mathematics

MDPI

Article

On Caputo–Riemann–Liouville Type Fractional Integro-Differential Equations with Multi-Point Sub-Strip Boundary Conditions

Ahmed Alsaedi [1], Amjad F. Albideewi [1], Sotiris K. Ntouyas [1,2] and Bashir Ahmad [1,*]

1 Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; aalsaedi@hotmail.com (A.A.); amjad.f.b@hotmail.com (A.F.A.); sntouyas@uoi.gr (S.K.N.)
2 Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
* Correspondence: bashirahmad_qau@yahoo.com or bahmad@kau.edu.sa

Received: 16 September 2020; Accepted: 26 October 2020; Published: 31 October 2020

Abstract: In this paper, we derive existence and uniqueness results for a nonlinear Caputo–Riemann–Liouville type fractional integro-differential boundary value problem with multi-point sub-strip boundary conditions, via Banach and Krasnosel'skiĭ's fixed point theorems. Examples are included for the illustration of the obtained results.

Keywords: Caputo derivative; Riemann–Liouville integral; multipoint and sub-strip boundary conditions; existence; fixed point theorem

MSC: 34A08; 34B10; 34B15

1. Introduction

The subject of fractional order boundary value problems has been addressed by many researchers in recent years. The interest in the subject owes to its extensive applications in natural and social sciences. Examples include bio-engineering [1], ecology [2], financial economics [3], chaos and fractional dynamics [4], etc. One can find many interesting results using boundary value problems dealing with Caputo, Riemann–Liouville and Hadamard type fractional derivatives and equipped with a variety of boundary conditions in [5–17].

Integro-differential equations constitute an important area of investigation due to their occurrence in several applied fields, such as heat transfer phenomena [18,19], fractional power law [20], etc. Fractional integro-differential equations complemented with different kinds of boundary conditions have also been studied by many researchers, for example, [21–28]. In a recent paper [29], a nonlocal boundary value problem containing left Caputo and right Riemann–Liouville fractional derivatives, and both left and right Riemann–Liouville fractional integral operators was discussed.

Motivated by aforementioned work on integro-differential equations, we introduce and investigate a nonlinear Caputo–Riemann–Liouville type fractional integro-differential boundary value problem involving multi-point sub-strip boundary conditions given by

$$\begin{cases} {}^cD^q x(t) + \sum_{i=1}^{k} I^{p_i} g_i(t, x(t)) = f(t, x(t)),\ 0 < t < 1, \\ x(0) = a,\ \ x'(0) = 0,\ \ x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \\ \alpha x(1) + \beta x'(1) = \gamma_1 \int_0^{\zeta} x(s)ds + \sum_{j=1}^{p} \alpha_j\, x(\eta_j) + \gamma_2 \int_{\xi}^{1} x(s)ds, \end{cases} \tag{1}$$

where ${}^cD^q$ represents the Caputo fractional derivative operator of order $q \in (m-1, m], m \in \mathbb{N}, m \geq 2$, $p_i > 0$, $0 < \zeta, \eta_1, \eta_2, \ldots, \eta_p, \xi < 1$, $f, g_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $(i = 1, \ldots, k)$ are continuous functions $a, \alpha, \beta, \gamma_1, \gamma_2 \in \mathbb{R}$ and $\alpha_j \in \mathbb{R}$, $j = 1, 2, \ldots, p$. Notice that the fixed/nonlocal points involved in the problem (1) are non-singular.

We emphasize that the problem considered in this paper is novel in the sense that the fractional integro-differential equation involves many finitely Riemann–Liouville fractional integral type nonlinearities together with a non-integral nonlinearity. In the literature, one can find results on linear integro-differential equations [30], fractional integro-differential equations with nonlinearity depending on the linear integral terms [31,32], and initial value problems involving two nonlinear integral terms [33]. In contrast to the aforementioned work, our problem contains many finitely nonlinear integral terms of fractional order, which reduce to the nonlinear integral terms by fixing $p_i = 1, \forall i = 1, \ldots, k$. For specific applications of integral-differential equations in the mathematical modeling of physical problems such as the spreading of disease by the dispersal of infectious individuals, and reaction-–diffusion models in ecology, see [1,2]. In particular, one can find more details on the topic in [34] and the references cited therein. For some recent work on fractional integro-differential equations, see [35,36]. In a more recent work [37], the authors studied the existence of solutions for a fractional integro-differential equation supplemented with dual anti-periodic boundary conditions. Concerning the boundary condition at the terminal position $t = 1$, the linear combination of the unknown function and its derivative is associated with the contribution due to two sub-strips $(0, \zeta)$ and $(\xi, 1)$ and finitely many nonlocal positions between them within the domain $[0, 1]$. This boundary condition covers many interesting situations, for example, it corresponds to the two-strip aperture condition for all $\alpha_j = 0, j = 1, \ldots, p$. By taking $\gamma_1 = 0 = \gamma_2$, this condition takes the form of a multi-point nonlocal boundary condition. It is interesting to note that the role of integral boundary conditions in studying practical problems such as blood flow problems [38] and bacterial self-regularization [39], etc., is crucial. For the application of strip conditions in engineering and real world problems, see [40,41]. On the other hand, the concept of nonlocal boundary conditions plays a significant role when physical, chemical or other processes depend on the interior positions (non-fixed points or segments) of the domain, for instance, see [42–45] and the references therein.

The rest of the paper is arranged as follows. Section 2 contains some related concepts of fractional calculus and an auxiliary result concerning a linear version of the problem (1). We prove the existence and uniqueness of solutions for the problem (1) by applying Banach and Krasnosel'skiǐ's fixed point theorems in Section 3. Finally, examples illustrating the main results are demonstrated in Section 4.

2. Preliminaries

Let us first outline some preliminary concepts of fractional calculus [5].

