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Existence of Positive Solutions for a System of Singular Fractional Boundary Value Problems with *p*-Laplacian Operators

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Abstract: We investigate the existence and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations with singular nonnegative nonlinearities and p-Laplacian operators, subject to nonlocal boundary conditions which contain fractional derivatives and Riemann–Stieltjes integrals.

Keywords: Riemann–Liouville fractional differential equations; nonlocal boundary conditions; positive solutions; existence; multiplicity

MSC: 34A08; 34B15; 45G15

1. Introduction

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t))) + f(t,u(t),v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}v(t))) + g(t,u(t),v(t)) = 0, & t \in (0,1), \end{cases} \tag{1}$$

with the nonlocal boundary conditions

$$\begin{cases} u^{(j)}(0) = 0,\ j = 0,\ldots,n-2;\ D_{0+}^{\beta_1}u(0) = 0,\ D_{0+}^{\gamma_0}u(1) = \sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}u(t)\,dH_i(t), \\ v^{(j)}(0) = 0,\ j = 0,\ldots,m-2;\ D_{0+}^{\beta_2}v(0) = 0,\ D_{0+}^{\delta_0}v(1) = \sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}v(t)\,dK_i(t), \end{cases} \tag{2}$$

where $\alpha_1,\ \alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, $n,\ m \in \mathbb{N}$, $n,\ m \geq 3$, $p,q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0,\ldots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0,\ldots,q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $r_1,\ r_2 > 1$, $\varphi_{r_i}(\tau) = |\tau|^{r_i-2}\tau$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{r_i}{r_i-1}$, $i = 1,2$, the functions f and g are nonnegative and they may be singular at $t = 0$ and/or $t = 1$, the integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with H_i, $i = 1,\ldots,p$ and K_j, $j = 1,\cdots,q$ functions of bounded variation, and $D_{0+}^{\theta}u$ denotes the Riemann–Liouville fractional derivative of order θ of function u (for $\theta = \alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \gamma_i$ for $i = 0,\ldots,p$, δ_j for $j = 0,\ldots,q$). The fractional derivative $D_{0+}^{\theta}u$ is defined by $D_{0+}^{\theta}u(t) = \frac{1}{\Gamma(r-\theta)}\left(\frac{d}{dt}\right)^r \int_0^t (t-s)^{r-\theta-1}u(s)\,ds$, $t > 0$, where $r = \lfloor\theta\rfloor + 1$, $\lfloor\theta\rfloor$ stands for the largest integer not greater than θ, and $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}\,dt$, $\zeta > 0$, is the gamma function (the Euler function of second type). This work is motivated by the application of p-Laplacian operator in several fields such as nonlinear elasticity, fluid flow through porous media, glaciology, nonlinear electrorheological fluids, etc., for details, see [1] and the references cited therein.

Under some assumptions on the functions f and g, we present existence and multiplicity results for the positive solutions of problem (1) and (2). By a positive solution of problem (1) and (2) we mean a pair of functions $(u, v) \in (C([0,1], \mathbb{R}_+))^2$, satisfying the system (1) and the boundary conditions (2), with $u(t) > 0$ for all $t \in (0,1]$, or $v(t) > 0$ for all $t \in (0,1]$, ($\mathbb{R}_+ = [0, \infty)$). In the proof of our main theorems we use the Guo–Krasnosel'skii fixed point theorem (see [2]). The existence and nonexistence of positive solutions for the system (1) with two positive parameters λ and μ, and nonsingular and nonnegative nonlinearities, supplemented with the multi-point boundary conditions

$$\begin{cases} u^{(j)}(0) = 0,\ j = 0, \ldots, n-2;\ D_{0+}^{\beta_1} u(0) = 0,\ D_{0+}^{p_1} u(1) = \sum_{i=1}^{N} a_i D_{0+}^{q_1} u(\xi_i), \\ v^{(j)}(0) = 0,\ j = 0, \ldots, m-2;\ D_{0+}^{\beta_2} v(0) = 0,\ D_{0+}^{p_2} v(1) = \sum_{i=1}^{M} b_i D_{0+}^{q_2} v(\eta_i), \end{cases}$$

where $p_1, p_2, q_1, q_2 \in \mathbb{R}$, $p_1 \in [1, n-2]$, $p_2 \in [1, m-2]$, $q_1 \in [0, p_1]$, $q_2 \in [0, p_2]$, $\xi_i, a_i \in \mathbb{R}$ for all $i = 1, \ldots, N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_N \leq 1$, $\eta_i, b_i \in \mathbb{R}$ for all $i = 1, \ldots, M$ ($M \in \mathbb{N}$), $0 < \eta_1 < \cdots < \eta_M \leq 1$, was investigated in [3], by applying the Guo–Krasnosel'skii theorem. In the paper [4], the authors studied the system (1) with positive parameters, and nonsingular and nonnegative nonlinearities, subject to the nonlocal coupled boundary conditions

$$\begin{cases} u^{(j)}(0) = 0,\ j = 0, \ldots, n-2;\ D_{0+}^{\beta_1} u(0) = 0,\ D_{0+}^{\gamma_0} u(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i} v(t)\, dH_i(t), \\ v^{(j)}(0) = 0,\ j = 0, \ldots, m-2;\ D_{0+}^{\beta_2} v(0) = 0,\ D_{0+}^{\delta_0} v(1) = \displaystyle\sum_{i=1}^{q} \int_0^1 D_{0+}^{\delta_i} u(t)\, dK_i(t), \end{cases}$$

where $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$.

In [5], by applying the fixed point theorem for mixed monotone operators, the authors proved the existence of positive solutions for the multi-point boundary value problem for nonlinear Riemann–Liouville fractional differential equations

$$\begin{cases} D_{0+}^{\beta} \varphi_p(D_{0+}^{\alpha} u(t)) = f(t, u(t)),\ 0 < t < 1, \\ u(0) = 0,\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma} u(\eta_i),\ D_{0+}^{\alpha} u(0) = 0, \\ \varphi_p(D_{0+}^{\alpha} u(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0+}^{\alpha} u(\eta_i)), \end{cases}$$

where $\alpha, \beta \in (1,2]$, $\gamma \in (0,1]$, $\xi_i, \eta_i, \zeta_i \in (0,1)$, $i = 1, \ldots, m-2$, and f is a nonnegative function which may be singular at $x = 0$. In [6], the authors investigated the existence and uniqueness of positive solutions for the fractional boundary value problem

$$\begin{cases} {}^cD_{0+}^{\alpha} \varphi_p\left(D_{0+}^{\beta} u(t) + \varphi_q(I_{0+}^{r} h(t, I_{0+}^{\rho_1} u(t), D_{0+}^{\gamma} u(t)))\right) \\ \quad + f(t, I_{0+}^{\rho_2} u(t), D_{0+}^{\gamma} u(t)) = 0,\ t \in (0,1), \\ u(0) = D_{0+}^{\delta_1} u(0) = \cdots = D_{0+}^{\delta_{n-2}} u(0) = D_{0+}^{\beta} u(0) = 0, \\ D_{0+}^{k_0} u(1) = \lambda_1 \int_0^1 l_1(\tau) D_{0+}^{k_1} u(\tau) dA_1(\tau) + \lambda_2 \int_0^{\zeta} l_2(\tau) D_{0+}^{k_2} u(\tau) dA_2(\tau) \\ \quad + \lambda_3 \sum_{i=1}^{\infty} \mu_i D_{0+}^{k_3} u(\eta_i), \end{cases}$$

where $\alpha \in (0,1]$, $\beta \in (n-1, n]$, $n \geq 3$, ${}^cD_{0+}^{\alpha} u$ denotes the Caputo fractional derivative of order α of function u defined by ${}^cD_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds,\ t > 0$, for $\alpha \in (0,1)$, and ${}^cD_{0+}^{\alpha} u(t) = u'(t),\ t > 0$, for $\alpha = 1$, and the nonlinear terms f and h may be singular on the time variable and space variables. The authors used in [6] the theory of mixed monotone operators, and they also discussed there the dependence of solutions upon a parameter.

Systems with fractional differential equations without p-Laplacian operators, with parameters or without parameters, subject to various multi-point or Riemann–Stieltjes integral boundary conditions were studied in the last years in [7–27]. For various applications of the fractional differential equations in many scientific and engineering domains we refer the reader to the books [28–34], and their references.

The paper is organized as follows. In Section 2, we study two nonlocal boundary value problems for fractional differential equations with p-Laplacian operators, and we present some properties of the associated Green functions. Section 3 contains the main existence theorems for the positive solutions for our problem (1) and (2), and in Section 4, we give two examples which illustrate our results.

