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The Nullity, Rank, and Invertibility of Linear Combinations of k-Potent Matrices

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Abstract: Baksalary et al. (Linear Algebra Appl., doi:10.1016/j.laa.2004.02.025, 2004) investigated the invertibility of a linear combination of idempotent matrices. This result was improved by Koliha et al. (Linear Algebra Appl., doi:10.1016/j.laa.2006.01.011, 2006) by showing that the rank of a linear combination of two idempotents is constant. In this paper, we consider similar problems for k-potent matrices. We study the rank and the nullity of a linear combination of two commuting k-potent matrices. Furthermore, the problem of the nonsingularity of linear combinations of two or three k-potent matrices is considered under some conditions. In these situations, we derive explicit formulae of their inverses.

Keywords: k-potent matrix; linear combination; nonsingularity; rank; nullity

1. Introduction

Let $\mathbb{C}^{n\times m}$ denote the set of all $n \times m$ complex matrices. The symbols $\mathcal{R}(A)$ and $\mathcal{N}(A)$ will denote the range (column space) and the null space of a matrix A, respectively. The rank of A, $r(A)$, is the dimension of $\mathcal{R}(A)$, and the nullity of A, $nul(A)$, is the dimension of $\mathcal{N}(A)$. By $\mathbb{C}_r^{n\times n}$, we will denote the set of all matrices from $\mathbb{C}^{n\times n}$ with a rank r. I_n denotes the identity matrix of order n. We say that integers k and l are congruent modulo the positive integer m, and we use the notation $k \equiv l(mod\ \ m)$, if $m|(k-l)$. For $T \in \mathbb{C}^{n\times n}$, the group inverse [1–4] of T is the unique (if it exists) matrix $T^\sharp \in \mathbb{C}^{n\times n}$ such that:

$$TT^\sharp = T^\sharp T, \qquad T^\sharp T T^\sharp = T^\sharp \qquad \text{and} \qquad T = TT^\sharp T.$$

The group inverse is a particular case of the generalized Drazin inverse. Recently, there has been interest in investigating the generalized Drazin inverse, for example [5,6].

In this paper, we will focus on the set of k-potent matrices, with k being a positive integer greater than one. This set of matrices is defined as $\{A \in \mathbb{C}^{n\times n} : A^k = A\}$. In particular, if $k = 2$ or $k = 3$, then $A \in \mathbb{C}^{n\times n}$ is called an idempotent (an oblique projector) or a tripotent matrix, respectively.

The research dealing with k-potent matrices, in particular idempotents and tripotents, is quite extensive. The fact that these matrices attract such attention is mostly due to their possible applications. Collections of results dealing with idempotent and tripotent matrices are available in several monographs emphasizing their usefulness in statistics, for instance [7] (Section 12.4), [8] (Chapter 7), and [9] (Sections 8.6, 8.7, and 20.5.3). In addition to the papers [10–20], each of which contains a systematic study over a selected topic concerning k-potent matrices, a collection of related isolated results was published in recent years in a number of independent articles.

Apart from the papers mentioned above, an inspiration for this paper was also the work of Baksalary et al. [21], where the authors discussed the nonsingularity of a linear combination of two idempotent matrices, and the work of Koliha et al. [22], where the authors considered the nullity and rank of linear combinations of two idempotent matrices. In this paper, we generalize the results given in [22] to the cases of commuting k-potent matrices. Furthermore, we give necessary and sufficient conditions for the nonsingularity of $c_1A + c_2B$ and $c_1A + c_2B + c_3C$, and we find some formulae for $(c_1A + c_2B)^{-1}$ and $(c_1A + c_2B + c_3C)^{-1}$ under various conditions, where $A, B, C \in \mathbb{C}^{n\times n}$ are k-potent matrices and c_1, c_2, c_3 are nonzero complex numbers.

2. The Nullity and Rank of a Linear Combination of Two k-Potent Matrices

In the following theorem, we investigate the null space of a linear combination of two commuting k-potent matrices.

Theorem 1. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 + c_2 \neq 0$. Then, $\mathcal{N}(c_1A + c_2B)$ is isomorphic to $\mathcal{N}((B - A)B^{k-1}) \cap \mathcal{N}(A)$.*

Proof. Let $\mathcal{N} = \mathcal{N}((B - A)B^{k-1}) \cap \mathcal{N}(A)$. First, we show that $\mathcal{N}(c_1A + c_2B) \subseteq \mathcal{N}$. Suppose that $x \in \mathcal{N}(c_1A + c_2B)$. Then, $Ax = -\frac{c_2}{c_1}Bx$. Furthermore, suppose that $x \in \mathcal{N}(A)$. Now, we get:

$$(B - A)B^{k-1}x = Bx - AB^{k-1}x = Bx + \frac{c_2}{c_1}Bx$$
$$= \frac{c_1 + c_2}{c_1c_2}(c_1A + c_2B)x = 0.$$

Thus, $x \in \mathcal{N}((B - A)B^{k-1})$, i.e., $x \in \mathcal{N}$, and the inclusion $\mathcal{N}(c_1A + c_2B) \subseteq \mathcal{N}$ is proven.