Definition 1. *The Caputo derivative for a function $h \in AC^n[a, b]$ of fractional order $q \in (n-1, n], n \in \mathbb{N}$, existing almost everywhere on $[a, b]$, is defined by*

$$ {}^cD^q h(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-u)^{n-q-1} h^{(n)}(u) du, \quad t \in [a, b]. $$

Definition 2. *The Riemann–Liouville fractional integral for a function $h \in L_1[a, b]$ of order $r > 0$, which exists almost everywhere on $[a, b]$, is defined by*

$$ I^r h(t) = \frac{1}{\Gamma(r)} \int_a^t \frac{h(u)}{(t-u)^{1-r}} du, \ t \in [a, b]. $$

Lemma 1. *For $m - 1 < r \leq m$, the general solution of the fractional differential equation ${}^cD^r x(t) = 0$ can be written as*

$$ x(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_{m-1} t^{m-1}, \tag{2} $$

where $b_i \in \mathbb{R}, i = 0, 1, 2, \ldots, m-1$.

It follows by Lemma 1 that

$$I^{r\,c}D^{r}x(t) = x(t) + b_0 + b_1 t + b_2 t^2 + \ldots + b_{m-1}t^{m-1}, \tag{3}$$

for some $b_i \in \mathbb{R}, i = 0, 1, 2, \ldots, m-1$, are arbitrary constants.

The following lemma deals with the linear version of the problem (1).

Lemma 2. *For a given function $h \in C([0,1], \mathbb{R})$ the unique solution of the following boundary value problem*

$$\begin{cases} {}^{c}D^{q}x(t) = h(t),\ 0 < t < 1, \\ x(0) = a,\ \ x'(0) = 0,\ \ x''(0) = 0, \ldots, x^{(m-2)}(0) = 0, \\ \alpha x(1) + \beta x'(1) = \gamma_1 \int_0^{\zeta} x(s)ds + \sum_{j=1}^{p} \alpha_j\, x(\eta_j) + \gamma_2 \int_{\xi}^{1} x(s)ds, \end{cases} \tag{4}$$

is given by

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + a - \frac{t^{m-1}}{\Lambda_1}\Bigg[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)ds \\ & - \gamma_1 \int_0^{\zeta} \int_0^{s} \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds - \sum_{j=1}^{p} \alpha_j \int_0^{\eta_j} \frac{(\eta_j - s)^{q-1}}{\Gamma(q)} h(s)ds \\ & - \gamma_2 \int_{\xi}^{1} \int_0^{s} \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds + \Lambda_2 \Bigg], \end{aligned} \tag{5}$$

where it is assumed that

$$\Lambda_1 = \left(\alpha + \beta(m-1) - \gamma_1 \left(\frac{\zeta^m}{m} \right) - \sum_{j=1}^{p} \alpha_j \eta_j^{m-1} - \gamma_2 \left(\frac{1-\xi^m}{m} \right) \right) \neq 0, \tag{6}$$

$$\Lambda_2 = a\Big[\alpha - \gamma_1 \zeta - \sum_{j=1}^{p} \alpha_j - \gamma_2(1-\xi)\Big]. \tag{7}$$

Proof. Using (3), we can write the general solution of the fractional differential equation in (4) as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds - c_0 - c_1 t - c_2 t^2 - \ldots - c_{m-1}t^{m-1}, \tag{8}$$

where $c_0, c_1, c_2, \ldots, c_{m-1} \in \mathbb{R}$ are arbitrary constants. From (8), we have

$$\begin{aligned} x'(t) &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)ds - c_1 - 2c_2 t - \ldots - (m-1)c_{m-1}t^{m-2}, \\ x''(t) &= \int_0^t \frac{(t-s)^{q-3}}{\Gamma(q-2)} h(s)ds - 2c_2 - \ldots - (m-1)(m-2)c_{m-1}t^{m-3}, \\ &\vdots \end{aligned} \tag{9}$$

Applying the conditions $x(0) = a, x'(0) = 0, \ldots, x^{(m-2)}(0) = 0$ in (8), it is found that $c_0 = -a$, $c_1 = 0, \ldots, c_{m-2} = 0$. Then (8) becomes

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + a - c_{m-1}t^{m-1}, \tag{10}$$

and

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)ds - (m-1)c_{m-1}t^{m-2}. \tag{11}$$

Combining (10) and (11) with the condition $\alpha x(1) + \beta x'(1) = \gamma_1 \int_0^{\zeta} x(s)ds + \sum_{j=1}^{p} \alpha_j\, x(\eta_j)$ $+ \gamma_2 \int_{\xi}^{1} x(s)ds$, we get

$$\begin{aligned} & \alpha \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds + a - c_{m-1} \right] + \beta \left[\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)ds - (m-1)c_{m-1} \right] \\ = \; & \gamma_1 \int_0^{\zeta} \left[\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)du + a - c_{m-1}s^{m-1} \right] ds + \sum_{j=1}^{p} \alpha_j \left[\int_0^{\eta_j} \frac{(\eta_j - s)^{q-1}}{\Gamma(q)} h(s)ds \right. \\ & \left. + a - c_{m-1}\eta_j^{m-1} \right] + \gamma_2 \int_{\xi}^{1} \left[\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)du + a - c_{m-1}s^{m-1} \right] ds, \end{aligned}$$

which, together with (6) and (7), yields

$$\begin{aligned} c_{m-1} = \; & \frac{1}{\Lambda_1} \left[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)ds \right. \\ & - \gamma_1 \int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds - \sum_{j=1}^{p} \alpha_j \int_0^{\eta_j} \frac{(\eta_j - s)^{q-1}}{\Gamma(q)} h(s)ds \\ & \left. - \gamma_2 \int_{\xi}^{1} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds + a\alpha - a\gamma_1\zeta - a\sum_{j=1}^{p} \alpha_j - a\gamma_2(1-\xi) \right]. \end{aligned}$$

Substituting the value of c_{m-1} in (10), we obtain

$$\begin{aligned} & x(t) \\ = \; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + a - \frac{t^{m-1}}{\Lambda_1} \left[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)ds \right. \\ & - \gamma_1 \int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds - \sum_{j=1}^{p} \alpha_j \int_0^{\eta_j} \frac{(\eta_i - s)^{q-1}}{\Gamma(q)} h(s)ds \\ & \left. - \gamma_2 \int_{\xi}^{1} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds + \Lambda_2 \right]. \end{aligned}$$