2. Auxiliary Results

We consider firstly the nonlinear fractional differential equation

$$D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t))) + h(t) = 0, \ \ t \in (0,1), \tag{3}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \ j = 0, \dots, n-2; \ \ D_{0+}^{\beta_1}u(0) = 0, \\ D_{0+}^{\gamma_0}u(1) = \sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}u(t)\, dH_i(t), \end{cases} \tag{4}$$

where $\alpha_1 \in (0,1]$, $\beta_1 \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, H_i, $i = 1, \dots, p$ are bounded variation functions, and $h \in C(0,1) \cap L^1(0,1)$. We denote by

$$\Delta_1 = \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_0)} - \sum_{i=1}^{p} \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 - \gamma_i)} \int_0^1 s^{\beta_1 - \gamma_i - 1}\, dH_i(s).$$

Lemma 1. *If $\Delta_1 \neq 0$, then the unique solution $u \in C[0,1]$ of problem (3) and (4) is given by*

$$u(t) = \int_0^1 \mathcal{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}h(s))\, ds, \ \ t \in [0,1], \tag{5}$$

where the Green function $\mathcal{G}_1$ is given by

$$\mathcal{G}_1(t,s) = g_1(t,s) + \frac{t^{\beta_1 - 1}}{\Delta_1} \sum_{i=1}^{p} \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right), \ \ t,s \in [0,1], \tag{6}$$

with

$$\begin{aligned} g_1(t,\zeta) &= \frac{1}{\Gamma(\beta_1)} \begin{cases} t^{\beta_1-1}(1-\zeta)^{\beta_1-\gamma_0-1} - (t-\zeta)^{\beta_1-1}, \ 0 \leq \zeta \leq t \leq 1, \\ t^{\beta_1-1}(1-\zeta)^{\beta_1-\gamma_0-1}, \ \ 0 \leq t \leq \zeta \leq 1, \end{cases} \\ g_{2i}(\tau,\zeta) &= \frac{1}{\Gamma(\beta_1 - \gamma_i)} \begin{cases} \tau^{\beta_1-\gamma_i-1}(1-\zeta)^{\beta_1-\gamma_0-1} - (\tau-\zeta)^{\beta_1-\gamma_i-1}, \\ \qquad 0 \leq \zeta \leq \tau \leq 1, \\ \tau^{\beta_1-\gamma_i-1}(1-\zeta)^{\beta_1-\gamma_0-1}, \ \ 0 \leq \tau \leq \zeta \leq 1, \end{cases} \\ & i = 1, \dots, p. \end{aligned} \tag{7}$$

Proof. We denote by $\varphi_{r_1}(D_{0+}^{\beta_1}u(t)) = x(t)$. Then problem (3) and (4) is equivalent to the following two boundary value problems

$$D_{0+}^{\alpha_1}x(t) + h(t) = 0, \ \ 0 < t < 1; \ \ x(0) = 0, \tag{8}$$

and

$$\begin{cases} D_{0+}^{\beta_1} u(t) = \varphi_{\varrho_1}(x(t)), \ \ 0 < t < 1; \\ u^{(j)}(0) = 0, \ j = 0, \ldots, n-2; \ \ D_{0+}^{\gamma_0} u(1) = \sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i} u(t)\, dH_i(t). \end{cases} \tag{9}$$

For the first problem (8), the function

$$x(t) = -I_{0+}^{\alpha_1} h(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} h(s)\, ds, \ \ t \in [0,1], \tag{10}$$

is the unique solution $x \in C[0,1]$ of (8). For the second problem (9), if $\Delta_1 \neq 0$, then by [7] (Lemma 2.2), we deduce that the function

$$u(t) = -\int_0^1 \mathcal{G}_1(t,s) \varphi_{\varrho_1}(x(s))\, ds, \ \ t \in [0,1], \tag{11}$$

where $\mathcal{G}_1$ is given by (6), is the unique solution $u \in C[0,1]$ of problem (9). Now, by using relations (10) and (11), we find formula (5) for the unique solution $u \in C[0,1]$ of problem (3) and (4). □

Next we consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2} v(t))) + k(t) = 0, \ \ t \in (0,1), \tag{12}$$

with the boundary conditions

$$\begin{cases} v^{(j)}(0) = 0, \ j = 0, \ldots, m-2; \ \ D_{0+}^{\beta_2} v(0) = 0, \\ D_{0+}^{\delta_0} v(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\delta_i} v(t)\, dK_i(t), \end{cases} \tag{13}$$

where $\alpha_2 \in (0,1]$, $\beta_2 \in (m-1, m]$, $m \in \mathbb{N}$, $m \geq 3$, $q \in \mathbb{N}$, $\delta_i \in \mathbb{R}$ for all $i = 0, \ldots, q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, K_i, $i = 1, \ldots, q$ are bounded variation functions, and $k \in C(0,1) \cap L^1(0,1)$. We denote by

$$\Delta_2 = \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_0)} - \sum_{i=1}^{q} \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 - \delta_i)} \int_0^1 s^{\beta_2 - \delta_i - 1}\, dK_i(s).$$

In a similar manner as above we obtain the following result.

Lemma 2. *If $\Delta_2 \neq 0$, then the unique solution $v \in C[0,1]$ of problem (12) and (13) is given by*

$$v(t) = \int_0^1 \mathcal{G}_2(t,s) \varphi_{\varrho_2}(I_{0+}^{\alpha_2} k(s))\, ds, \ \ t \in [0,1], \tag{14}$$

where the Green function $\mathcal{G}_2$ is given by

$$\mathcal{G}_2(t,s) = g_3(t,s) + \frac{t^{\beta_2 - 1}}{\Delta_2} \sum_{i=1}^{q} \left(\int_0^1 g_{4i}(\tau, s)\, dK_i(\tau) \right), \ \ t, s \in [0,1], \tag{15}$$

with

$$\begin{aligned} g_3(t,\zeta) &= \frac{1}{\Gamma(\beta_2)} \begin{cases} t^{\beta_2 - 1}(1-\zeta)^{\beta_2 - \delta_0 - 1} - (t-\zeta)^{\beta_2 - 1}, \ 0 \leq \zeta \leq t \leq 1, \\ t^{\beta_2 - 1}(1-\zeta)^{\beta_2 - \delta_0 - 1}, \ \ 0 \leq t \leq \zeta \leq 1, \end{cases} \\ g_{4i}(\tau,\zeta) &= \frac{1}{\Gamma(\beta_2 - \delta_i)} \begin{cases} \tau^{\beta_2 - \delta_i - 1}(1-\zeta)^{\beta_2 - \delta_0 - 1} - (\tau - \zeta)^{\beta_2 - \delta_i - 1}, \\ \qquad 0 \leq \zeta \leq \tau \leq 1, \\ \tau^{\beta_2 - \delta_i - 1}(1-\zeta)^{\beta_2 - \delta_0 - 1}, \ \ 0 \leq \tau \leq \zeta \leq 1, \end{cases} \\ &\quad i = 1, \ldots, q. \end{aligned} \tag{16}$$

By using the properties of the functions g_1, g_{2i}, $i = 1, \ldots, p$, g_3, g_{4i}, $i = 1, \ldots, q$ given by (7) and (16) (see [7,17]), we obtain the following properties of the Green functions $\mathcal{G}_1$ and $\mathcal{G}_2$ that we will use in the next section.

Lemma 3. *Assume that $H_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, p$, and $K_j : [0,1] \to \mathbb{R}$, $j = 1, \ldots, q$ are nondecreasing functions and $\Delta_1 > 0$, $\Delta_2 > 0$. Then the Green functions $\mathcal{G}_1$ and $\mathcal{G}_2$ given by (6) and (15) have the properties:*

(a) $\mathcal{G}_1, \mathcal{G}_2 : [0,1] \times [0,1] \to [0, \infty)$ *are continuous functions;*
(b) $\mathcal{G}_1(t,s) \le \mathcal{J}_1(s)$ *for all* $t, s \in [0,1]$, *where*
$\mathcal{J}_1(s) = h_1(s) + \frac{1}{\Delta_1} \sum_{i=1}^{p} \int_0^1 g_{2i}(\tau, s)\, dH_i(\tau)$, *with*
$h_1(s) = \frac{1}{\Gamma(\beta_1)} [(1-s)^{\beta_1 - \gamma_0 - 1} - (1-s)^{\beta_1 - 1}]$, $s \in [0,1]$;
(c) $\mathcal{G}_1(t,s) \ge t^{\beta_1 - 1} \mathcal{J}_1(s)$ *for all* $t, s \in [0,1]$;
(d) $\mathcal{G}_2(t,s) \le \mathcal{J}_2(s)$ *for all* $t, s \in [0,1]$, *where*
$\mathcal{J}_2(s) = h_2(s) + \frac{1}{\Delta_2} \sum_{i=1}^{q} \int_0^1 g_{4i}(\tau, s)\, dK_i(\tau)$, *with*
$h_2(s) = \frac{1}{\Gamma(\beta_2)} [(1-s)^{\beta_2 - \delta_0 - 1} - (1-s)^{\beta_2 - 1}]$, $s \in [0,1]$;
(e) $\mathcal{G}_2(t,s) \ge t^{\beta_2 - 1} \mathcal{J}_2(s)$ *for all* $t, s \in [0,1]$.

By similar arguments used in the proof of [17] (Lemma 2.5), we deduce the next lemma.

Lemma 4. *Assume that $H_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, p$ and $K_j : [0,1] \to \mathbb{R}$, $j = 1, \ldots, q$ are nondecreasing functions, $\Delta_1 > 0$, $\Delta_2 > 0$, $h \in C(0,1) \cap L^1(0,1)$, $k \in C(0,1) \cap L^1(0,1)$, $h(t) \ge 0$ for all $t \in (0,1)$, $k(t) \ge 0$ for all $t \in (0,1)$. Then the solutions u and v of problems (3), (4), (12) and (13), respectively, satisfy the inequalities $u(t) \ge 0$, $v(t) \ge 0$ for all $t \in [0,1]$. In addition, we have the inequalities $u(t) \ge t^{\beta_1 - 1} u(\tau)$, $v(t) \ge t^{\beta_2 - 1} v(\tau)$ for all $t, \tau \in [0,1]$.*

3. Existence of Positive Solutions

In this section, we investigate the existence of positive solutions for problem (1) and (2) under various assumptions on the functions f and g which may be singular at $t = 0$ and/or $t = 1$. We present the basic assumptions that we will use in the main theorems.

$(I1)$ $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1, n]$, $\beta_2 \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \ge 3$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, \ldots, p$, $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_p \le \gamma_0 < \beta_1 - 1$, $\gamma_0 \ge 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, \ldots, q$, $0 \le \delta_1 < \delta_2 < \cdots < \delta_q \le \delta_0 < \beta_2 - 1$, $\delta_0 \ge 1$, H_i, $i = 1, \ldots, p$, K_j, $j = 1, \ldots, q$ are nondecreasing functions, $\Delta_1 > 0$, $\Delta_2 > 0$, $r_i > 1$, $\varphi_{r_i}(s) = |s|^{r_i - 2} s$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{r_i}{r_i - 1}$, $i = 1, 2$.