Next, we prove that $\mathcal{N}$ is isomorphic to $(cI_n - B^{k-1})\mathcal{N}$ and $(cI_n - B^{k-1})\mathcal{N} \subseteq \mathcal{N}(c_1A + c_2B)$ for some $c \neq 0$.

Let $x \in \mathcal{N}$ and $(cI_n - B^{k-1})x = 0$. Then, $Ax = 0$ and $(B - A)B^{k-1}x = 0$, i.e., $Bx = AB^{k-1}x$. Furthermore, the equality $B^{k-1}x = cx$ holds. The previous relations give $Bx = Acx = 0$ and $cx = B^{k-1}x = 0$. Therefore, we have $x = 0$. This proves the item $\mathcal{N} \cong (cI_n - B^{k-1})\mathcal{N}$.

Now, we show the inclusion $(cI_n - B^{k-1})\mathcal{N} \subseteq \mathcal{N}(c_1A + c_2B)$ for $c = 1 + \frac{c_1}{c_2}$. Suppose that $x \in \mathcal{N}$. Applying $B = AB^{k-1}$ and $B = B^k$, we obtain:

$$(c_1A + c_2B)(cI_n - B^{k-1})x = c_1cAx - c_1AB^{k-1}x + c_2cBx - c_2B^kx$$
$$= (c_1 + c_2)Bx - (c_1 + c_2)Bx = 0,$$

i.e., $(cI_n - B^{k-1})x \in \mathcal{N}(c_1A + c_2B)$. Hence, $(cI_n - B^{k-1})\mathcal{N} \subseteq \mathcal{N}(c_1A + c_2B)$ is satisfied. From the previous inclusion and $\mathcal{N} \cong (cI_n - B^{k-1})\mathcal{N}$, it follows that $\mathcal{N} \subseteq \mathcal{N}(c_1A + c_2B)$. □

Following this, the results present that the nullity and the rank of a linear combination of two commuting k-potent matrices are constant.

Corollary 1. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 + c_2 \neq 0$. Then, $nul(c_1A + c_2B) = dim[\mathcal{N}((B - A)B^{k-1}) \cap \mathcal{N}(A)]$.*

Proof. This follows from Theorem 1. □

Corollary 2. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 + c_2 \neq 0$. Then, $r(c_1A + c_2B) = n - dim[\mathcal{N}((B - A)B^{k-1}) \cap \mathcal{N}(A)]$.*

Proof. This follows from the well-known general equality between the range of the column space and the null space of a matrix A, $r(c_1A + c_2B) = n - nul(c_1A + c_2B)$, and Corollary 1. □

Remark 1. *Since $M \in \mathbb{C}^{n\times n}$ is invertible if and only if $nul(M) = 0$, by Corollary 1, we conclude that the invertibility of the linear combination $c_1A + c_2B$ of two commuting k-potent matrices does not depend on the choice of the constants c_1, c_2, where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 + c_2 \neq 0$.*

3. Invertibility of A Linear Combination of Two k-Potent Matrices

In this section, we discuss the invertibility of a linear combination of two commuting k-potent matrices. The next representation will be useful for further results. Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. Since A is k-potent, this matrix is diagonalizable, and its spectrum is contained in $\{0\} \cup \sqrt[k-1]{1}$ (see [23]). Therefore, $A \in \mathbb{C}_r^{n\times n}$ can be written as:

$$A = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^{-1}, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is nonsingular, $K = diag(\lambda_1, \ldots, \lambda_r)$, $\lambda_i^{k-1} = 1$ for $i = 1, \ldots, r$, $r(A) = r$. Note that $K \in \mathbb{C}^{r\times r}$ is nonsingular and $K^{k-1} = I_r$. Furthermore, we can write $B \in \mathbb{C}^{n\times n}$ as follows:

$$B = U \begin{bmatrix} D & F \\ E & G \end{bmatrix} U^{-1}, \tag{2}$$

where $D \in \mathbb{C}^{r\times r}$ and $G \in \mathbb{C}^{(n-r)\times(n-r)}$.

The following fact will be used very often: If $X, Y \in \mathbb{C}^{n\times n}$ and $XY = YX$, then for $k \geq 2$:

$$(X + Y) \sum_{i=0}^{k-2} (-1)^i X^{k-2-i} Y^i = X^{k-1} + (-1)^k Y^{k-1}. \tag{3}$$

Since $A^l = A^s$, where A is a k-potent matrix, $l \equiv s(mod\ (k-1))$, $l, s \in \mathbb{N}$, we give the following results.

