial f_i(x_0)$, such that

$$\max_{1\leq i\leq p} \max_{x^*_{0_i} \in \partial f_i(x_0)} \langle x^*_{0_i}, \eta(x, x_0)\rangle < \tau\|x - x_0\|.$$

Hence, we have,

$$\langle x^*_{0_i}, \eta(x, x_0)\rangle < \tau\|x - x_0\|,$$

and we can rewrite as

$$\langle -x^*_{0_i}, \eta(x, x_0)\rangle + \tau\|x - x_0\| > 0. \tag{10}$$

Since $-f_i$ is a strictly approximate η-pseudoconvex type-II at $x_0 \in X$, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$ such that, for $\delta := \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$-f_i(x) > -f_i(x_0), \forall x \in B(x_0, \delta) \cap X.$$

Therefore, we have

$$f_i(x) - f_i(x_0) < 0 \leq \tau d(x, \bar{X}),$$

this implies that

$$\max_{1\leq i\leq p} \{f_i(x) - f_i(x_0)\} < \tau d(x, \bar{X}), \ \forall x \in B(x_0, \delta) \cap X. \tag{11}$$

Therefore, we show that x_0 is a local weak sharp efficient solution of (VOP). This completes the proof. □

4. Minty Local Sharp Vector Variational Type Inequalities

In this section, we present relationship between the solutions of Minty local sharp vector variational type inequalities (MLSVVTI) and local sharp (or weak local sharp) efficient solutions of vector optimization problem (VOP).

Now, we consider Minty local sharp and Minty weak local sharp formulations of vector variational type inequality problems as follows:

(MLSVVTI): Finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x_i^* \in \partial f_i(x)} \langle x_i^*, \eta(x, x_0) \rangle \ge \tau \|x - x_0\|, \ \ \forall x_i^* \in \partial f_i(x). \tag{12}$$

(MWLSVVTI): For finding $x_0 \in X$, there exists a δ-neighborhood of x_0 and any $\tau > 0$, such that $x \in B(x_0, \delta) \cap X$ and

$$\max_{1 \le i \le p} \max_{x_i^* \in \partial f_i(x)} \langle x_i^*, \eta(x, x_0) \rangle \ge \tau d(x, \bar{X}), \ \ \forall x_i^* \in \partial f_i(x), \tag{13}$$

where $\bar{X} = \{x \in X \mid f(x) = f(x_0)\} = X \cap f^{-1}(f(x_0))$.

Theorem 8. *For each $i = 1, \cdots, p$, let η and f_i be same as in Theorem 4, $-f_i$ be approximate η-convex at $x_0 \in X$, and satisfies the condition (C). If x_0 solves (MLSVVTI), then x_0 is a local sharp efficient solution of (VOP).*

Proof. Suppose that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$, such that

$$\max_{1 \le i \le p} \{f_i(x) - f_i(x_0)\} < \frac{\tau}{2} \|x - x_0\|,$$

it implies,

$$f_i(x) - f_i(x_0) < \frac{\tau}{2} \|x - x_0\|. \tag{14}$$

Since $-f_i$ is approximate η-convex at $x_0 \in X$, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$, such that, for $\delta := \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$-f_i(x_0) \ge -f_i(x) + \langle -x_i^*, \eta(x_0, x) \rangle - \frac{\tau}{2} \|x_0 - x\|, \ \forall x \in B(x_0, \delta) \cap X \text{ and } -x_i^* \in -\partial f_i(x). \tag{15}$$

It follows from (14) and (15), we have

$$\frac{\tau}{2} \|x - x_0\| > \langle -x_i^*, \eta(x_0, x) \rangle - \frac{\tau}{2} \|x_0 - x\|, \forall x \in B(x_0, \delta) \cap X \text{ and } -x_i^* \in -\partial f_i(x),$$

that is,

$$\tau \|x - x_0\| > \langle -x_i^*, \eta(x_0, x) \rangle,$$

implies that

$$\max_{1 \le i \le p} \max_{x_i^* \in \partial f_i(x)} \langle x_i^*, \eta(x, x_0) \rangle < \tau \|x - x_0\|, \forall x \in B(x_0, \delta) \cap X \text{ and } x_i^* \in \partial f_i(x),$$

which is a contradiction to the fact that x_0 solves (MLSVVTI). This completes the proof. □

In following theorem, we can get the converse result of Theorem 8 by assuming the approximate η-convexity of f_i instead of $-f_i$.