$(I2)$ The functions $f, g \in C((0,1) \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and there exist the functions $\zeta_i \in C((0,1), \mathbb{R}_+)$ and $\chi_i \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, with $\Lambda_1, \Lambda_2 \in (0, \infty)$ such that

$$f(t,x,y) \le \zeta_1(t) \chi_1(t,x,y), \;\; g(t,x,y) \le \zeta_2(t) \chi_2(t,x,y), \;\; \forall\, t \in (0,1), \;\; x, y \in \mathbb{R}_+, \tag{17}$$

where $\Lambda_1 = \int_0^1 (1-s)^{\beta_1 - \gamma_0 - 1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1} \zeta_1(s))\, ds$, $\Lambda_2 = \int_0^1 (1-s)^{\beta_2 - \delta_0 - 1} \varphi_{\varrho_2}(I_{0+}^{\alpha_2} \zeta_2(s))\, ds$.

Remark 1. *We present below two cases in which $\Lambda_1, \Lambda_2 \in (0, \infty)$; for other cases see the examples from Section 4.*
a) If $f, g \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, that is $\zeta_i(s) = 1$ for all $s \in [0,1]$, $i = 1, 2$, $\chi_1 = f$, $\chi_2 = g$, then the inequalities (17) are satisfied with equality. In addition, the conditions $\Lambda_1, \Lambda_2 \in (0, \infty)$ are also satisfied, because in this nonsingular case, we obtain

$$
\begin{aligned}
\Lambda_1 &= \int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = \int_0^1 (1-s)^{\beta_1-\gamma_0-1}(I_{0+}^{\alpha_1}\zeta_1(s))^{\varrho_1-1}\,ds \\
&= \frac{1}{(\Gamma(\alpha_1))^{\varrho_1-1}}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\left(\int_0^s (s-\tau)^{\alpha_1-1}\,d\tau\right)^{\varrho_1-1} ds \\
&= \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}s^{\alpha_1(\varrho_1-1)}\,ds \\
&= \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}B(\alpha_1(\varrho_1-1)+1,\beta_1-\gamma_0) \in (0,\infty),
\end{aligned}
$$

where $B(\theta_1,\theta_2) = \int_0^1 t^{\theta_1-1}(1-t)^{\theta_2-1}\,dt$ is the beta function (the Euler function of first type), with $\theta_1,\theta_2 > 0$. In a similar manner we have $\Lambda_2 = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}B(\alpha_2(\varrho_2-1)+1,\beta_2-\delta_0) \in (0,\infty)$.
b) If $\zeta_1,\ \zeta_2 \in L^2(0,1)$, $\zeta_1 \not\equiv 0$, $\zeta_2 \not\equiv 0$, and $\alpha_1,\alpha_2 \in (1/2,1]$, then by using the Cauchy inequality we find

$$
\begin{aligned}
0 < \Lambda_1 &\le \frac{1}{(\Gamma(\alpha_1))^{\varrho_1-1}}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\left(\int_0^s (s-\tau)^{2(\alpha_1-1)}\,d\tau\right)^{\frac{\varrho_1-1}{2}}\left(\int_0^s \zeta_1^2(\tau)\,d\tau\right)^{\frac{\varrho_1-1}{2}} ds \\
&\le \frac{\|\zeta_1\|_2^{\varrho_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}(2\alpha_1-1)^{\frac{\varrho_1-1}{2}}}\int_0^1 s^{\frac{(2\alpha_1-1)(\varrho_1-1)}{2}}(1-s)^{\beta_1-\gamma_0-1}\,ds \\
&= \frac{\|\zeta_1\|_2^{\varrho_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}(2\alpha_1-1)^{\frac{\varrho_1-1}{2}}}B\left(\frac{(2\alpha_1-1)(\varrho_1-1)}{2}+1,\beta_1-\gamma_0\right) < \infty,
\end{aligned}
$$

where $\|\zeta_1\|_2$ is the norm of ζ_1 in the space $L^2(0,1)$. In a similar manner we obtain $\Lambda_2 \in (0,\infty)$.

By using Lemmas 1 and 2 (the relations (5) and (14)), (u,v) is a solution of problem (1) and (2) if and only if (u,v) is a solution of the nonlinear system of integral equations

$$
\begin{cases}
u(t) = \int_0^1 \mathcal{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds, & t \in [0,1], \\
v(t) = \int_0^1 \mathcal{G}_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds, & t \in [0,1].
\end{cases}
$$

We consider the Banach space $\mathcal{X} = C[0,1]$ with supremum norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$, and the Banach space $\mathcal{Y} = \mathcal{X}\times\mathcal{X}$ with the norm $\|(u,v)\|_{\mathcal{Y}} = \|u\|+\|v\|$. We define the cone $\mathcal{Q} \subset \mathcal{Y}$ by

$$
\mathcal{Q} = \{(u,v) \in \mathcal{Y},\ u(t) \ge 0,\ v(t) \ge 0,\ \forall\, t \in [0,1]\}.
$$

We also define the operators $\mathcal{A}_1,\ \mathcal{A}_2 : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ by

$$
\begin{cases}
\mathcal{A}_1(u,v)(t) = \int_0^1 \mathcal{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds, & t \in [0,1], \\
\mathcal{A}_2(u,v)(t) = \int_0^1 \mathcal{G}_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds, & t \in [0,1],
\end{cases}
$$

and $\mathcal{A}(u,v) = (\mathcal{A}_1(u,v),\mathcal{A}_2(u,v)),\ (u,v) \in \mathcal{Y}$. Then (u,v) is a solution of problem (1) and (2) if and only if (u,v) is a fixed point of operator $\mathcal{A}$.

Lemma 5. *Assume that $(I1)$ and $(I2)$ hold. Then $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}$ is a completely continuous operator (continuous, and it maps bounded sets into relatively compact sets).*

Proof. We denote by $M_i = \int_0^1 \mathcal{J}_i(s)\varphi_{\varrho_i}(I_{0+}^{\alpha_i}\zeta_i(s))\,ds$, $i=1,2$. Using $(I2)$ and Lemma 3, we deduce that $M_i > 0$, $i=1,2$. In addition, we find

$$\begin{aligned}
M_1 &= \int_0^1 \left[h_1(s) + \frac{1}{\Delta_1}\sum_{i=1}^{p}\int_0^1 g_{2i}(\tau,s)\,dH_i(\tau)\right]\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\
&= \int_0^1 \frac{1}{\Gamma(\beta_1)}(1-s)^{\beta_1-\gamma_0-1}(1-(1-s)^{\gamma_0})\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\
&\quad + \frac{1}{\Delta_1}\int_0^1\left(\sum_{i=1}^{p}\int_0^1 g_{2i}(\tau,s)\,dH_i(\tau)\right)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\
&\le \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\
&\quad + \frac{1}{\Delta_1}\int_0^1\left(\sum_{i=1}^{p}\int_0^1 \frac{1}{\Gamma(\beta_1-\gamma_i)}\tau^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1}\,dH_i(\tau)\right)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\
&= \Lambda_1\left(\frac{1}{\Gamma(\beta_1)} + \frac{1}{\Delta_1}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta_1-\gamma_i)}\int_0^1 \tau^{\beta_1-\gamma_i-1}\,dH_i(\tau)\right) < \infty,\\
M_2 &= \int_0^1 \left[h_2(s) + \frac{1}{\Delta_2}\sum_{i=1}^{q}\int_0^1 g_{4i}(\tau,s)\,dK_i(\tau)\right]\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds\\
&= \int_0^1 \frac{1}{\Gamma(\beta_2)}(1-s)^{\beta_2-\delta_0-1}(1-(1-s)^{\delta_0})\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds\\
&\quad + \frac{1}{\Delta_2}\int_0^1\left(\sum_{i=1}^{q}\int_0^1 g_{4i}(\tau,s)\,dK_i(\tau)\right)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds\\
&\le \frac{1}{\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-\delta_0-1}\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds\\
&\quad + \frac{1}{\Delta_2}\int_0^1\left(\sum_{i=1}^{q}\int_0^1 \frac{1}{\Gamma(\beta_2-\delta_i)}\tau^{\beta_2-\delta_i-1}(1-s)^{\beta_2-\delta_0-1}\,dK_i(\tau)\right)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds\\
&= \Lambda_2\left(\frac{1}{\Gamma(\beta_2)} + \frac{1}{\Delta_2}\sum_{i=1}^{q}\frac{1}{\Gamma(\beta_2-\delta_i)}\int_0^1 \tau^{\beta_2-\delta_i-1}\,dK_i(\tau)\right) < \infty.
\end{aligned}$$

By Lemma 3 we conclude that $\mathcal{A}$ maps $\mathcal{Q}$ into $\mathcal{Q}$.

We will show that $\mathcal{A}$ maps bounded sets into relatively compact sets. Suppose $\mathcal{S} \subset \mathcal{Q}$ is an arbitrary bounded set. Then there exists $L_1 > 0$ such that $\|(u,v)\|_{\mathcal{Y}} \le L_1$ for all $(u,v) \in \mathcal{S}$. By the continuity of χ_1 and χ_2 we deduce that there exists $L_2 > 0$ such that $L_2 = \max\{\sup_{t\in[0,1],\,u,v\in[0,L_1]}\chi_1(t,u,v), \sup_{t\in[0,1],\,u,v\in[0,L_1]}\chi_2(t,u,v)\}$. By using Lemma 3, for any $(u,v) \in \mathcal{S}$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
\mathcal{A}_1(u,v)(t) &\le \int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}f(s,u(s),v(s)))\,ds\\
&\le \int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\zeta_1(\tau)\chi_1(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\le L_2^{\varrho_1-1}\int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\zeta_1(\tau)\,d\tau\right)ds\\
&= L_2^{\varrho_1-1}\int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = M_1 L_2^{\varrho_1-1},\\
\mathcal{A}_2(u,v)(t) &\le \int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}g(s,u(s),v(s)))\,ds\\
&\le \int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1}\zeta_2(\tau)\chi_2(\tau,u(\tau),v(\tau))\,d\tau\right)ds\\
&\le L_2^{\varrho_2-1}\int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}\left(\frac{1}{\Gamma(\alpha_2)}\int_0^s (s-\tau)^{\alpha_2-1}\zeta_2(\tau)\,d\tau\right)ds\\
&= L_2^{\varrho_2-1}\int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds = M_2 L_2^{\varrho_2-1}.
\end{aligned}$$

Then $\|\mathcal{A}_1(u,v)\| \le M_1 L_2^{\varrho_1-1}$, $\|\mathcal{A}_2(u,v)\| \le M_2 L_2^{\varrho_2-1}$ for all $(u,v) \in \mathcal{S}$, and so $\mathcal{A}_1(\mathcal{S})$, $\mathcal{A}_2(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})$ are bounded.