Theorem 2. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$.*

(i) *If $AB = A^s$, $s \in \{0, 1, 3, \ldots, k-2\}$, and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, then $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + B(I_n - A^{k-1})$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = A_1 &- B^{k-2}(I_n - A^{k-1})^2 BA^{k-1}A_1 \\ &+ \frac{1}{c_2} B^{k-2}(I_n - A^{k-1}),\end{aligned} \tag{4}$$

where:

$$A_1 = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i+(s-1)i}.$$

(ii) *If $AB = A^2$ and $c_1 + c_2 \neq 0$, then $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + B(I_n - A^{k-1})$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = &\frac{1}{c_1 + c_2} A^{k-2} - \frac{1}{c_1 + c_2} B^{k-2}(I_n - A^{k-1})^2 BA^{k-1}A^{k-2} \\ &+ \frac{1}{c_2} B^{k-2}(I_n - A^{k-1}).\end{aligned} \tag{5}$$

Proof. Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $r(A) = r$.

(i) By $AB = A^s$, $s \in \{0, 1, 3, \dots, k-2\}$, it follows that B has the form:

$$B = U \begin{bmatrix} K^{s-1} & 0 \\ E & G \end{bmatrix} U^{-1}, \tag{6}$$

where $G \in \mathbb{C}^{(n-r)\times(n-r)}$. A simple induction argument shows there exists a sequence $(E_m)_{m=1}^{\infty}$ such that:

$$B^m = U \begin{bmatrix} K^{m(s-1)} & 0 \\ E_m & G^m \end{bmatrix} U^{-1}, \ \forall m \in \mathbb{N}.$$

Since $B^k = B$, we get that $G^k = G$. Thus, $G \in \mathbb{C}^{(n-r)\times(n-r)}$ is the k-potent matrix. Now, $c_1A + c_2B$ has the form:

$$c_1A + c_2B = U \begin{bmatrix} c_1K + c_2K^{s-1} & 0 \\ c_2E & c_2G \end{bmatrix} U^{-1}.$$

Using (3), we have:

$$\begin{aligned}
&(c_1K + c_2K^{s-1}) \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i K^{k-2-i} (K^{s-1})^i \\
&= c_1^{k-1} K^{k-1} + (-1)^k c_2^{k-1} K^{(s-1)(k-1)} \\
&= (c_1^{k-1} + (-1)^k c_2^{k-1}) I_r
\end{aligned}$$

and:

$$(c_1K + c_2K^{s-1})^{-1} = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i K^{k-2-i+(s-1)i}.$$

Therefore, the linear combination $c_1K + c_2K^s$ is invertible for all constants $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. Now, we conclude that $c_1A + c_2B$ is nonsingular if and only if G is nonsingular. Since:

$$A^{k-1} = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{-1},$$

$$B(I_n - A^{k-1}) = U \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} U^{-1}$$

and:

$$A^{k-1} + B(I_n - A^{k-1}) = U \begin{bmatrix} I_r & 0 \\ 0 & G \end{bmatrix} U^{-1}, \tag{7}$$

we conclude that $A^{k-1} + B(I_n - A^{k-1})$ is nonsingular if and only if G is nonsingular. Thus, $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + B(I_n - A^{k-1})$ is nonsigular and in case the inverse of $c_1A + c_2B$ is defined by:

$$(c_1A+c_2B)^{-1} = U\begin{bmatrix} (c_1K+c_2K^{s-1})^{-1} & 0 \\ -G^{-1}E(c_1K+c_2K^{s-1})^{-1} & c_2^{-1}G^{-1} \end{bmatrix} U^{-1},$$

i.e.,

$$(c_1A+c_2B)^{-1} = U\begin{bmatrix} (c_1K+c_2K^{s-1})^{-1} & 0 \\ -G^{k-2}E(c_1K+c_2K^{s-1})^{-1} & c_2^{-1}G^{k-2} \end{bmatrix} U^{-1},$$

where $(c_1K+c_2K^{s-1})^{-1}$ is given by (3). We observe that:

$$B^{k-2}(I_n-A^{k-1}) = U\begin{bmatrix} 0 & 0 \\ 0 & G^{k-2} \end{bmatrix} U^{-1}$$

and:

$$(I_n-A^{k-1})BA^{k-1} = U\begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix} U^{-1}.$$

Thus, the inverse $(c_1A+c_2B)^{-1}$ has the form (4).