We can obtain the converse of this lemma by direct computation. This finishes the proof. □

Using Lemma 2, we can transform problem (1) into a fixed point problem as $x = \mathcal{F}x$, where the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\mathcal{F}x(t) \;=\; \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i(s, x(s))ds + a$$

$$
\begin{aligned}
&-\frac{t^{m-1}}{\Lambda_1}\left[\alpha\int_0^1\left(\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))-\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))\right)ds\right.\\
&+\beta\int_0^1\left(\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))-\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)}g_i(s,x(s))\right)ds\\
&-\gamma_1\int_0^{\zeta}\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))\,du\,ds \qquad (12)\\
&+\gamma_1\sum_{i=1}^{k}\int_0^{\zeta}\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^u\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}g_i(w,x(w))dw\,du\,ds\\
&-\sum_{j=1}^{p}\alpha_j\left(\int_0^{\eta_j}\frac{(\eta_j-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\sum_{i=1}^{k}\int_0^{\eta_j}\frac{(\eta_j-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))ds\right)\\
&-\gamma_2\int_{\xi}^1\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))\,du\,ds\\
&\left.+\gamma_2\sum_{i=1}^{k}\int_{\xi}^1\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^u\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}g_i(w,x(w))dw\,du\,ds+\Lambda_2\right].
\end{aligned}
$$

Here $\mathcal{C}$, represents the Banach space of all continuous functions $x:[0,1]\to\mathbb{R}$ equipped with the norm $\|x\|=\sup_{t\in[0,1]}|x(t)|$.

By a solution of (1), we mean a function $x\in\mathcal{C}$ of class $\mathcal{C}^m[0,1]$ satisfying the nonlocal integro-multipoint boundary value problem (1).

For computational convenience, we set

$$
\begin{aligned}
\Omega &= \left[\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+1)}+\frac{|\beta|}{\Gamma(q)}+\frac{|\gamma_1|\zeta^{q+1}}{\Gamma(q+2)}+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^q}{\Gamma(q+1)}\right.\right.\\
&\qquad\left.\left.+\frac{|\gamma_2||1-\xi^{q+1}|}{\Gamma(q+2)}\right)\right], \qquad (13)\\
\Omega_i &= \left[\frac{1}{\Gamma(q+p_i+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+p_i+1)}+\frac{|\beta|}{\Gamma(q+p_i)}+\frac{|\gamma_1|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}\right.\right.\\
&\qquad\left.\left.+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{|\gamma_2||1-\xi^{q+p_i+1}|}{\Gamma(q+p_i+2)}\right)\right],\quad i=1,2,\ldots,k. \qquad (14)
\end{aligned}
$$

3. Existence and Uniqueness Results

In the following theorem, we make use of Banach's fixed point theorem.

Theorem 1. *Let $f,g_i:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous functions and let there exist constants $L,L_i>0$, $(i=1,\ldots,k)$ such that:*

(A_1) $|f(t,x)-f(t,y)|\le L\,|x-y|$, and $|g_i(t,x)-g_i(t,y)|\le L_i\,|x-y|$ for all $t\in[0,1]$, $x,y\in\mathbb{R}$.

Then, the boundary value problem (1) has a unique solution on $[0,1]$ if

$$L\Omega+\sum_{i=1}^{k}L_i\Omega_i<1, \qquad (15)$$

where $\Omega,\Omega_i,i=1,2,\ldots,k$ are given by (13) and (14), respectively.

Proof. The proof will be given in two steps.

Step 1. We show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ with $r \ge \Big(M\Omega + \sum_{i=1}^{k} M_i\Omega_i + |a| + (\Lambda_2/\Lambda_1)\Big/\Big(1 - (L\Omega + \sum_{i=1}^{k} L_i\Omega_i)\Big)$ and M, M_i are positive numbers such that $M = \sup_{t\in[0,1]} |f(t,0)|$ and $M_i = \sup_{t\in[0,1]} |g_i(t,0)|\,, i = 1,2,\ldots,k$.

For $x \in B_r$ and $t \in [0,1]$, it follows by (A_1) that

$$|f(t,x(t))| \le |f(t,x(t)) - f(t,0)| + |f(t,0)| \le L\|x\| + M \le Lr + M. \tag{16}$$

In a similar manner, we have $|g_i(t,x(t))| \le L_i\, r + M_i,\ i = 1,2,\ldots,k$. Then

$$\begin{aligned}
&\|\mathcal{F}x\| \\
\le\ & \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s))| ds + |a| \\
& + \frac{t^{m-1}}{|\Lambda_1|} \Bigg[|\alpha| \int_0^1 \left(\frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s))| \right) ds \\
& + |\beta| \int_0^1 \left(\frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} |g_i(s,x(s))| \right) ds \\
& + |\gamma_1| \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u))|\, du\, ds \\
& + |\gamma_1| \sum_{i=1}^{k} \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} |g_i(w,x(w))|\, dw\, du\, ds \\
& + \sum_{j=1}^{p} |\alpha_j| \left(\int_0^{\eta_j} \frac{(\eta_j-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \sum_{i=1}^{k} \int_0^{\eta_j} \frac{(\eta_j-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s))| ds \right) \\
& + |\gamma_2| \int_\zeta^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u))|\, du\, ds \\
& + |\gamma_2| \sum_{i=1}^{k} \int_\zeta^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} |g_i(w,x(w))|\, dw\, du\, ds + |\Lambda_2| \Bigg] \Bigg\} \\
\le\ & (Lr+M) \Bigg[\sup_{t\in[0,1]} \Bigg\{ \frac{t^q}{\Gamma(q+1)} + \frac{t^{m-1}}{|\Lambda_1|} \Bigg(\frac{|\alpha|}{\Gamma(q+1)} + \frac{|\beta|}{\Gamma(q)} + \frac{|\gamma_1|\zeta^{q+1}}{\Gamma(q+2)} + \frac{\sum_{j=1}^{p} |\alpha_j| \eta_j^q}{\Gamma(q+1)} \\
& + \frac{|\gamma_2|(1-\zeta^{q+1})}{\Gamma(q+2)} \Bigg) \Bigg] + \sum_{i=1}^{k} (L_i r + M_i) \Bigg[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)} + \frac{t^{m-1}}{|\Lambda_1|} \Bigg(\frac{|\alpha|}{\Gamma(q+p_i+1)} \\
& + \frac{|\beta|}{\Gamma(q+p_i)} + \frac{|\gamma_1|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)} + \frac{\sum_{j=1}^{p} |\alpha_i| \eta_j^{q+p_i}}{\Gamma(q+p_i+1)} + \frac{|\gamma_2|(1-\zeta^{q+p_i+1})}{\Gamma(q+p_i+2)} \Bigg) \Bigg] \\
& + |a| + \frac{t^{m-1}|\Lambda_2|}{|\Lambda_1|} \Bigg\} \\
=\ & (Lr+M)\Omega + \sum_{i=1}^{k} (L_i r + M_i)\Omega_i + |a| + \left| \frac{\Lambda_2}{\Lambda_1} \right| \\
=\ & \Big(L\Omega + \sum_{i=1}^{k} L_i\Omega_i \Big) r + M\Omega + \sum_{i=1}^{k} M_i\Omega_i + |a| + \left| \frac{\Lambda_2}{\Lambda_1} \right| \le r,
\end{aligned}$$