We will prove next that $\mathcal{A}(\mathcal{S})$ is equicontinuous. By using Lemma 1, for $(u,v) \in \mathcal{S}$ and $t \in [0,1]$ we deduce

$$\begin{aligned}
\mathcal{A}_1(u,v)(t) &= \int_0^1 \left(g_1(t,s) + \frac{t^{\beta_1-1}}{\Delta_1} \sum_{i=1}^p \int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds \\
&= \int_0^t \frac{1}{\Gamma(\beta_1)} \left[t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1} - (t-s)^{\beta_1-1} \right] \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds \\
&\quad + \int_t^1 \frac{1}{\Gamma(\beta_1)} t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds \\
&\quad + \frac{t^{\beta_1-1}}{\Delta_1} \int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds.
\end{aligned}$$

Hence for any $t \in (0,1)$ we find

$$\begin{aligned}
(\mathcal{A}_1(u,v))'(t) &= \int_0^t \frac{1}{\Gamma(\beta_1)} \left[(\beta_1-1)t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} - (\beta_1-1)(t-s)^{\beta_1-2} \right] \\
&\quad \times \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds \\
&\quad + \int_t^1 \frac{1}{\Gamma(\beta_1)} (\beta_1-1)t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds \\
&\quad + \frac{(\beta_1-1)t^{\beta_1-2}}{\Delta_1} \int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1} f(s,u(s),v(s)))\, ds.
\end{aligned}$$

Then for any $t \in (0,1)$ we obtain

$$\begin{aligned}
|(\mathcal{A}_1(u,v))'(t)| &\le \frac{1}{\Gamma(\beta_1-1)} \int_0^t [t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} + (t-s)^{\beta_1-2}] \\
&\quad \times \varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\zeta_1(s)\chi_1(s,u(s),v(s))))\, ds \\
&\quad + \frac{1}{\Gamma(\beta_1-1)} \int_t^1 t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\zeta_1(s)\chi_1(s,u(s),v(s))))\, ds \\
&\quad + \frac{(\beta_1-1)t^{\beta_1-2}}{\Delta_1} \int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\zeta_1(s)\chi_1(s,u(s),v(s))))\, ds.
\end{aligned}$$

Therefore for any $t \in (0,1)$ we deduce

$$\begin{aligned}
|(\mathcal{A}_1(u,v))'(t)| &\le L_2^{\varrho_1-1} \Big[\frac{1}{\Gamma(\beta_1-1)} \int_0^t [t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} + (t-s)^{\beta_1-2}] \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds \\
&\quad + \frac{1}{\Gamma(\beta_1-1)} \int_t^1 t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds \\
&\quad + \frac{(\beta_1-1)t^{\beta_1-2}}{\Delta_1} \int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds \Big].
\end{aligned} \tag{18}$$

We denote by

$$\begin{aligned}
\theta_1(t) &= \frac{1}{\Gamma(\beta_1-1)} \int_0^t [t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} + (t-s)^{\beta_1-2}] \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds \\
&\quad + \frac{1}{\Gamma(\beta_1-1)} \int_t^1 t^{\beta_1-2}(1-s)^{\beta_1-\gamma_0-1} \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds, \\
\theta_2(t) &= \theta_1(t) + \frac{(\beta_1-1)t^{\beta_1-2}}{\Delta_1} \int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right) \varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\, ds.
\end{aligned}$$

We compute the integral of function θ_1, by exchanging the order of integration, and we have

$$\int_0^1 \theta_1(t)\,dt = \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}(1+(1-s)^{\gamma_0})\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds$$
$$\le \frac{2}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = \frac{2\Lambda_1}{\Gamma(\beta_1)} < \infty.$$

For the integral of the function θ_2, we obtain

$$\begin{aligned}\int_0^1 \theta_2(t)\,dt &= \int_0^1 \theta_1(t)\,dt + \left(\int_0^1 \frac{(\beta_1-1)t^{\beta_1-2}}{\Delta_1}dt\right)\\ &\quad\times\left(\int_0^1 \sum_{i=1}^p \left(\int_0^1 g_{2i}(\tau,s)\,dH_i(\tau)\right)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\right)\\ &\le \frac{2}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\ &\quad+\frac{1}{\Delta_1}\left(\int_0^1 \sum_{i=1}^p\left(\int_0^1 \frac{1}{\Gamma(\beta_1-\gamma_i)}\tau^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1}\,dH_i(\tau)\right)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\right)\\ &= \frac{2}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\\ &\quad+\frac{1}{\Delta_1}\left(\int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds\right)\left(\sum_{i=1}^p \frac{1}{\Gamma(\beta_1-\gamma_i)}\int_0^1 \tau^{\beta_1-\gamma_i-1}dH_i(\tau)\right).\end{aligned}$$

Then we deduce

$$\int_0^1 \theta_2(t)\,dt \le \Lambda_1\left(\frac{2}{\Gamma(\beta_1)} + \frac{1}{\Delta_1}\sum_{i=1}^p \frac{1}{\Gamma(\beta_1-\gamma_i)}\int_0^1 \tau^{\beta_1-\gamma_i-1}\,dH_i(\tau)\right) < \infty. \tag{19}$$

We conclude that $\theta_2 \in L^1(0,1)$. Hence for any $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$ and $(u,v) \in \mathcal{S}$, by (18) and (19), we find

$$|\mathcal{A}_1(u,v)(t_1) - \mathcal{A}_1(u,v)(t_2)| = \left|\int_{t_1}^{t_2} (\mathcal{A}_1(u,v))'(t)\,dt\right| \le L_2^{\varrho_1-1}\int_{t_1}^{t_2}\theta_2(t)\,dt. \tag{20}$$

By (19), (20) and the absolute continuity of the integral function, we deduce that $\mathcal{A}_1(\mathcal{S})$ is equicontinuous. By a similar approach, we obtain that $\mathcal{A}_2(\mathcal{S})$ is also equicontinuous, and so $\mathcal{A}(\mathcal{S})$ is equicontinuous. Using the Ascoli–Arzela theorem, we conclude that $\mathcal{A}_1(\mathcal{S})$ and $\mathcal{A}_2(\mathcal{S})$ are relatively compact sets, and so $\mathcal{A}(\mathcal{S})$ is also relatively compact. Besides, we can prove that $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$ are continuous on $\mathcal{Q}$ (see [16] (Lemma 1.4.1)). Then $\mathcal{A}$ is a completely continuous operator on $\mathcal{Q}$. □

We define now the cone

$$\mathcal{Q}_0 = \{(u,v) \in \mathcal{Q},\ \min_{t\in[0,1]} u(t) \ge t^{\beta_1-1}\|u\|,\ \min_{t\in[0,1]} v(t) \ge t^{\beta_2-1}\|v\|\}.$$

Under the assumptions $(I1)$ and $(I2)$, by using Lemma 4, we obtain $\mathcal{A}(\mathcal{Q}) \subset \mathcal{Q}_0$, and so $\mathcal{A}|_{\mathcal{Q}_0} : \mathcal{Q}_0 \to \mathcal{Q}_0$ (denoted again by $\mathcal{A}$) is also a completely continuous operator. For $r > 0$ we denote by B_r the open ball centered at zero of radius r, and by $\overline{B}_r$ and ∂B_r its closure and its boundary, respectively.

Theorem 1. *Assume that $(I1)$ and $(I2)$ hold. In addition, the functions χ_1, χ_2, f and g satisfy the conditions*

$(I3)$ *There exist $\mu_1 \ge 1$ and $\mu_2 \ge 1$ such that*

$$\chi_{10} = \lim_{\substack{x+y\to 0\\ x,y\ge 0}} \sup_{t\in[0,1]} \frac{\chi_1(t,x,y)}{\varphi_{r_1}((x+y)^{\mu_1})} = 0 \ \ \textit{and}\ \ \chi_{20} = \lim_{\substack{x+y\to 0\\ x,y\ge 0}} \sup_{t\in[0,1]} \frac{\chi_2(t,x,y)}{\varphi_{r_2}((x+y)^{\mu_2})} = 0;$$

(*I*4) *There exists* $[a_1, a_2] \subset [0,1]$, $0 < a_1 < a_2 < 1$ *such that*

$$f^i_\infty = \lim_{\substack{x+y\to\infty \\ x,y\geq 0}} \inf_{t\in[a_1,a_2]} \frac{f(t,x,y)}{\varphi_{r_1}(x+y)} = \infty \text{ or } g^i_\infty = \lim_{\substack{x+y\to\infty \\ x,y\geq 0}} \inf_{t\in[a_1,a_2]} \frac{g(t,x,y)}{\varphi_{r_2}(x+y)} = \infty.$$

Then problem (1) and (2) has at least one positive solution $(u(t), v(t))$, $t \in [0,1]$.