Theorem 9. *For each $i = 1, \cdots, p$, let η and f_i be same as in Theorem 8, $f_i : X \to \mathbb{R}$ be approximate η-convex at $x_0 \in X$, and satisfies the condition (C). If x_0 is a local sharp efficient solution of (VOP), then x_0 solves (MLSVVTI).*

Proof. Suppose that $x_0 \in X$ is not a solution of the (MLSVVTI). Then, for any $\delta_0 > 0$ and $\frac{\tau}{2} > 0$, there exists $x \in B(x_0, \delta_0) \cap X$ and $x_i^* \in \partial f_i(x)$, such that

$$\max_{1 \le i \le p} \max_{x_i^* \in \partial f_i(x)} \langle x_i^*, \eta(x, x_0) \rangle < \frac{\tau}{2} \|x - x_0\|,$$

it implies,

$$\langle x_i^*, \eta(x, x_0) \rangle < \frac{\tau}{2} \|x - x_0\|. \tag{16}$$

Since f_i is approximate η-convex at $x_0 \in X$, for any $\frac{\tau}{2} > 0$, there exists $\bar{\delta}_i > 0$, such that, for $\delta := \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$f_i(x_0) \ge f_i(x) + \langle x_i^*, \eta(x_0, x) \rangle - \frac{\tau}{2} \|x_0 - x\|, \forall x \in B(x_0, \delta) \cap X \text{ and } x_i^* \in \partial f_i(x),$$

we can rewrite as

$$\langle x_i^*, \eta(x, x_0) \rangle \ge f_i(x) - f_i(x_0) - \frac{\tau}{2} \|x - x_0\|. \tag{17}$$

Combining (16) and (17), we have

$$\frac{\tau}{2} \|x - x_0\| > f_i(x) - f_i(x_0) - \frac{\tau}{2} \|x - x_0\|, \forall x \in B(x_0, \delta) \cap X.$$

Hence, we have

$$f_i(x) - f_i(x_0) < \tau \|x - x_0\|,$$

implies that

$$\max_{1 \le i \le p} \{f_i(x) - f_i(x_0)\} < \tau \|x - x_0\|, \forall x \in B(x_0, \delta) \cap X. \tag{18}$$

This is a contradiction to the fact that x_0 is a local sharp efficient solution of (VOP). □

In following theorem, we can get same result of Theorem 8 by assuming the strictly approximate η-quasiconvex type-II condition insted of approximate η-convexity on $-f_i$.

Theorem 10. *For each $i = 1, \cdots, p$, let η and f_i be same as in Theorem 8, $-f_i$ be a strictly approximate η-quasiconvex type-II at $x_0 \in X$, and satisfies the condition (C). If x_0 solves (MLSVVTI), then x_0 is a local sharp efficient solution of (VOP).*

Proof. Assume that $x_0 \in X$ is not a local sharp efficient solution of (VOP). Then, for any $\delta_0 > 0$ and $\tau > 0$, there exists $x \in B(x_0, \delta_0) \cap X$, such that

$$\max_{1 \le i \le p} \{f_i(x) - f_i(x_0)\} < \tau \|x - x_0\|,$$

it implies,

$$f_i(x) - f_i(x_0) < \tau \|x - x_0\|.$$

Hence, we can rewrite as

$$-f_i(x_0) - (-f_i(x)) < \tau \|x_0 - x\|. \tag{19}$$

Since $-f_i$ is a strictly approximate η-quasiconvex type-II at $x_0 \in X$, for any $\tau > 0$, there exists $\bar{\delta}_i > 0$ such that, for $\delta := \min\{\delta_0, \bar{\delta}_i : i = 1, \cdots, p\}$, we have

$$\langle -x_i^*, \eta(x_0, x) \rangle \le 0, \forall x \in B(x_0, \delta) \cap X \text{ and } -x_i^* \in -\partial f_i(x).$$