(ii) From $AB=A^2$, we can conclude that B has the form:

$$B = U\begin{bmatrix} K & 0 \\ E & G \end{bmatrix} U^{-1}, \tag{8}$$

where $G\in\mathbb{C}^{(n-r)\times(n-r)}$. A simple induction argument shows that there exists a sequence $(E_m)_{m=1}^{\infty}$ such that:

$$B^m = U\begin{bmatrix} K^m & 0 \\ E_m & G^m \end{bmatrix} U^{-1}, \;\; \forall m\in\mathbb{N}.$$

Since $B^k=B$, we get that $G^k=G$. Hence, $G\in\mathbb{C}^{(n-r)\times(n-r)}$ is the k-potent matrix. Therefore,

$$c_1A+c_2B = U\begin{bmatrix} (c_1+c_2)K & 0 \\ c_2E & c_2G \end{bmatrix} U^{-1}.$$

Since $(c_1+c_2)K$ is nonsingular for all $c_1,c_2\in\mathbb{C}\setminus\{0\}$ such that $c_1+c_2\neq 0$, then c_1A+c_2B is nonsingular if and only if G is nonsingular.

Analogously, as in Part (i), we conclude that $A^{k-1}+B(I_n-A^{k-1})$ has the form (7) and that $A^{k-1}+B(I_n-A^{k-1})$ is nonsingular if and only if G is nonsingular. Thus, the necessary and sufficient condition for the invertibility of c_1A+c_2B is the invertibility of $A^{k-1}+B(I_n-A^{k-1})$.

Furthermore,

$$(c_1A+c_2B)^{-1} = U\begin{bmatrix} (c_1+c_2)^{-1}K^{-1} & 0 \\ -G^{-1}E(c_1+c_2)^{-1}K^{-1} & c_2^{-1}G^{-1} \end{bmatrix} U^{-1},$$

and (5) holds.

□

Theorem 3. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$.*

(i) *If $BA = A^s$, $s \in \{0, 1, 3, \ldots, k-2\}$, and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, then $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + (I_n - A^{k-1})B$ is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = A_1 - A_1A^{k-1}B(I_n - A^{k-1})^2B^{k-2} + \frac{1}{c_2}(I_n - A^{k-1})B^{k-2},$$

where:

$$A_1 = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i+(s-1)i}.$$

(ii) *If $BA = A^2$ and $c_1 + c_2 \neq 0$, then $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + (I_n - A^{k-1})B$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = &\frac{1}{c_1 + c_2}A^{k-2} - \frac{1}{c_1 + c_2}A^{k-2}A^{k-1}B(I_n - A^{k-1})^2B^{k-2}\\ &+ \frac{1}{c_2}(I_n - A^{k-1})B^{k-2}.\end{aligned}$$

Proof. This follows along the same lines as Theorem 2. □

In the next theorem, we investigate the invertibility of $c_1A + c_2B$, in the case when A and B are k-potent matrices such that $AB = B$ or $BA = B$. The following lemma is very important in the proof of this result.

Lemma 1. *Let $A \in \mathbb{C}_r^{n\times n}$ be a k-potent matrix, and let $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. Then, $c_1I_n + c_2A$ is nonsingular and:*

$$(c_1I_n + c_2A)^{-1} = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^i + \frac{1}{c_1}(I_n - A^{k-1}). \quad (9)$$

Proof. Let $A \in \mathbb{C}_r^{n\times n}$ be of the form (1) and $r(A) = r$. Then:

$$c_1I_n + c_2A = U \begin{bmatrix} c_1I_r + c_2K & 0 \\ 0 & c_1I_{n-r} \end{bmatrix} U^{-1}. \quad (10)$$

Obviously, $c_1I_n + c_2A$ is nonsingular if and only if $c_1I_r + c_2K$ is nonsingular. Now, by (3), it follows that:

$$\begin{aligned}&(c_1I_r + c_2K) \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i K^i = c_1^{k-1}I_r + (-1)^k c_2^{k-1}K^{k-1}\\ &= (c_1^{k-1} + (-1)^k c_2^{k-1})I_r.\end{aligned}$$

Hence, $c_1I_r + c_2K$ is nonsingular for all $c_1, c_2 \in \mathbb{C}$ such that $c_1 \neq 0$ and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and:

$$(c_1I_r + c_2K)^{-1} = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i K^i.$$

Thus, the matrix $c_1I_n + c_2A$ is nonsingular for all $c_1, c_2 \in \mathbb{C}$ such that $c_1 \neq 0$ and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and (9) holds. □

Theorem 4. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. If $AB = B$ or $BA = B$, then $c_1A + c_2B$ is nonsingular if and only if A is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = A^{-1}\left(\frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i B^i + \frac{1}{c_1}(I_n - B^{k-1})\right). \tag{11}$$