Proof. We consider the above cone $\mathcal{Q}_0$. By $(I3)$ we deduce that for $\epsilon_1 = \frac{1}{(2M_1)^{r_1-1}}$ and $\epsilon_2 = \frac{1}{(2M_2)^{r_2-1}}$, there exists $R_1 \in (0,1)$ such that

$$\chi_i(t,x,y) \leq \epsilon_i (x+y)^{\mu_i(r_i-1)}, \ \forall\, t\in[0,1], \ x+y\leq R_1, \ i=1,2, \tag{21}$$

where M_i, $i=1,2$ are defined in the proof of Lemma 5. Then by (21) and Lemma 3, for any $(u,v) \in \partial B_{R_1} \cap \mathcal{Q}_0$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
\mathcal{A}_i(u,v)(t) &\leq \int_0^1 \mathcal{J}_i(s)\varphi_{\varrho_i}(I^{\alpha_i}_{0+}(\zeta_i(s)\chi_i(s,u(s),v(s))))\,ds \\
&\leq \int_0^1 \mathcal{J}_i(s)\varphi_{\varrho_i}(I^{\alpha_i}_{0+}(\zeta_i(s)\epsilon_i(u(s)+v(s))^{\mu_i(r_i-1)}))\,ds \\
&\leq \epsilon_i^{\varrho_i-1}\int_0^1 \mathcal{J}_i(s)\varphi_{\varrho_i}(I^{\alpha_i}_{0+}\zeta_i(s)(\|u\|+\|v\|)^{\mu_i(r_i-1)})\,ds \\
&= \epsilon_i^{\varrho_i-1}\|(u,v)\|^{\mu_i}_{\mathcal{Y}}\int_0^1 \mathcal{J}_i(s)\varphi_{\varrho_i}(I^{\alpha_i}_{0+}\zeta_i(s))\,ds \\
&= M_i\epsilon_i^{\varrho_i-1}\|(u,v)\|^{\mu_i}_{\mathcal{Y}} \leq M_i\epsilon_i^{\varrho_i-1}\|(u,v)\|_{\mathcal{Y}} = \tfrac{1}{2}\|(u,v)\|_{\mathcal{Y}}, \ i=1,2.
\end{aligned}$$

So we deduce that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} = \|\mathcal{A}_1(u,v)\| + \|\mathcal{A}_2(u,v)\| \leq \|(u,v)\|_{\mathcal{Y}}, \ \forall\,(u,v)\in\partial B_{R_1}\cap\mathcal{Q}_0. \tag{22}$$

By $(I4)$, we suppose that $f^i_\infty = \infty$ (in a similar manner we can study the case $g^i_\infty = \infty$). Then for $\epsilon_3 = 2(A\min\{a_1^{\beta_1-1}, a_1^{\beta_2-1}\})^{1-r_1}$, where $A = \frac{a_1^{\beta_1-1}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\int_{a_1}^{a_2}\mathcal{J}_1(s)(s-a_1)^{\alpha_1(\varrho_1-1)}\,ds$, there exists $C_1 > 0$ such that

$$f(t,x,y) \geq \epsilon_3(x+y)^{r_1-1} - C_1, \ \forall\, t\in[a_1,a_2], \ x,y\geq 0. \tag{23}$$

Then by (23), for any $(u,v) \in \mathcal{Q}_0$ and $t \in [a_1,a_2]$, we find

$$\begin{aligned}
\mathcal{A}_1(u,v)(t) &\geq \int_{a_1}^{a_2}\mathcal{G}_1(t,s)\left(\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^s (s-\tau)^{\alpha_1-1}f(\tau,u(\tau),v(\tau))\,d\tau\right)^{\varrho_1-1} ds \\
&\geq a_1^{\beta_1-1}\int_{a_1}^{a_2}\frac{\mathcal{J}_1(s)}{(\Gamma(\alpha_1))^{\varrho_1-1}}\left(\int_{a_1}^s (s-\tau)^{\alpha_1-1}(\epsilon_3(u(\tau)+v(\tau))^{r_1-1}-C_1)\,d\tau\right)^{\varrho_1-1} ds \\
&= a_1^{\beta_1-1}\int_{a_1}^{a_2}\frac{\mathcal{J}_1(s)}{(\Gamma(\alpha_1))^{\varrho_1-1}}\left(\int_{a_1}^s (s-\tau)^{\alpha_1-1}(\epsilon_3(a_1^{\beta_1-1}\|u\|+a_1^{\beta_2-1}\|v\|)^{r_1-1}-C_1)d\tau\right)^{\varrho_1-1} ds \\
&= \frac{a_1^{\beta_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}}\int_{a_1}^{a_2}\mathcal{J}_1(s)\left[\epsilon_3(a_1^{\beta_1-1}\|u\|+a_1^{\beta_2-1}\|v\|)^{r_1-1}-C_1\right]^{\varrho_1-1}\frac{(s-a_1)^{\alpha_1(\varrho_1-1)}}{\alpha_1^{\varrho_1-1}}\,ds \\
&\geq \frac{a_1^{\beta_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}}\left[\epsilon_3\left(\min\{a_1^{\beta_1-1},a_1^{\beta_2-1}\}\right)^{r_1-1}\|(u,v)\|^{r_1-1}_{\mathcal{Y}}-C_1\right]^{\varrho_1-1} \\
&\quad\times\int_{a_1}^{a_2}\mathcal{J}_1(s)\frac{(s-a_1)^{\alpha_1(\varrho_1-1)}}{\alpha_1^{\varrho_1-1}}\,ds \\
&= \left[A^{\frac{1}{\varrho_1-1}}\epsilon_3\left(\min\{a_1^{\beta_1-1},a_1^{\beta_2-1}\}\right)^{r_1-1}\|(u,v)\|^{r_1-1}_{\mathcal{Y}} - A^{\frac{1}{\varrho_1-1}}C_1\right]^{\varrho_1-1} \\
&= \left(2\|(u,v)\|^{r_1-1}_{\mathcal{Y}} - C_2\right)^{\varrho_1-1}, \ C_2 = A^{r_1-1}C_1.
\end{aligned}$$

Hence we deduce

$$\|\mathcal{A}_1(u,v)\| \geq (2\|(u,v)\|_{\mathcal{Y}}^{r_1-1} - C_2)^{\varrho_1-1}, \ \ \forall\,(u,v) \in \mathcal{Q}_0.$$

We can choose $R_2 \geq \max\left\{1, C_2^{\varrho_1-1}\right\}$ and then we conclude

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \geq \|\mathcal{A}_1(u,v)\| \geq \|(u,v)\||_{\mathcal{Y}}, \ \ \forall\,(u,v) \in \partial B_{R_2} \cap \mathcal{Q}_0. \tag{24}$$

By using Lemma 5, the relations (22), (24), and the Guo–Krasnosel'skii fixed point theorem, we deduce that $\mathcal{A}$ has a fixed point $(u,v) \in (\overline{B}_{R_2} \setminus B_{R_1}) \cap \mathcal{Q}_0$, that is $R_1 \leq \|(u,v)\|_{\mathcal{Y}} \leq R_2$, and $u(t) \geq t^{\beta_1-1}\|u\|$ and $v(t) \geq t^{\beta_2-1}\|v\|$ for all $t \in [0,1]$. Then $\|u\| > 0$ or $\|v\| > 0$, that is $u(t) > 0$ for all $t \in (0,1]$ or $v(t) > 0$ for all $t \in (0,1]$. Hence $(u(t),v(t)),\ t \in [0,1]$ is a positive solution of problem (1) and (2). □

Remark 2. *Theorem 1 remains valid if the functions $\chi_1,\ \chi_2$ and f satisfy the inequalities (21) and (23), instead of $(I3)$ and $(I4)$.*

Theorem 2. *Assume that $(I1)$ and $(I2)$ hold. In addition the functions $\chi_1,\ \chi_2,\ f$ and g satisfy the conditions*

$(I5)$
$$\chi_{1\infty} = \lim_{\substack{x+y\to\infty\\ x,y\geq 0}} \sup_{t\in[0,1]} \frac{\chi_1(t,x,y)}{\varphi_{r_1}(x+y)} = 0 \ \ \textit{and} \ \ \chi_{2\infty} = \lim_{\substack{x+y\to\infty\\ x,y\geq 0}} \sup_{t\in[0,1]} \frac{\chi_2(t,x,y)}{\varphi_{r_2}(x+y)} = 0;$$

$(I6)$ *There exist $[a_1,a_2] \subset [0,1]$, $0 < a_1 < a_2 < 1$, $\nu_1 \in (0,1]$ and $\nu_2 \in (0,1]$ such that*

$$f_0^i = \lim_{\substack{x+y\to 0\\ x,y\geq 0}} \inf_{t\in[a_1,a_2]} \frac{f(t,x,y)}{\varphi_{r_1}((x+y)^{\nu_1})} = \infty \ \ \textit{or} \ \ g_0^i = \lim_{\substack{x+y\to 0\\ x,y\geq 0}} \inf_{t\in[a_1,a_2]} \frac{g(t,x,y)}{\varphi_{r_2}((x+y)^{\nu_2})} = \infty.$$

Then problem (1) and (2) has at least one positive solution $(u(t),v(t)),\ t \in [0,1]$.

Proof. We consider again the cone $\mathcal{Q}_0$. By $(I5)$ we deduce that for $0 < \epsilon_4 < \min\left\{\frac{1}{(2M_1)^{r_1-1}}, \frac{1}{2M_1^{r_1-1}}\right\}$, $0 < \epsilon_5 < \min\left\{\frac{1}{(2M_2)^{r_2-1}}, \frac{1}{2M_2^{r_2-1}}\right\}$, there exist $C_3 > 0$, $C_4 > 0$ such that

$$\chi_1(t,x,y) \leq \epsilon_4(x+y)^{r_1-1} + C_3,\ \ \chi_2(t,x,y) \leq \epsilon_5(x+y)^{r_2-1} + C_4,\ \ \forall\, t \in [0,1],\ \ x,y \geq 0. \tag{25}$$