which shows that $\mathcal{F}B_r \subset B_r$.

Step 2. We show that $\mathcal{F}$ is a contraction. For $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
& \|\mathcal{F}x - \mathcal{F}y\| \\
\leq\ & \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s)) - g_i(s,y(s))| ds \\
& + \frac{t^{m-1}}{|\Lambda_1|} \Bigg[|\alpha| \int_0^1 \Bigg(\frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| \\
& + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s)) - g_i(s,y(s))| \Bigg) ds \\
& + |\beta| \int_0^1 \Bigg(\frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s)) - f(s,y(s))| \\
& + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} |g_i(s,x(s)) - g_i(s,y(s))| \Bigg) ds \\
& + |\gamma_1| \int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))|\, du\, ds \\
& + |\gamma_1| \sum_{i=1}^{k} \int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} |g_i(w,x(w)) - g_i(w,y(w))| dw du ds \\
& + \sum_{j=1}^{p} |\alpha_j| \Bigg(\int_0^{\eta_j} \frac{(\eta_j-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
& + \sum_{i=1}^{k} \int_0^{\eta_i} \frac{(\eta_i-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s)) - g_i(x,y(s))| ds \Bigg) \\
& + |\gamma_2| \int_{\xi}^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))|\, du\, ds \\
& + |\gamma_2| \sum_{i=1}^{k} \int_{\xi}^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} |g_i(w,x(w)) - g_i(w,y(w))|\, dw\, du\, ds \Bigg] \Bigg\} \\
\leq\ & L \Bigg[\sup_{t\in[0,1]} \Bigg\{ \frac{t^q}{\Gamma(q+1)} + \frac{t^{m-1}}{|\Lambda_1|} \Bigg(\frac{|\alpha|}{\Gamma(q+1)} + \frac{|\beta|}{\Gamma(q)} + \frac{|\gamma_1|\zeta^{q+1}}{\Gamma(q+2)} + \frac{\sum_{j=1}^{p} |\alpha_j| \eta_j^q}{\Gamma(q+1)} \\
& + \frac{|\gamma_2|(1-\xi^{q+1})}{\Gamma(q+2)} \Bigg) \Bigg\} \Bigg] \|x-y\| + \sum_{i=1}^{k} L_i \Bigg[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)} + \frac{t^{m-1}}{|\Lambda_1|} \Bigg(\frac{|\alpha|}{\Gamma(q+p_i+1)} \\
& + \frac{|\beta|}{\Gamma(q+p_i)} + \frac{|\gamma_1|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)} + \frac{\sum_{j=1}^{p} |\alpha_j| \eta_j^{q+p_i}}{\Gamma(q+p_i+1)} + \frac{|\gamma_2|(1-\xi^{q+p_i+1})}{\Gamma(q+p_i+2)} \Bigg) \Bigg] \|x-y\| \\
\leq\ & \Big(L\Omega + \sum_{i=1}^{k} L_i \Omega_i \Big) \|x-y\|,
\end{aligned}$$

which, by the condition (15), implies that $\mathcal{F}$ is a contraction. Thus the conclusion of the Banach contraction mapping principle applies and hence the operator $\mathcal{F}$ has a unique fixed point. Therefore, there exists a unique solution for the boundary value problem (1) on $[0,1]$. □

Next, we prove an existence result for the boundary value problem (1), which relies on Krasnosel'skiĭ's fixed point theorem [46].

Theorem 2. *Let $f, g_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $(i = 1, \ldots, k)$ be continuous functions satisfying the condition (A_1). In addition, we assume that:*

(A_2) $|f(t,x)| \le \mu(t)$, $|g_i(t,x)| \le \mu_i(t)$, *for all* $(t,x) \in [0,1] \times \mathbb{R}$, $\mu, \mu_i \in C([0,1], \mathbb{R}^+)$.

Then, the boundary value problem (1) has at least one solution on $[0,1]$, *provided that*

$$L\left[\Omega - \left(\frac{1}{\Gamma(q+1)}\right)\right] + \sum_{i=1}^{k} L_i\left[\Omega_i - \left(\frac{1}{\Gamma(q+p_i+1)}\right)\right] < 1, \tag{17}$$

where Ω, $\Omega_i, i = 1, 2, \ldots, k$ *are given by (13) and (14), respectively.*