Proof. First suppose that the condition $AB = B$ holds. Furthermore, suppose that A has the form (1) and B has the form given by (2). From $AB = B$, we get:

$$B = U \begin{bmatrix} D & F \\ 0 & 0 \end{bmatrix} U^{-1}, \tag{12}$$

where $D \in \mathbb{C}^{r\times r}$, $KD = D$, and $KF = F$. A simple induction argument shows that B^m has the form:

$$B^m = U \begin{bmatrix} D^m & D^{m-1}F \\ 0 & 0 \end{bmatrix} U^{-1}, \forall m \in \mathbb{N}.$$

Since $B^k = B$, we have that $D^k = D$. Hence, D is a k-potent. Now, $c_1A + c_2B$ has the form:

$$c_1A + c_2B = U \begin{bmatrix} c_1K + c_2D & c_2F \\ 0 & 0 \end{bmatrix} U^{-1},$$

i.e.,

$$c_1A + c_2B = U \begin{bmatrix} K(c_1I_r + c_2D) & c_2F \\ 0 & 0 \end{bmatrix} U^{-1}.$$

Since D is a k-potent matrix, then by Lemma 1, it follows that $c_1I_r + c_2D$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^{k-1} + (-1)^{k-2} c_2^{k-1} \neq 0$, and:

$$(c_1I_r + c_2D)^{-1} = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i D^i + \frac{1}{c_1}(I_r - D^{k-1}). \tag{13}$$

Now, $K(c_1I_r + c_2D)$ is nonsingular. Since $r((K(c_1I_r + c_2D)) = r$, then $c_1A + c_2B$ is nonsingular if and only if $r = n$, i.e., $c_1A + c_2B$ is nonsingular if and only if A is nonsingular. Then:

$$(c_1A + c_2B)^{-1} = U(c_1I_n + c_2D)^{-1}K^{-1}U^{-1},$$

where $(c_1I_n + c_2D)^{-1}$ is given by (13). Thus, we can conclude that (11) holds. The proof under the condition $BA = B$ is similar to the proof given in the first part, so we omit it. □

The following theorem presents necessary and sufficient conditions for the nonsingularity of linear combinations of two commuting k-potent matrices and presents the form of its inverse.

Theorem 5. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. Then, $c_1A + c_2B$ is nonsingular if and only if $A^{k-1} + (I_n - A^{k-1})B$ is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i} c_2^i A^{k-2-i}(A^{k-1}B)^i + \frac{1}{c_2}(I_n - A^{k-1})B^{k-2}. \tag{14}$$

Proof. Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $r(A) = r$. We get that the condition $AB = BA$ is equivalent to the fact that B has the form:

$$B = U \begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix} U^{-1}, \tag{15}$$

where $D \in \mathbb{C}^{r\times r}$ such that $KD = DK$ and $G \in \mathbb{C}^{(n-r)\times(n-r)}$. A simple induction argument shows that B^m has the form:

$$B = U \begin{bmatrix} D^m & 0 \\ 0 & G^m \end{bmatrix} U^{-1}, \forall m \in \mathbb{N}.$$

Since $B^k = B$, we get that $D^k = D$ and $G^k = G$. Hence, D and G are k-potent matrices. Now,

$$c_1A + c_2B = U \begin{bmatrix} c_1K + c_2D & 0 \\ 0 & c_2G \end{bmatrix} U^{-1}.$$

Using (3), we derive:

$$(c_1K + c_2D)\sum_{i=0}^{k-2}(-1)^i (c_1K)^{k-2-i}(c_2D)^i = (c_1K)^{k-1} + (-1)^k (c_2D)^{k-1}. \tag{16}$$

Since D^{k-1} is a projector and:

$$(c_1K)^{k-1} + (-1)^k (c_2D)^{k-1} = c_1^{k-1} I_r + (-1)^k c_2^{k-1} D^{k-1}$$

we get that $(c_1K)^{k-1} + (-1)^k(c_2D)^{k-1}$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. Based on (16) and the invertibility of $(c_1K)^{k-1} + (-1)^{k-2}(c_2D)^{k-1}$, we conclude that $c_1K + c_2D$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and:

$$(c_1K + c_2D)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i} c_2^i K^{k-2-i} D^i. \tag{17}$$

Now, $c_1A + c_2B$ is nonsingular if and only if G is nonsingular. We observe that $(I_n - A^{k-1})B + A^{k-1}$ can be represented as (7) and that $(I_n - A^{k-1})B + A^{k-1}$ is nonsingular if and only if G is nonsingular. Hence, $c_1A + c_2B$ is invertible if and only if $(I_n - A^{k-1})B + A^{k-1}$ is invertible. In this case, the inverse $(c_1A + c_2B)^{-1}$ has the form:

$$(c_1A + c_2B)^{-1} = U \begin{bmatrix} (c_1K + c_2D)^{-1} & 0 \\ 0 & c_2^{-1}G^{-1} \end{bmatrix} U^{-1},$$

i.e.,

$$(c_1A + c_2B)^{-1} = U \begin{bmatrix} (c_1K + c_2D)^{-1} & 0 \\ 0 & c_2^{-1}G^{k-2} \end{bmatrix} U^{-1},$$

where $(c_1K + c_2D)^{-1}$ is given by (17). Therefore, (14) holds. □

The following corollary is very useful in Section 4.