By using $(I2)$ and (25), for any $(u,v) \in \mathcal{Q}_0$, we obtain

$$\begin{aligned}
&\mathcal{A}_1(u,v)(t) \leq \int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\zeta_1(s)\chi_1(s,u(s),v(s))))\,ds \\
&\leq \int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\zeta_1(s)(\epsilon_4(u(s)+v(s))^{r_1-1} + C_3)))\,ds \\
&\leq \int_0^1 \mathcal{J}_1(s)\frac{1}{(\Gamma(\alpha_1))^{\varrho_1-1}}\left(\epsilon_4\|(u,v)\|_{\mathcal{Y}}^{r_1-1} + C_3\right)^{\varrho_1-1}\left(\int_0^s (s-\tau)^{\alpha_1-1}\zeta_1(\tau)\,d\tau\right)^{\varrho_1-1} ds \\
&= M_1\left(\epsilon_4\|(u,v)\|_{\mathcal{Y}}^{r_1-1} + C_3\right)^{\varrho_1-1}, \ \ \forall\, t \in [0,1], \\
&\mathcal{A}_2(u,v)(t) \leq \int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}(\zeta_2(s)\chi_2(s,u(s),v(s))))\,ds \\
&\leq \int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}(\zeta_2(s)(\epsilon_5(u(s)+v(s))^{r_2-1} + C_4)))\,ds \\
&\leq \int_0^1 \mathcal{J}_2(s)\frac{1}{(\Gamma(\alpha_2))^{\varrho_2-1}}\left(\epsilon_5\|(u,v)\|_{\mathcal{Y}}^{r_2-1} + C_4\right)^{\varrho_2-1}\left(\int_0^s (s-\tau)^{\alpha_2-1}\zeta_2(\tau)\,d\tau\right)^{\varrho_2-1} ds \\
&= M_2\left(\epsilon_5\|(u,v)\|_{\mathcal{Y}}^{r_2-1} + C_4\right)^{\varrho_2-1}, \ \ \forall\, t \in [0,1].
\end{aligned}$$

Then we find

$$\|\mathcal{A}_1(u,v)\| \le M_1 \left(\epsilon_4 \|(u,v)\|_{\mathcal{Y}}^{r_1-1} + C_3\right)^{\varrho_1-1},$$
$$\|\mathcal{A}_2(u,v)\| \le M_2 \left(\epsilon_5 \|(u,v)\|_{\mathcal{Y}}^{r_2-1} + C_4\right)^{\varrho_2-1},$$

and so

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \le M_1 \left(\epsilon_4 \|(u,v)\|_{\mathcal{Y}}^{r_1-1} + C_3\right)^{\varrho_1-1} + M_2 \left(\epsilon_5 \|(u,v)\|_{\mathcal{Y}}^{r_2-1} + C_4\right)^{\varrho_2-1},$$

for all $(u,v) \in \mathcal{Q}_0$. We choose

$$R_3 > \max\left\{1, \frac{M_1 C_3^{\varrho_1-1} + M_2 C_4^{\varrho_2-1}}{1-\left(M_1 \epsilon_4^{\varrho_1-1} + M_2 \epsilon_5^{\varrho_2-1}\right)}, \frac{M_1 2^{\varrho_1-2} C_3^{\varrho_1-1} + M_2 2^{\varrho_2-2} C_4^{\varrho_2-1}}{1-\left(M_1 2^{\varrho_1-2}\epsilon_4^{\varrho_1-1} + M_2 2^{\varrho_2-2}\epsilon_5^{\varrho_2-1}\right)}, \frac{M_1 C_3^{\varrho_1-1} + M_2 2^{\varrho_2-2} C_4^{\varrho_2-1}}{1-\left(M_1 \epsilon_4^{\varrho_1-1} + M_2 2^{\varrho_2-2}\epsilon_5^{\varrho_2-1}\right)}, \frac{M_1 2^{\varrho_1-2} C_3^{\varrho_1-1} + M_2 C_4^{\varrho_2-1}}{1-\left(M_1 2^{\varrho_1-2}\epsilon_4^{\varrho_1-1} + M_2 \epsilon_5^{\varrho_2-1}\right)}\right\}, \tag{26}$$

and then we deduce

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \le \|(u,v)\|_{\mathcal{Y}}, \ \ \forall\,(u,v) \in \partial B_{R_3} \cap \mathcal{Q}_0. \tag{27}$$

The choosing of R_3 above is based on the inequalities $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for $p \ge 1$ and $a,b \ge 0$, and $(a+b)^p \le a^p + b^p$ for $p \in (0,1]$ and $a,b \ge 0$. Here $p = \varrho_1 - 1$ or $\varrho_2 - 1$. We explain the above inequality (27) in one case, namely $\varrho_1 \ge 2$ and $\varrho_2 \ge 2$. In this situation, by using (26), and the relations $M_1 2^{\varrho_1-2}\epsilon_4^{\varrho_1-1} < \frac{1}{2}$, $M_2 2^{\varrho_2-2}\epsilon_5^{\varrho_2-1} < \frac{1}{2}$ (from the definition of ϵ_4 and ϵ_5), we have the inequalities

$$\begin{aligned}
&M_1(\epsilon_4 R_3^{r_1-1} + C_3)^{\varrho_1-1} + M_2(\epsilon_5 R_3^{r_2-1} + C_4)^{\varrho_2-1} \\
&\le M_1 2^{\varrho_1-2}(\epsilon_4^{\varrho_1-1} R_3 + C_3^{\varrho_1-1}) + M_2 2^{\varrho_2-2}(\epsilon_5^{\varrho_2-1} R_3 + C_4^{\varrho_2-1}) \\
&= (M_1 2^{\varrho_1-2}\epsilon_4^{\varrho_1-1} + M_2 2^{\varrho_2-2}\epsilon_5^{\varrho_2-1}) R_3 + M_1 2^{\varrho_1-2} C_3^{\varrho_1-1} + M_2 2^{\varrho_2-1} C_4^{\varrho_2-1} < R_3.
\end{aligned}$$

In a similar manner we treat the cases: $\varrho_1 \in (1,2]$ and $\varrho_2 \in (1,2]$; $\varrho_1 \ge 2$ and $\varrho_2 \in (1,2]$; $\varrho_1 \in (1,2]$ and $\varrho_2 \ge 2$.

By $(I6)$, we suppose that $g_0^i = \infty$ (in a similar manner we can study the case $f_0^i = \infty$). We deduce that for $\epsilon_6 = (\min\{a_1^{\beta_1-1}, a_1^{\beta_2-1}\})^{\nu_2(1-r_2)} \widetilde{A}^{1-r_2}$, where $\widetilde{A} = \frac{a_1^{\beta_2-1}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{a_1}^{a_2} (s-a_1)^{\alpha_2(\varrho_2-1)} \mathcal{J}_2(s)\,ds$, there exists $R_4 \in (0,1]$ such that

$$g(t,x,y) \ge \epsilon_6 (x+y)^{\nu_2(r_2-1)}, \ \ \forall\, t \in [a_1,a_2], \ x,y \ge 0, \ x+y \le R_4. \tag{28}$$

Then by using (28), for any $(u,v) \in \partial B_{R_4} \cap \mathcal{Q}_0$ and $t \in [a_1,a_2]$, we find

$$\begin{aligned}
&\mathcal{A}_2(u,v)(t) \ge \int_{a_1}^{a_2} \mathcal{G}_2(t,s) \left(\frac{1}{\Gamma(\alpha_2)} \int_{a_1}^{s} (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,d\tau\right)^{\varrho_2-1} ds \\
&\ge a_1^{\beta_2-1} \int_{a_1}^{a_2} \mathcal{J}_2(s) \frac{1}{(\Gamma(\alpha_2))^{\varrho_2-1}} \left(\int_{a_1}^{s} (s-\tau)^{\alpha_2-1} \epsilon_6 (u(\tau)+v(\tau))^{\nu_2(r_2-1)} d\tau\right)^{\varrho_2-1} ds \\
&\ge a_1^{\beta_2-1} \int_{a_1}^{a_2} \mathcal{J}_2(s) \frac{1}{(\Gamma(\alpha_2))^{\varrho_2-1}} \left(\int_{a_1}^{s} (s-\tau)^{\alpha_2-1} \epsilon_6 (a_1^{\beta_1-1}\|u\| + a_1^{\beta_2-1}\|v\|)^{\nu_2(r_2-1)} d\tau\right)^{\varrho_2-1} ds \\
&\ge a_1^{\beta_2-1} \epsilon_6^{\varrho_2-1} \left(\min\{a_1^{\beta_1-1}, a_1^{\beta_2-1}\}\right)^{\nu_2} \frac{\|(u,v)\|_{\mathcal{Y}}^{\nu_2}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{a_1}^{a_2} (s-a_1)^{\alpha_2(\varrho_2-1)} \mathcal{J}_2(s)\,ds \\
&= \|(u,v)\|_{\mathcal{Y}}^{\nu_2} \ge \|(u,v)\|_{\mathcal{Y}}.
\end{aligned}$$

Therefore $\|\mathcal{A}_2(u,v)\| \ge \|(u,v)\|_{\mathcal{Y}}$ for all $(u,v) \in \partial B_{R_4} \cap \mathcal{Q}_0$, and then

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \ge \|\mathcal{A}_2(u,v)\| \ge \|(u,v)\|_{\mathcal{Y}}, \ \ \forall\,(u,v) \in \partial B_{R_4} \cap \mathcal{Q}_0. \tag{29}$$

By Lemma 5, the relations (27), (29), and the Guo–Krasnosl'skii fixed point theorem, we conclude that $\mathcal{A}$ has at least one fixed point $(u,v) \in (\overline{B}_{R_3} \setminus B_{R_4}) \cap \mathcal{Q}_0$, that is $R_4 \le \|(u,v)\|_{\mathcal{Y}} \le R_3$, which is a positive solution of problem (1) and (2). □

Remark 3. *Theorem 2 remains valid if the functions χ_1, χ_2 and g satisfy the inequalities (25) and (28), instead of $(I5)$ and $(I6)$.*

Theorem 3. *Assume that $(I1)$, $(I2)$, $(I4)$ and $(I6)$ hold. In addition, the functions χ_1 and χ_2 satisfy the condition*

$(I7)$ $D_0^{\varrho_1-1}M_1 < \frac{1}{2}$, $D_0^{\varrho_2-1}M_2 < \frac{1}{2}$, *where*

$$D_0 = \max\{\max_{t,x,y\in[0,1]} \chi_1(t,x,y), \max_{t,x,y\in[0,1]} \chi_2(t,x,y)\}.$$

Then problem (1) and (2) has at least two positive solutions $(u_1(t),v_1(t))$, $(u_2(t),v_2(t))$, $t \in [0,1]$.