Proof. Consider $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$, $\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|$, $\|\mu_i\| = \sup_{t\in[0,1]} |\mu_i(t)|$, $i = 1, 2, \ldots, k$ with $\rho \ge \|\mu\|\Omega + \sum_{i=1}^{k} \|\mu_i\|\Omega_i + |a| + |\Lambda_2|/|\Lambda_1|$. Then, we define the operators Φ and Ψ on B_ρ as

$$\begin{aligned}
\Phi x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i(s,x(s))ds, \ t \in [0,1], \\
\Psi x(t) &= a - \frac{t^{m-1}}{\Lambda_1}\Bigg[\alpha \int_0^1 \left(\frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s)) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i(s,x(s))\right) ds \\
&\quad + \beta \int_0^1 \left(\frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s)) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} g_i(s,x(s))\right) ds \\
&\quad - \gamma_1 \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))\,du\,ds \\
&\quad + \gamma_1 \sum_{i=1}^{k} \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} g_i(w,x(w))dw\,du\,ds \\
&\quad - \sum_{j=1}^{p} \alpha_j \left(\int_0^{\eta_j} \frac{(\eta_j - s)}{\Gamma(q)} f(s,x(s))ds - \sum_{i=1}^{k} \int_0^{\eta_j} \frac{(\eta_j-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i(s,x(s))ds\right) \\
&\quad - \gamma_2 \int_\xi^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))\,du\,ds \\
&\quad + \gamma_2 \sum_{i=1}^{k} \int_\xi^1 \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} g_i(w,x(w))dwduds + \Lambda_2\Bigg], t \in [0,1].
\end{aligned}$$

We complete the proof in three steps.

Step 1. We show that $\Phi x + \Psi y \in B_\rho$. For $x, y \in B_\rho$, we find that

$$\begin{aligned}
&\|\Phi x + \Psi y\| \\
\le\ & \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds + \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,x(s))|ds + |a| \\
& + \frac{t^{m-1}}{|\Lambda_1|}\Bigg[|\alpha| \int_0^1 \left(\frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,y(s))| + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} |g_i(s,y(s))|\right) ds \\
& + |\beta| \int_0^1 \left(\frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s,y(s))| + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} |g_i(s,y(s))|\right) ds \\
& + |\gamma_1| \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,y(u))|\,du\,ds
\end{aligned}$$

$$
\begin{aligned}
&+|\gamma_1|\sum_{i=1}^{k}\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}|g_i(w,y(w))|\,dw\,du\,ds\\
&+\sum_{j=1}^{p}|\alpha_j|\left(\int_0^{\eta_j}\frac{(\eta_j-s)^{q-1}}{\Gamma(q)}|f(s,y(s))|ds+\sum_{i=1}^{k}\int_0^{\eta_j}\frac{(\eta_j-s)^{q+p_i-1}}{\Gamma(q+p_i)}|g_i(s,y(s))|ds\right)\\
&+|\gamma_2|\int_{\zeta}^{1}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|\,du\,ds\\
&+|\gamma_2|\sum_{i=1}^{k}\int_{\zeta}^{1}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}|g_i(w,y(w))|\,dw\,du\,ds+|\Lambda_2|\Bigg]\Bigg\}\\
\leq\ & \|\mu\|\left[\sup_{t\in[0,1]}\left\{\frac{t^q}{\Gamma(q+1)}+\frac{t^{m-1}}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+1)}+\frac{|\beta|}{\Gamma(q)}+\frac{|\gamma_1|\zeta^{q+1}}{\Gamma(q+2)}+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^{q}}{\Gamma(q+1)}\right.\right.\right.\\
&\left.\left.\left.+\frac{|\gamma_2|(1-\xi^{q+1})}{\Gamma(q+2)}\right)\right\}\right]+\sum_{i=1}^{k}\|\mu_i\|\left[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{t^{m-1}}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+p_i+1)}+\frac{|\beta|}{\Gamma(q+p_i)}\right.\right.\\
&\left.\left.+\frac{|\gamma_1|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{|\gamma_2|(1-\xi^{q+p_i+1})}{\Gamma(q+p_i+2)}\right)\right]+\left(|a|+\left|\frac{\Lambda_2}{\Lambda_1}\right|\right)\\
=\ & \|\mu\|\Omega+\sum_{i=1}^{k}\|\mu_i\|\Omega_i+|a|+\left|\frac{\Lambda_2}{\Lambda_1}\right|\leq\rho.
\end{aligned}
$$

Thus, $\Phi x+\Psi y\in B_\rho$.

Step 2. We show that Ψ is a contraction mapping. For that, let $x,y\in\mathcal{C}$. Then, for each $t\in[0,1]$, we have

$$
\begin{aligned}
&\|\Psi x-\Psi y\|\\
\leq\ & \sup_{t\in[0,1]}\left\{\frac{t^{m-1}}{|\Lambda_1|}\left[|\alpha|\int_0^1\left(\frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|\right.\right.\right.\\
&\left.+\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)}|g_i(s,x(s))-g_i(s,y(s))|\right)ds\\
&+|\beta|\int_0^1\left(\frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|\right.\\
&\left.+\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)}|g_i(s,x(s))-g_i(s,y(s))|\right)ds\\
&+|\gamma_1|\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|\,du\,ds\\
&+|\gamma_1|\sum_{i=1}^{k}\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}|g_i(w,x(w))-g_i(w,y(w))|\,dw\,du\,ds\\
&+\sum_{j=1}^{p}|\alpha_j|\left(\int_0^{\eta_j}\frac{(\eta_j-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\right.\\
&\left.+\sum_{i=1}^{k}\int_0^{\eta_j}\frac{(\eta_j-s)^{q+p_i-1}}{\Gamma(q+p_i)}|g_i(s,x(s))-g_i(x,y(s))|ds\right)\\
&+|\gamma_2|\int_{\xi}^{1}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|\,du\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+|\gamma_2|\sum_{i=1}^{k}\int_{\xi}^{1}\int_{0}^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_{0}^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}|g_i(w,x(w))-g_i(w,y(w))|\,dw\,du\,ds\Bigg]\Bigg\}\\
\leq\;& L\left[\sup_{t\in[0,1]}\left\{\frac{t^{m-1}}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+1)}+\frac{|\beta|}{\Gamma(q)}+\frac{|\gamma_1|\zeta^{q+1}}{\Gamma(q+2)}+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^{q}}{\Gamma(q+1)}+\frac{|\gamma_2|(1-\xi^{q+1})}{\Gamma(q+2)}\right)\right\}\right]\\
&\times\|x-y\|+\sum_{i=1}^{k}L_i\left[\frac{t^{m-1}}{|\Lambda_1|}\left(\frac{|\alpha|}{\Gamma(q+p_i+1)}+\frac{|\beta|}{\Gamma(q+p_i)}+\frac{|\gamma_1|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}\right.\right.\\
&\left.\left.+\frac{\sum_{j=1}^{p}|\alpha_j|\eta_j^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{|\gamma_2|(1-\xi^{q+p_i+1})}{\Gamma(q+p_i+2)}\right)\right]\|x-y\|\\
\leq\;&\left(L\left[\Omega-\left(\frac{1}{\Gamma(q+1)}\right)\right]+\sum_{i=1}^{k}L_i\left[\Omega_i-\left(\frac{1}{\Gamma(q+p_i+1)}\right)\right]\right)\|x-y\|,
\end{aligned}
$$

which is a contraction by the condition (17).