Corollary 3. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. Then, $c_1A + c_2B$ is nonsingular if and only if $A + B$ is nonsingular.*

Remark 2. *The results in this section show us that the nonsingularity of the linear combination of two k-potent matrices does not depend on the choice of nonzero complex constants as long as the assumptions of the previous theorems are met.*

4. Invertibility of a Linear Combination of Three k-Potent Matrices

In this section, we study the invertibility of a linear combination of three k-potent matrices. We use the notions from previous sections. We also use the next representation for $C \in \mathbb{C}^{n\times n}$:

$$C = U \begin{bmatrix} M & L \\ S & T \end{bmatrix} U^{-1}, \tag{18}$$

where $M \in \mathbb{C}^{r\times r}$ and $T \in \mathbb{C}^{(n-r)\times(n-r)}$. We consider the invertibility of $c_1A + c_2B + c_3C$, where $A, C \in \mathbb{C}^{n\times n}$ are k-potent matrices such that $AC = 0 = CA$. Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $r(A) = r$. Using $AC = 0 = CA$ and the nonsingularity of K leads to $M = L = S = 0$. Hence,

$$C = U \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} U^{-1}, \tag{19}$$

where $T \in \mathbb{C}^{(n-r)\times(n-r)}$. A simple induction argument shows that C^m has the form:

$$C^m = U \begin{bmatrix} 0 & 0 \\ 0 & T^m \end{bmatrix} U^{-1}, \forall m \in \mathbb{N}.$$

Since C is a k-potent matrix, then T is also a k-potent matrix. Furthermore, the conditions from the theorems of the previous section for k-potent matrices $A, B \in \mathbb{C}^{n\times n}$ apply.

Theorem 6. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B, C \in \mathbb{C}^{n\times n}$ be k-potent matrices such that $AC = 0 = CA$ and $BC = CB$, and let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$.*

(i) *If $AB = A^s, s \in \{0, 1, 3, \ldots, k-2\}$, and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, then $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (B + C)(I_n - A^{k-1})$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = A_1 - \frac{1}{c_2}[(c_2B + c_3C)(I_n - A^{k-1})]^{\sharp}(I_n - A^{k-1})BA^{k-1}A_1 \\ + [(c_2B + c_3C)(I_n - A^{k-1})]^{\sharp},\end{aligned} \tag{20}$$

where:

$$A_1 = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i+(s-1)i}.$$

(ii) *If $AB = A^2$, then $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (B + C)(I_n - A^{k-1})$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A+c_2B)^{-1} &= \frac{1}{c_1+c_2}A^{k-2}\\ &-\frac{1}{c^2(c_1+c_2)}[(c_2B+c_3C)(I_n-A^{k-1})]^{\sharp}(I_n-A^{k-1})BA^{k-1}A^{k-2}\\ &+[(c_2B+c_3C)(I_n-A^{k-1})]^{\sharp}.\end{aligned} \tag{21}$$

Proof. Suppose that $A, C \in \mathbb{C}^{n\times n}$ have the form (1) and (19), respectively, and $r(A) = r$.

(i) Since $AB = A^s$, $s \in \{0, 1, 3, \ldots, k-2\}$, we can write B as in (6). We note that $c_1A + c_2B + c_3C$ can be represented as:

$$c_1A+c_2B+c_3C = U\begin{bmatrix} c_1K+c_2K^{s-1} & 0 \\ c_2E & c_2G+c_3T \end{bmatrix}U^{-1},$$

where $G, T \in \mathbb{C}^{(n-r)\times(n-r)}$ are k-potent matrices. Since $BC = CB$, then $GT = TG$. Analogous to the proof to Theorem 2, we conclude that $c_1K + c_2K^s$ is invertible for all constants $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$ and that the inverse of $c_1K + c_2K^s$ has the form (3).