Proof. We consider the operators $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}$, and the cone $\mathcal{Q}_0$ defined in this section. If $(I1)$, $(I2)$ and $(I4)$ hold, then by the proof of Theorem 1, we deduce that there exists $R_2 > 1$ (we can consider $R_2 > 1$) such that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \ge \|(u,v)\|_{\mathcal{Y}}, \ \forall (u,v) \in \partial B_{R_2} \cap \mathcal{Q}_0. \tag{30}$$

If $(I1)$, $(I2)$ and $(I6)$ hold, then by the proof of Theorem 2 we find that there exists $R_4 < 1$ (we can consider $R_4 < 1$) such that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \ge \|(u,v)\|_{\mathcal{Y}}, \ \forall (u,v) \in \partial B_{R_4} \cap \mathcal{Q}_0. \tag{31}$$

We consider now the set $B_1 = \{(u,v) \in \mathcal{Y}, \ \|(u,v)\|_{\mathcal{Y}} < 1\}$. By $(I7)$, for any $(u,v) \in \partial B_1 \cap \mathcal{Q}_0$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
\mathcal{A}_i(u,v)(t) &\le \int_0^1 \mathcal{J}_i(s) \left(\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i-1} \zeta_i(\tau) \chi_i(\tau, u(\tau), v(\tau))\, d\tau \right)^{\varrho_i-1} ds \\
&\le D_0^{\varrho_i-1} \int_0^1 \mathcal{J}_i(s) \left(\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i-1} \zeta_i(\tau)\, d\tau \right)^{\varrho_i-1} ds \\
&= D_0^{\varrho_i-1} \int_0^1 \mathcal{J}_i(s) \varphi_{\varrho_i}(I_{0+}^{\alpha_i} \zeta_i(s))\, ds = D_0^{\varrho_i-1} M_i < \frac{1}{2}, \ i = 1,2.
\end{aligned}$$

So $\|\mathcal{A}_i(u,v)\| < \frac{1}{2}$, for all $(u,v) \in \partial B_1 \cap \mathcal{Q}_0$, $i = 1,2$. Then

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} = \|\mathcal{A}_1(u,v)\| + \|\mathcal{A}_2(u,v)\| < 1 = \|(u,v)\|_{\mathcal{Y}}, \ \forall (u,v) \in \partial B_1 \cap \mathcal{Q}_0. \tag{32}$$

Therefore, by (30), (32) and the Guo–Krasnosel'skii fixed point theorem, we conclude that problem (1), (2) has one positive solution $(u_1,v_1) \in \mathcal{Q}_0$ with $1 < \|(u_1,v_1)\|_{\mathcal{Y}} \le R_2$. By (31), (32) and the Guo–Krasnosel'skii fixed point theorem, we deduce that problem (1), (2) has another positive solution $(u_2,v_2) \in \mathcal{Q}_0$ with $R_4 \le \|(u_2,v_2)\|_{\mathcal{Y}} < 1$. Then problem (1) and (2) has at least two positive solutions $(u_1(t),v_1(t))$, $(u_2(t),v_2(t))$, $t \in [0,1]$. □

Remark 4. *Theorem 3 remains valid if the functions f and g satisfy the inequalities (23) and (28), instead of $(I4)$ and $(I6)$.*

4. Examples

Let $\alpha_1 = 1/3$, $\alpha_2 = 1/2$, $\beta_1 = 5/2$, $(n = 3)$, $\beta_2 = 13/4$, $(m = 4)$, $r_1 = 4$, $\varrho_1 = 4/3$, $r_2 = 5$, $\varrho_2 = 5/4$, $p = 2$, $q = 1$, $\gamma_0 = 4/3$, $\gamma_1 = 1/4$, $\gamma_2 = 6/5$, $\delta_0 = 11/5$, $\delta_1 = 7/6$, $H_1(t) = t/3$ for all $t \in [0,1]$, $H_2(t) = \{1/6, \ t \in [0,2/3); 2/3, \ t \in [2/3,1]\}$, $K_1(t) = \{1/4, \ t \in [0,1/3); 9/4, \ t \in [1/3,1]\}$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{1/3}(\varphi_4(D_{0+}^{5/2}u(t))) + f(t,u(t),v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{1/2}(\varphi_5(D_{0+}^{13/4}v(t))) + g(t,u(t),v(t)) = 0, \ \ t \in (0,1), \end{cases} \tag{33}$$

with the nonlocal boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \ \ D_{0+}^{5/2}u(0) = 0, \ \ D_{0+}^{4/3}u(1) = \dfrac{1}{3}\displaystyle\int_0^1 D_{0+}^{1/4}u(t)\,dt + \dfrac{1}{2}D_{0+}^{6/5}u\left(\dfrac{2}{3}\right), \\ v(0) = v'(0) = v''(0) = 0, \ \ D_{0+}^{13/4}v(0) = 0, \ \ D_{0+}^{11/5}v(1) = 2D_{0+}^{7/6}v\left(\dfrac{1}{3}\right). \end{cases} \tag{34}$$

We obtain here $\Delta_1 \approx 0.60331103 > 0$ and $\Delta_2 \approx 1.12479609 > 0$. We also find

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(5/2)}\begin{cases} t^{3/2}(1-s)^{1/6} - (t-s)^{3/2}, \ \ 0 \le s \le t \le 1, \\ t^{3/2}(1-s)^{1/6}, \ \ 0 \le t \le s \le 1, \end{cases} \\
g_{21}(t,s) &= \frac{1}{\Gamma(9/4)}\begin{cases} t^{5/4}(1-s)^{1/6} - (t-s)^{5/4}, \ \ 0 \le s \le t \le 1, \\ t^{5/4}(1-s)^{1/6}, \ \ 0 \le t \le s \le 1, \end{cases} \\
g_{22}(t,s) &= \frac{1}{\Gamma(13/10)}\begin{cases} t^{3/10}(1-s)^{1/6} - (t-s)^{3/10}, \ \ 0 \le s \le t \le 1, \\ t^{3/10}(1-s)^{1/6}, \ \ 0 \le t \le s \le 1, \end{cases} \\
g_3(t,s) &= \frac{1}{\Gamma(13/4)}\begin{cases} t^{9/4}(1-s)^{1/20} - (t-s)^{9/4}, \ \ 0 \le s \le t \le 1, \\ t^{9/4}(1-s)^{1/20}, \ \ 0 \le t \le s \le 1, \end{cases} \\
g_{41}(t,s) &= \frac{1}{\Gamma(25/12)}\begin{cases} t^{13/12}(1-s)^{1/20} - (t-s)^{13/12}, \ \ 0 \le s \le t \le 1, \\ t^{13/12}(1-s)^{1/20}, \ \ 0 \le t \le s \le 1, \end{cases} \\
\mathcal{G}_1(t,s) &= g_1(t,s) + \frac{t^{3/2}}{\Delta_1}\left[\frac{1}{3}\int_0^1 g_{21}(\tau,s)\,d\tau + \frac{1}{2}g_{22}\left(\frac{2}{3},s\right)\right], \ \ t,s \in [0,1], \\
\mathcal{G}_2(t,s) &= g_3(t,s) + \frac{2t^{9/4}}{\Delta_2}g_{41}\left(\frac{1}{3},s\right), \ \ t,s \in [0,1], \\
h_1(s) &= \frac{1}{\Gamma(5/2)}[(1-s)^{1/6} - (1-s)^{3/2}], \ \ s \in [0,1], \\
h_2(s) &= \frac{1}{\Gamma(13/4)}[(1-s)^{1/20} - (1-s)^{9/4}], \ \ s \in [0,1].
\end{aligned}$$

In addition we deduce

$$\begin{aligned}
\mathcal{J}_1(s) &= \begin{cases} h_1(s) + \frac{1}{\Delta_1}\Big\{\frac{4}{27\Gamma(9/4)}(1-s)^{1/6} - \frac{4}{27\Gamma(9/4)}(1-s)^{9/4} + \frac{1}{2\Gamma(13/10)} \\ \quad \times\left[\left(\frac{2}{3}\right)^{3/10}(1-s)^{1/6} - \left(\frac{2}{3}-s\right)^{3/10}\right]\Big\}, \ \ 0 \le s < \frac{2}{3}, \\ h_1(s) + \frac{1}{\Delta_1}\Big[\frac{4}{27\Gamma(9/4)}(1-s)^{1/6} - \frac{4}{27\Gamma(9/4)}(1-s)^{9/4} + \frac{1}{2\Gamma(13/10)} \\ \quad \times\left(\frac{2}{3}\right)^{3/10}(1-s)^{1/6}\Big], \ \ 2/3 \le s \le 1, \end{cases} \\
\mathcal{J}_2(s) &= \begin{cases} h_2(s) + \frac{2}{\Delta_2\Gamma(25/12)}\left[\left(\frac{1}{3}\right)^{13/12}(1-s)^{1/20} - \left(\frac{1}{3}-s\right)^{13/12}\right], \ \ 0 \le s < \frac{1}{3}, \\ h_2(s) + \frac{2}{\Delta_2\Gamma(25/12)}\left(\frac{1}{3}\right)^{13/12}(1-s)^{1/20}, \ \ \frac{1}{3} \le s \le 1. \end{cases}
\end{aligned}$$