Step 3. We show that Φ is compact and continuous.

(i) Observe that the continuity of the operator Φ follows from that of f and $g_i, i=1,\ldots,k$.

(ii) Φ is uniformly bounded on B_ρ as:

$$
\begin{aligned}
\|\Phi x\| &\leq \sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+\sum_{i=1}^{k}\int_0^t\frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)}|g_i(s,x(s))|ds\right\}\\
&\leq \|\mu\|\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds\right\}+\|\mu_i\|\sup_{t\in[0,1]}\left\{\sum_{i=1}^{k}\int_0^t\frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)}ds\right\}\\
&\leq \frac{\|\mu\|}{\Gamma(q+1)}+\sum_{i=1}^{k}\frac{\|\mu_i\|}{\Gamma(q+p_i+1)}.
\end{aligned}
$$

(iii) Φ is equicontinuous.

Let us set $\max_{(t,x)\in[0,1]\times B_\rho}|f(t,x)|=\widehat{f}$ and $\max_{(t,x)\in[0,1]\times B_\rho}|g_i(t,x)|=\widehat{g}_i,\ i=1,2,\ldots,m$. Then, for $t_1,t_2\in[0,1], t_1>t_2$, we have

$$
\begin{aligned}
&|\Phi x(t_1)-\Phi x(t_2)|\\
=\;&\left|\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds-\sum_{i=1}^{k}\int_0^{t_1}\frac{(t_1-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))ds\right.\\
&\left.-\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\sum_{i=1}^{k}\int_0^{t_2}\frac{(t_2-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))ds\right|\\
\leq\;&\left|\int_0^{t_2}\frac{(t_1-s)^{q-1}-(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\right|+\left|\int_{t_2}^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\right|\\
&+\left|\sum_{i=1}^{k}\int_0^{t_2}\frac{(t_2-s)^{q+p_i-1}-(t_1-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))ds\right|\\
&+\left|\int_{t_2}^{t_1}\frac{(t_1-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i(s,x(s))ds\right|\\
\leq\;&\frac{\widehat{f}}{\Gamma(q+1)}\left\{|(t_1-t_2)^q|+|t_1^q-t_2^q|+|(t_1-t_2)^q|\right\}
\end{aligned}
$$

$$+\frac{\widehat{g_i}}{\Gamma(q+p_i+1)}\left\{|t_2^{q+p_i}-t_1^{q+p_i}|+|(t_1-t_2)^{q+p_i}|+|(t_1-t_2)^{q+p_i}|\right\}$$
$$\leq \frac{\widehat{f}}{\Gamma(q+1)}\left|2\,(t_1-t_2)^q+t_1^q-t_2^q\right|+\frac{\widehat{g_i}}{\Gamma(q+p_i+1)}|2\,(t_1-t_2)^{q+p_i}-t_1^{q+p_i}+t_2^{q+p_i}|,$$

which tends to zero, independent of x, as $t_1-t_2\to 0$. So, Φ is equicontinuous. Hence, we deduce by the Arzelá–Ascoli Theorem that Φ is compact on B_r. So, the hypothesis (Steps 1–3) of Krasnosel'skiǐ's fixed point theorem [46] holds true. Consequently, there exists at least one solution for the boundary value problem (1) on $[0,1]$. The proof is completed. □

4. Examples

Here, we illustrate the applicability of our results by constructing numerical examples.

Example 1. *Consider the following boundary value problem:*

$$\begin{cases} {}^cD^{13/4}x(t)+\sum\limits_{i=1}^{2} I^{p_i}g_i(t,x(t))=f(t,x(t)),\ \ t\in[0,1],\\ x(0)=0,\ x'(0)=0,\ x''(0)=0,\\ \alpha x(1)+\beta x'(1)=\gamma_1\int_0^{1/4}x(s)ds+\sum\limits_{j=1}^{3}\alpha_j\,x(\eta_j)+\gamma_2\int_{1/2}^{1}x(s)ds. \end{cases} \tag{18}$$

Here, $m=4,\ q=13/14,\ p_1=10/14,\ p_2=11/14,\ p_3=12/14,\ \alpha=\beta=\gamma_1=\gamma_2=1,\ \zeta=1/4,$ $\alpha_1=1/2,\ \alpha_2=3/4,\ \alpha_3=1,\ \eta_1=1/7,\ \eta_2=2/7,\ \eta_3=3/7,\ \xi=1/2.$

(i) Let $f(t,x)=\dfrac{e^{2t}}{70}(\arctan x+\sin 5t)$, $g_1(t,x)=\dfrac{1}{4}\left(\dfrac{e^{-t}\cos x+t^2+1}{\sqrt{t^2+49}}\right)$, and $g_2(t,x)=\dfrac{1}{34}(\sin x+e^{-t}\sqrt{57})$. It is easy to see that (A_1) is satisfied with $L=e^2/70$, $L_1=1/28$, and $L_2=1/34$.

Using the given data, we have $\Omega\approx 1.932128$, $\Omega_1\approx 0.677039$, $\Omega_2\approx 1.237301$, and

$$\Lambda_1=\left(\alpha+\beta(m-1)-\gamma_1\left(\frac{\zeta^m}{m}\right)-\sum_{j=1}^{3}\alpha_j\eta_j^{m-1}-\gamma_2\left(\frac{1-\xi^m}{m}\right)\right)\approx 3.666981.$$

Then, $L\Omega+\sum\limits_{i=1}^{2}L_i\Omega_i\approx 0.203951+0.060571<1$. Thus, by Theorem 1, the boundary value problem (18) has a unique solution on $[0,1]$.