Therefore, the linear combination $c_1A + c_2B + c_3C$ is nonsingular if and only if $c_2G + c_3T$ is nonsingular. Since $G, T \in \mathbb{C}^{(n-r)\times(n-r)}$ are commuting k-potent matrices, we deduce that $c_2G + c_3T$ is nonsingular if and only if $G + T$ is nonsingular for all constants $c_2, c_3 \in \mathbb{C}\setminus\{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$ by Corollary 3. Furthermore,

$$A^{k-1}+(B+C)(I_n-A^{k-1}) = U\begin{bmatrix} I_r & 0 \\ 0 & G+T \end{bmatrix}U^{-1}, \tag{22}$$

and $A^{k-1} + (B + C)(I_n - A^{k-1})$ is nonsingular if and only if $G + T$ is nonsingular. Thus, the necessary and sufficient condition for the invertibility of and $c_1A + c_2B + c_3C$ is the invertibility of $A^{k-1} + (B + C)(I_n - A^{k-1})$.

By a direct computation, we get:

$$(c_1A+c_2B+c_3C)^{-1} = U\begin{bmatrix} (c_1K+c_2K^{s-1})^{-1} & 0 \\ -c_2(c_2G+c_3T)^{-1}E(c_1K+c_2K^{s-1})^{-1} & (c_2G+c_3T)^{-1} \end{bmatrix}U^{-1},$$

where $(c_1K + c_2K^{s-1})^{-1}$ is given by (3). Furthermore,

$$(c_2B+c_3C)(I_n-A^{k-1}) = U\begin{bmatrix} 0 & 0 \\ 0 & c_2G+c_3T \end{bmatrix}U^{-1}$$

and:

$$(I_n-A^{k-1})BA^{k-1} = U\begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix}U^{-1}.$$

It is noteworthy that the group inverse of $(c_2B + c_3C)(I_n - A^{k-1})$ is:

$$[(c_2B+c_3C)(I_n-A^{k-1})]^{\sharp} = U\begin{bmatrix} 0 & 0 \\ 0 & (c_2G+c_3T)^{-1} \end{bmatrix}U^{-1}. \tag{23}$$

Hence, the formula (20) holds.

(ii) The proof is similar to the one in Item (i). We use the form (8) for $B \in \mathbb{C}^{n\times n}$. Now,

$$c_1A + c_2B + c_3C = U \begin{bmatrix} (c_1+c_2)K & 0 \\ c_2E & c_2G + c_3T \end{bmatrix} U^{-1},$$

where $G, T \in \mathbb{C}^{(n-r)\times(n-r)}$ are k-potent matrices. Since $BC = CB$, then $GT = TG$. Note that $(c_1+c_2)K$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $c_1 + c_2 \neq 0$. Analogous to the proof of Item (i), we conclude that $c_2G + c_3T$ is nonsingular if and only if if $G + T$ is nonsingular for all constants $c_2, c_3 \in \mathbb{C}\setminus\{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. We observe that $A^{k-1} + (B+C)(I_n - A^{k-1})$ can be represented as (22) and that $A^{k-1} + (B+C)(I_n - A^{k-1})$ is nonsingular if and only if $G + T$ is nonsingular. Therefore, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (B+C)(I_n - A^{k-1})$ is nonsingular for all constants $c_2, c_3 \in \mathbb{C}\setminus\{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. The inverse of $c_1A + c_2B + c_3C$ is:

$$(c_1A + c_2B + c_3C)^{-1} = U \begin{bmatrix} (c_1+c_2)^{-1}K^{-1} & 0 \\ -c_2(c_1+c_2)^{-1}(c_2G+c_3T)^{-1}EK^{-1} & (c_2G+c_3T)^{-1} \end{bmatrix} U^{-1},$$

and can be represented as (21). The rest of the proof is similar to the one in Item (i).

□

Theorem 7. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B, C \in \mathbb{C}^{n\times n}$ be k-potent matrices such that $AC = 0 = CA$, and let $c_1, c_2, c_3 \in \mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. If $AB = B$ or $BA = B$, then $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + C(I_n - A^{k-1})$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = A^{k-2}\Big(&\frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i B^i \\ &+ \frac{1}{c_1}(I_n - B^{k-1})\Big) + \frac{1}{c_3}C^{k-2}.\end{aligned} \tag{24}$$

Proof. We only consider the case when $AB = B$. The remaining case can be proven in the same way. Conditions $AB = B$ and $AC = 0 = CA$ imply that $A, B, C \in$ can be represented as (1), (12), and (19), respectively. Then, we have:

$$c_1A + c_2B + c_3C = U \begin{bmatrix} c_1K + c_2D & c_2F \\ 0 & c_{3G} \end{bmatrix} U^{-1}.$$