That is,

$$\langle x_i^*, \eta(x, x_0) \rangle \le 0 < \tau \|x - x_0\|,$$

implies that

$$\max_{1\leq i\leq p}\max_{x_i^*\in\partial f_i(x)}\langle x_i^*,\eta(x,x_0)\rangle<\tau\|x-x_0\|,\forall x\in B(x_0,\delta)\cap X \text{ and } x_i^*\in\partial f_i(x), \tag{20}$$

which is a contradiction to the fact that x_0 solves (MLSVVTI). □

The following theorem is an improvement of the Theorem 9 for the weak local sharp efficient solution of (VOP).

Theorem 11. *For each $i=1,\cdots,p$, let η and f_i be same as in Theorem 8, $f_i: X\to\mathbb{R}$ be a strictly approximate η-pseudoconvex type-II at $x_0\in X$, and satisfies the condition (C). If x_0 is a weak local sharp efficient solution of (VOP), then x_0 solves (MLSVVTI).*

Proof. On the contrary, assume that $x_0\in X$ is not a solution of (MLSVVTI). Then, for any $\delta_0>0$ and $\tau>0$, there exists $x\in B(x_0,\delta_0)\cap X$ and $x_i^*\in\partial f_i(x)$, such that

$$\max_{1\leq i\leq p}\max_{x_i^*\in\partial f_i(x)}\langle x_i^*,\eta(x,x_0)\rangle<\tau\|x-x_0\|.$$

Hence, we obtain

$$\langle x_i^*,\eta(x,x_0)\rangle<\tau\|x-x_0\|,$$

it implies,

$$\langle x_i^*,\eta(x_0,x)\rangle+\tau\|x_0-x\|>0.$$

Since f_i is a strictly approximate η-pseudoconvex type-II at $x_0\in X$, for any $\tau>0$, there exists $\bar{\delta}_i>0$ such that, for $\delta:=\min\{\delta_0,\bar{\delta}_i: i=1,\cdots,p\}$, we have

$$f_i(x_0)>f_i(x),\forall x\in B(x_0,\delta)\cap X.$$

This implies that

$$f_i(x)-f_i(x_0)<0\leq\tau d(x,\bar{X}),$$

hence, we have

$$\max_{1\leq i\leq p}\{f_i(x)-f_i(x_0)\}<\tau\ d(x,\bar{X}),\ \forall x\in B(x_0,\delta)\cap X. \tag{21}$$

This is a contradiction to the fact that x_0 is a weak local sharp efficient solution of (VOP). □

5. Conclusions

In this paper, we formulate local (Minty local) sharp vector variational type inequality problems and establish the relationship between local (Minty local) sharp vector variational type inequality and vector optimization problems involving locally Lipschitzian functions; that is, in Theorems 4–7, we give the necessary or sufficient conditions between the local sharp vector variational type inequality (LSVVTI) and vector optimization problems (VOP), and in Theorems 8–11, we give the necessary or sufficient conditions between the Minty local sharp vector variational type inequality (MLSVVTI) and vector optimization problems (VOP), by using the approximate η-convexity, strictly approximate η-quasiconvex type-II condition, and strictly approximate η-pseudoconvex type-II condition at $x_0\in X$,

The results of our research in this paper are generalized, extended, and improved studies of concepts of ϵ-efficient solutions for vector minimization problems [13], ϵ-optimality for scalar problems to vector maximization problems, or efficiency problems [14], weak sharp minima for scalar optimization problem [15], weak local sharp efficient solution of a constrained multi-objective optimization, and the local and global weak sharp efficient solutions of such a multi-objective optimization problem [16].

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