(ii) We choose the following functions in problem (18) for illustrating Theorem 2:

$$f(t,x)=\frac{2}{17}(\sin x+e^{-t}\cos 7t),\ g_1(t,x)=\frac{3}{32}\left(\frac{|x|}{1+|x|}\right)+2t,\ g_2(t,x)=\frac{1}{34}(\sin x+e^{-t}\sqrt{32}). \tag{19}$$

Here $L=2/17, L_1=3/32$ and $L_2=1/34$ as $|f(t,x)-f(t,y)|\leq\dfrac{2}{17}|x-y|$, $|g_1(t,x)-g_1(t,y)|\leq\dfrac{3}{32}|x-y|$ and $|g_2(t,x)-g_2(t,y)|\leq\dfrac{1}{34}|x-y|$.

Further,

$$\|f(t,x)\| \leq \frac{2}{17}|\sin x|+e^{-t}|\cos 7t|\leq\frac{2}{17}+e^{-t}\cos 7t=\mu(t),$$
$$\|g_1(t,x)\| \leq \frac{3}{32}+2t=\mu_1(t),$$

and

$$\|g_2(t,x)\|\leq\frac{1}{34}|\sin x|+e^{-t}\sqrt{32}\leq\frac{1}{34}+e^{-t}\sqrt{32}=\mu_2(t).$$

Obviously, $\|\mu\| = 19/17$, $\|\mu_1\| = 67/32$ and $\|\mu_2\| = 5.686266$. Moreover, we have

$$\left(L\left[\Omega - \left(\frac{1}{\Gamma(q+1)}\right)\right] + \sum_{i=1}^{2} L_i \left[\Omega_i - \left(\frac{1}{\Gamma(q+p_i+1)}\right)\right]\right) \approx 0.1824334 < 1.$$

As the hypothesis of Theorem 2 holds true, so there exists least one solution for problem (18) with the functions given by (19).

5. Conclusions

We have studied a nonlinear fractional integro-differential equation involving many finitely Riemann–Liouville fractional integral type nonlinearities together with a non-integral nonlinearity complemented by multi-point sub-strip boundary conditions. In fact, we considered a more general situation by considering the fractional order nonlinear integral terms in the integro-differential equation at hand, which reduce to the usual nonlinear integral terms for $p_i = 1, \forall i = 1, \ldots, k$. Under appropriate assumptions, the existence and uniqueness results for the given problem are proved by applying the standard tools of the fixed point theory. The results obtained in this paper are not only new, but they also lead to some new results associated with the particular choices of the parameters involved in the problem. For example, our results correspond to the two-strip aperture (ζ, ξ) boundary value problem when $\alpha_j = 0, \forall j = 1, \ldots, p$. On the other hand, by letting $\gamma_1 = 0 = \gamma_2$ in the the results of this paper, we obtain the ones for a nonlinear Caputo–Riemann–Liouville type fractional integro-differential equation with multi-point boundary conditions. Thus, the work presented in this paper significantly contributes to the existing literature on the topic.

Author Contributions: Conceptualization, B.A.; formal analysis, A.A., A.F.A., S.K.N. and B.A.; funding acquisition, A.A.; methodology, A.A., A.F.A., S.K.N. and B.A. All authors have read and agreed to the published version of the manuscript.

Funding: The Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, funded this project, under grant no. (FP-20-42).

Acknowledgments: The Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, funded this project, under grant no. (FP-20-42). The authors, therefore, acknowledge the DSR, with thanks for the technical and financial support provided. The authors thank the reviewers for their constructive remarks on our work.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Redding, CT, USA, 2006.
2. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system. *Ecol. Model.* **2015**, *318*, 8–18. [CrossRef]
3. Fallahgoul, H.A.; Focardi, S.M.; Fabozzi, F.J. *Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application*; Elsevier/Academic Press: London, UK, 2017.
4. Zaslavsky, G.M. *Hamiltonian Chaos and Fractional Dynamics*; Oxford University Press: Oxford, UK, 2005.
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
6. Sabatier, J.; Agrawal, O.P.; Machado, J.A.T. (Eds.) *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*; Springer: Dordrecht, The Netherlands, 2007.
7. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
8. Henderson, J.; Kosmatov, N. Eigenvalue comparison for fractional boundary value problems with the Caputo derivative. *Fract. Calc. Appl. Anal.* **2014**, *17*, 872–880. [CrossRef]
9. Peng, L.; Zhou, Y. Bifurcation from interval and positive solutions of the three-point boundary value problem for fractional differential equations. *Appl. Math. Comput.* **2015**, *257*, 458–466. [CrossRef]