Example 1. *We consider the functions*

$$f(t,x,y) = \frac{(x+y)^{3a}}{t^{\eta_1}(1-t)^{\eta_2}}, \ \ g(t,x,y) = \frac{(x+y)^{4b}}{t^{\eta_3}(1-t)^{\eta_4}}, \ \ t \in (0,1), \ \ x,y \ge 0, \tag{35}$$

where $a > 1$, $b > 1$, $\eta_1, \eta_2 \in (0,1/4)$, $\eta_3, \eta_4 \in (0,1/3)$. Here $f(t,x,y) = \zeta_1(t)\chi_1(t,x,y)$, $g(t,x,y) = \zeta_2(t)\chi_2(t,x,y)$, $\zeta_1(t) = \frac{1}{t^{\eta_1}(1-t)^{\eta_2}}$, $\zeta_2(t) = \frac{1}{t^{\eta_3}(1-t)^{\eta_4}}$ for all $t \in (0,1)$, $\chi_1(t,x,y) = (x+y)^{3a}$, $\chi_2(t,x,y) = (x+y)^{4b}$ for all $t \in [0,1]$, $x,y \ge 0$. By using the Hölder inequality, we obtain

$$
\begin{aligned}
0 < \Lambda_1 &= \int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = \int_0^1 (1-s)^{1/6}\left(I_{0+}^{1/3}\zeta_1(s)\right)^{1/3} ds \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 (1-s)^{1/6}\left(\int_0^s (s-\tau)^{-2/3}\frac{1}{\tau^{\eta_1}(1-\tau)^{\eta_2}}\,d\tau\right)^{1/3} ds \\
&\le \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 (1-s)^{1/6}\left[\left(\int_0^s (s-\tau)^{-8/9}d\tau\right)^{3/4}\left(\int_0^s\left(\frac{1}{\tau^{\eta_1}(1-\tau)^{\eta_2}}\right)^4 d\tau\right)^{1/4}\right]^{1/3} ds \\
&\le \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 (1-s)^{1/6}\left[(9s^{1/9})^{3/4}(B(1-4\eta_1,1-4\eta_2))^{1/4}\right]^{1/3} ds \\
&= \frac{3^{1/2}}{(\Gamma(1/3))^{1/3}}(B(1-4\eta_1,1-4\eta_2))^{1/12}B\left(\frac{37}{36},\frac{7}{6}\right) < \infty, \\
0 < \Lambda_2 &= \int_0^1 (1-s)^{\beta_2-\delta_0-1}\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds = \int_0^1 (1-s)^{1/20}\left(I_{0+}^{1/2}\zeta_2(s)\right)^{1/4} ds \\
&= \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left(\int_0^s (s-\tau)^{-1/2}\frac{1}{\tau^{\eta_3}(1-\tau)^{\eta_4}}d\tau\right)^{1/4} ds \\
&\le \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left[\left(\int_0^s (s-\tau)^{-3/4}d\tau\right)^{2/3}\left(\int_0^s \tau^{-3\eta_3}(1-\tau)^{-3\eta_4}d\tau\right)^{1/3}\right]^{1/4} ds \\
&\le \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}[(4s^{1/4})^{2/3}(B(1-3\eta_3,1-3\eta_4))^{1/3}]^{1/4}\,ds \\
&= \frac{2^{1/3}}{(\Gamma(1/2))^{1/4}}(B(1-3\eta_3,1-3\eta_4))^{1/12}B\left(\frac{25}{24},\frac{21}{20}\right) < \infty.
\end{aligned}
$$

Hence assumptions $(I1)$ and $(I2)$ are satisfied.

In addition, in $(I3)$, for $\mu_1=\mu_2=1$, we obtain $\chi_{10}=\chi_{20}=0$, and in $(I4)$ for $[a_1,a_2]\subset(0,1)$ we have $f_\infty^i=\infty$ (and $g_\infty^i=\infty$). Then by Theorem 1, we conclude that problem (33) and (34) with the nonlinearities (35) has at least one positive solution $(u(t),v(t)),\ t\in[0,1]$.

Example 2. *We consider the functions*

$$
\begin{aligned}
f(t,x,y) &= \frac{c_0(t+1)}{(t^2+4)\sqrt[5]{t}}[(x+y)^{\sigma_1}+(x+y)^{\sigma_2}],\ \ t\in(0,1],\ \ x,y\ge 0, \\
g(t,x,y) &= \frac{d_0(2+\sin t)}{(t+3)^4\sqrt[4]{1-t}}(x^{\sigma_3}+y^{\sigma_4}),\ \ t\in[0,1),\ \ x,y\ge 0,
\end{aligned}
\tag{36}
$$

where $c_0>0,\ d_0>0,\ \sigma_1>3,\ \sigma_2\in(0,3),\ \sigma_3>0,\ \sigma_4>0.$ *Here we have* $\zeta_1(t)=\frac{1}{\sqrt[5]{t}},\ t\in(0,1],$ $\chi_1(t,x,y)=\frac{c_0(t+1)}{t^2+4}[(x+y)^{\sigma_1}+(x+y)^{\sigma_2}],\ t\in[0,1],\ x,y\ge 0,\ \zeta_2(t)=\frac{1}{\sqrt[4]{1-t}},\ t\in[0,1),\ \chi_2(t,x,y)=$ $\frac{d_0(2+\sin t)}{(t+3)^4}(x^{\sigma_3}+y^{\sigma_4}),\ t\in[0,1],\ x,y\ge 0.$ *By using a computer program, we obtain*

$$
\begin{aligned}
\Lambda_1 &= \int_0^1 (1-s)^{\beta_1-\gamma_0-1}\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = \int_0^1 (1-s)^{1/6}(I_{0+}^{1/3}\zeta_1(s))^{1/3}\,ds \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 (1-s)^{1/6}\left(\int_0^s \tau^{-1/5}(s-\tau)^{-2/3}\,d\tau\right)^{1/3} ds \\
&\overset{\tau=sx}{=} \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 (1-s)^{1/6}\left(\int_0^1 (sx)^{-1/5}(s-sx)^{-2/3}s\,dx\right)^{1/3} ds \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}\left(\int_0^1 s^{2/45}(1-s)^{1/6}\,ds\right)\left(\int_0^1 x^{-1/5}(1-x)^{-2/3}dx\right)^{1/3} \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}B\left(\frac{47}{45},\frac{7}{6}\right)\left(B\left(\frac{4}{5},\frac{1}{3}\right)\right)^{1/3} \approx 0.877777,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2 &= \int_0^1 (1-s)^{\beta_2-\delta_0-1}\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds = \int_0^1 (1-s)^{1/20}(I_{0+}^{1/2}\zeta_2(s))^{1/4}\,ds \\
&= \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left(\int_0^s (s-\tau)^{-1/2}(1-\tau)^{-1/4}d\tau\right)^{1/4} ds \\
&\overset{\tau=sx}{=} \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left(\int_0^1 (s-sx)^{-1/2}(1-sx)^{-1/4}s\,dx\right)^{1/4} ds \\
&= \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left(s^{1/2}\int_0^1 (1-x)^{-1/2}(1-sx)^{-1/4}\,dx\right)^{1/4} ds \\
&= \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 (1-s)^{1/20}\left(s^{1/2}\sqrt{\pi}\,{}_2F_1\left[\frac{1}{4},1,\frac{3}{2},s\right]\right)^{1/4} ds \approx 0.901313,
\end{aligned}
$$

where ${}_2F_1[a,b,c,z] = \frac{1}{\Gamma(b)\Gamma(c-b)}\int_0^1 s^{b-1}(1-s)^{c-b-1}(1-sz)^{-a}\,ds$ *is the regularized hypergeometric function. So* $\Lambda_i \in (0,\infty),\ i=1,2$, *and then assumptions* $(I1)$ *and* $(I2)$ *are satisfied.*

For $[a_1,a_2]\subset(0,1)$, we find $f_\infty^i=\infty$, and if we consider $0<\nu_1\le 1,\ 3\nu_1>\sigma_2$ we obtain $f_0^i=\infty$, and then assumptions $(I4)$ and $(I6)$ are satisfied. After some computations we deduce

$$
\begin{aligned}
M_1 &= \int_0^1 \mathcal{J}_1(s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\zeta_1(s))\,ds = \int_0^1 \mathcal{J}_1(s)\varphi_{4/3}\left(I_{0+}^{1/3}\frac{1}{\sqrt[5]{s}}\right)ds \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 \mathcal{J}_1(s)\left(\int_0^s (s-\tau)^{-2/3}\tau^{-1/5}d\tau\right)^{1/3} ds \\
&= \frac{1}{(\Gamma(1/3))^{1/3}}\int_0^1 \mathcal{J}_1(s)\left(s^{2/15}B\left(\frac{4}{5},\frac{1}{3}\right)\right)^{1/3} ds \approx 0.78160052, \\
M_2 &= \int_0^1 \mathcal{J}_2(s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\zeta_2(s))\,ds = \int_0^1 \mathcal{J}_2(s)\varphi_{5/4}(I_{0+}^{1/2}\zeta_2(s))\,ds \\
&= \int_0^1 \mathcal{J}_2(s)\left(\frac{1}{\Gamma(1/2)}\int_0^s (s-\tau)^{-1/2}(1-\tau)^{-1/4}d\tau\right)^{1/4} ds \\
&= \frac{1}{(\Gamma(1/2))^{1/4}}\int_0^1 \mathcal{J}_2(s)\left(s^{1/2}\sqrt{\pi}\,{}_2F_1\left[\frac{1}{4},1,\frac{3}{2},s\right]\right)^{1/4} ds \approx 0.65997289.
\end{aligned}
$$

In addition, we find $D_0=\max\left\{\frac{2c_0}{5}(2^{\sigma_1}+2^{\sigma_2}),\frac{4d_0}{81}\right\}$. If $c_0<\min\left\{\frac{5}{16M_1^3(2^{\sigma_1}+2^{\sigma_2})},\frac{5}{32M_2^4(2^{\sigma_1}+2^{\sigma_2})}\right\}$ and $d_0<\min\left\{\frac{81}{32M_1^3},\frac{81}{64M_2^4}\right\}$, then the inequalities $D_0^{1/3}M_1<\frac{1}{2}$ and $D_0^{1/4}M_2<\frac{1}{2}$ are satisfied (that is, assumption $(I7)$ is satisfied). For example, if $\sigma_1=4$ and $\sigma_2=2$, and $c_0\le 0.032$ and $d_0\le 5.301$, then the above inequalities are satisfied. By Theorem 3, we conclude that problem (33) and (34) with the nonlinearities (36) has at least two positive solutions $(u_1(t),v_1(t)),\ (u_2(t),v_2(t)),\ t\in[0,1]$.

5. Conclusions

In this paper, we have discussed the existence and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations with singular nonnegative nonlinearities and p-Laplacian operators, complemented with nonlocal boundary conditions involving fractional derivatives and Riemann–Stieltjes integrals. Some properties of the associated Green functions are also presented. Two examples are constructed for the illustration of the obtained results.

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