10. Agarwal, R.P.; Ahmad, B.; Alsaedi, A. Fractional-order differential equations with anti-periodic boundary conditions: A survey. *Bound. Value Probl.* **2017**, *173*, 27. [CrossRef]
11. Cui, Y.; Ma, W.; Sun, Q.; Su, X. New uniqueness results for boundary value problem of fractional differential equation. *Nonlinear Anal. Model. Control* **2018**, *23*, 31–39. [CrossRef]
12. Baghani, H.; Nieto, J.J. On fractional Langevin equation involving two fractional orders in different intervals. *Nonlinear Anal. Model. Control* **2019**, *24*, 884–897. [CrossRef]
13. Alsaedi, A.; Ahmad, B.; Alghanmi, M. Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions. *Appl. Math. Lett.* **2019**, *91*, 113–120. [CrossRef]
14. Ahmad, B.; Alsaedi, A.; Alruwaily, Y.; Ntouyas, S.K. Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions. *AIMS Math.* **2020**, *5*, 1446–1461. [CrossRef]
15. Liang, S.; Wang, L.; Yin, G. Fractional differential equation approach for convex optimization with convergence rate analysis. *Optim. Lett.* **2020**, *14*, 145–155. [CrossRef]
16. Iskenderoglu, G.; Kaya, D. Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense. *Chaos Solitons Fractals* **2020**, *134*, 109684. [CrossRef]
17. Cen, Z.; Liu, L.-B.; Huang, J. A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann-Liouville fractional derivative. *Appl. Math. Lett.* **2020**, *102*, 106086. [CrossRef]
18. Laitinen, M.; Tiihonen, T. Heat transfer in conducting and radiating bodies. *Appl. Math. Lett.* **1997**, *10*, 5–8. [CrossRef]
19. Laitinen, M.; Tiihonen, T. Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials. *Math. Methods Appl. Sci.* **1998**, *21*, 375–392. [CrossRef]
20. Tarasov, V.E. Fractional integro-differential equations for electromagnetic waves in dielectric media. *Theor. Math. Phys.* **2009**, *158*, 355–359. [CrossRef]
21. Ahmad, B.; Nieto, J.J. Boundary value problems for a class of sequential integrodifferential equations of fractional order. *J. Funct. Spaces Appl.* **2013**, 149659. [CrossRef]
22. Debbouche, A.; Nieto, J.J. Relaxation in controlled systems described by fractional integro-differential equations with nonlocal control conditions. *Electron. J. Differ. Equ.* **2015**, *89*, 18.
23. Ahmad, B.; Luca, R. Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. *Appl. Math. Comput.* **2018**, *339*, 516–534. [CrossRef]
24. Alsaedi, A.; Ahmad, B.; Aljoudi, S.; Ntouyas, S.K. A study of a fully coupled two-parameter system of sequential fractional integro-differential equations with nonlocal integro-multipoint boundary conditions. *Acta Math. Sci. Ser. B* **2019**, *39*, 927–944. [CrossRef]
25. Zhou, Y.; Suganya, S.; Arjunan, M.M.; Ahmad, B. Approximate controllability of impulsive fractional integro-differential equation with state-dependent delay in Hilbert spaces. *IMA J. Math. Control Inform.* **2019**, *36*, 603–622. [CrossRef]
26. Ahmad, B.; Alsaedi, A.; Aljoudi, S.; Ntouyas, S.K. A six-point nonlocal boundary value problem of nonlinear coupled sequential fractional integro-differential equations and coupled integral boundary conditions. *J. Appl. Math. Comput.* **2018**, *56*, 367–389. [CrossRef]
27. Liu, J.; Zhao, K. Existence of mild solution for a class of coupled systems of neutral fractional integro-differential equations with infinite delay in Banach space. *Adv. Differ. Equ.* **2019**, *2019*, 284. [CrossRef]
28. Ravichandran, C.; Valliammal, N.; Nieto, J.J. New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces. *J. Franklin Inst.* **2019**, *356*, 1535–1565. [CrossRef]
29. Ahmad, B.; Broom, A.; Alsaedi, A.; Ntouyas, S.K. Nonlinear integro-differential equations involving mixed right and left fractional derivatives and integrals with nonlocal boundary data. *Mathematics* **2020**, *8*, 336. [CrossRef]
30. Huang, L.; Li, X.-F.; Zhao, Y.; Duan, X.-Y. Approximate solution of fractional integro-differential equations by Taylor expansion method. *Comput. Math. Appl.* **2011**, *62*, 1127–1134. [CrossRef]
31. Zhang, L.; Ahmad, B.; Wang, G.; Agarwal, R.P. Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. *J. Comput. Appl. Math.* **2013**, *249*, 51–56. [CrossRef]
32. Tate, S.; Kharat, V.V.; Dinde, H.T. On nonlinear mixed fractional integro-differential equations with positive constant coefficient. *Filomat* **2019**, *33*, 5623–5638. [CrossRef]
33. Jalilian, Y.; Ghasemi, M. On the solutions of a nonlinear fractional integro-differential equation of pantograph type. *Mediterr. J. Math.* **2017**, *14*, 194. [CrossRef]

34. Ahmad, B. A quasilinearization method for a class of integro-differential equations with mixed nonlinearities. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 997–1004. [CrossRef]
35. Hussain, K.H.; Hamoud, A.A.; Mohammed, N.M. Some new uniqueness results for fractional integro-differential equations. *Nonlinear Funct. Anal. Appl.* **2019**, *24*, 827–836.
36. Hamoud, A.A.; Ghadle, K.; Atshan, S. The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method. *Khayyam J. Math.* **2019**, *5*, 21–39.
37. Ahmad, B.; Alruwaily, Y.; Alsaedi, A.; Nieto, J.J. Fractional integro-differential equations with dual anti-periodic boundary conditions. *Differ. Integral Equ.* **2020**, *33*, 181–206.
38. Ahmad, B.; Alsaedi, A.; Alghamdi, B.S. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. *Nonlinear Anal. Real World Appl.* **2008**, *9*, 1727–1740. [CrossRef]
39. Ciegis, R.; Bugajev, A. Numerical approximation of one model of the bacterial self-organization. *Nonlinear Anal. Model. Control* **2012**, *17*, 253–270. [CrossRef]
40. Ahmad, B.; Asghar, S.; Hayat, T. Diffraction of a plane wave by an elastic knife-edge adjacent to a rigid strip. *Canad. Appl. Math. Quart.* **2001**, *9*, 303–316.
41. Yusufoglu, E.; Turhan, I. A mixed boundary value problem in orthotropic strip containing a crack. *J. Franklin Inst.* **2012**, *349*, 2750–2769. [CrossRef]
42. Bitsadze, A.; Samarskii, A. On some simple generalizations of linear elliptic boundary problems. *Russ. Acad. Sci. Dokl. Math.* **1969**, *10*, 398–400.
43. Byszewski, L.; Lakshmikantham, V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.* **1991**, *40*, 11–19. [CrossRef]
44. Byszewski, L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **1991**, *162*, 494–505. [CrossRef]
45. Groza, G.; Pop, N. Approximate solution of multipoint boundary value problems for linear differential equations by polynomial functions. *J. Differ. Equ. Appl.* **2008**, *14*, 1289–1309. [CrossRef]
46. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: Cambridge, UK, 1974.