Since $c_1K + c_2D$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^{k-1} + (-1)^{k-2} c_2^{k-1} \neq 0$ (see the proof of Theorem 4, then $c_1A + c_2B + c_3C$ is nonsingular if and only if G is nonsingular. We observe that:

$$A^{k-1} + C(I_n - A^{k-1}) = U \begin{bmatrix} I_r & \\ 0 & G \end{bmatrix} U^{-1}$$

and that $A^{k-1} + C(I_n - A^{k-1})$ is invertible if and only if G is invertible. Therefore, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + C(I_n - A^{k-1})$ is nonsigular. In this case, the inverse of $c_1A + c_2B + c_3C$ has the form:

$$(c_1A + c_2B + c_3C)^{-1} = U \begin{bmatrix} (c_1K + c_2D)^{-1} & -c_2c_3^{-1}(c_1K + c_2D)^{-1}FG^{-1} \\ 0 & c_3^{-1}G^{-1} \end{bmatrix} U^{-1},$$

where $(c_1K + c_2D)^{-1} = (c_1I_n + c_2D)^{-1}K^{-1}$ (see the proof of Theorem 4) and $(c_1I_n + c_2D)^{-1}$ is given by (13). Thus, (24) is satisfied. □

Theorem 8. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B, C \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices such that $AC = 0 = CA$, and let: $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Then, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (I_n - A^{k-1})(B + C)$ is nonsingular. Furthermore,*

$$\begin{aligned}(c_1A + c_2B)^{-1} = &\frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i}(A^{k-1}B)^i \\ &+ \frac{1}{c_2}(I_n - A^{k-1})B^{k-2} + [(c_2B + c_3C)(I_n - A^{k-1})]^\sharp.\end{aligned} \tag{25}$$

Proof. Let $A, B, C \in \mathbb{C}^{n\times n}$ be of the form (1), (15), and (19), respectively, and $r(A) = r$. Now,

$$c_1A + c_2B + c_3C = U \begin{bmatrix} c_1K + c_2D & 0 \\ 0 & c_2G + c_3T \end{bmatrix} U^{-1},$$

where $D \in \mathbb{C}^{r\times r}$ and $G, T \in \mathbb{C}^{(n-r)\times(n-r)}$ are k-potent matrices such that $KD = DK$ and $GT = TG$. As in the proof of Theorem 5, we derive that $c_1K + c_2D$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and the inverse of $c_1K + c_2D$ can be represented as in (17). Hence, $c_1A + c_2B + c_3C$ is nonsingular if and only if $c_2G + c_3T$ is nonsingular. By Corollary 3, the nonsingularity of $c_2G + c_3T$ is equivalent to the nonsingularity of $G + T$ for all constants $c_2, c_3 \in \mathbb{C} \setminus \{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$, i.e., $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (I_n - A^{k-1})(B + C)$ is nonsingular. Using (17) and (23), the form (25) follows immediately. □

Remark 3. *The results in this section show us that the nonsingularity of the linear combination of two k-potent matrices does not depend on the choice of nonzero complex constants as long as the assumptions of the previous theorems are met.*

5. Conclusions

In this paper, we used an elegant representation of k-potent matrices and the matrix rank to investigate the invertibility of a linear combination of two or three k-potent matrices under some conditions. Similar to [21,22] for idempotents, we proved that the invertibility of a linear combination of two commuting k-potent matrices is independent of the choice of the nonzero complex constants. Furthermore, we proved that the invertibility of a linear combination of two or three k-potent matrices is independent of the choice of the nonzero complex constants under various conditions. An open problem is if the given conclusion also holds for the invertibility of the linear combination of two or three arbitrary k-potent matrices.

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References

1. Ben-Israel A.; Greville, T.N.E. *Generalized Inverses: Theory and Applications*, 2nd ed.; Springer: New York, NY, USA, 2003.
2. Campbell S.L.; Meyer C.D. *Generalized Inverses of Linear Transformations*; SIAM, Society of Industrial and Applied Mathematics: Philadelphia, PA, USA, 2009.
3. Rao, C.R.; Mitra S.K. *Generalized Inverse of Matrices and its Applications*; Wiley: New York, NY, USA, 1971.
4. Wang, W.; Xu, S.; Benítez, J. Rank Equalities Related to the Generalized Inverses $A\|(B_1, C_1)$, $D\|(B_2, C_2)$ of Two matrices A and D. *Symmetry* **2019**, *11*, 539. [CrossRef]
5. Ahmed, D.; Hama, M.; Jwamer, K.H.F.; Shateyi, S. A Seventh-Order Scheme for Computing the Generalized Drazin Inverse. *Mathematics* **2019**, *7*, 622. [CrossRef]
6. Qin, Y.; Liu, X.; Benítez, J. Some Results on the Symmetric Representation of the Generalized Drazin Inverse in a Banach Algebra. *Symmetry* **2019**, *11*, 105. [CrossRef]
7. Graybill, F.A. *Matrices with Applications in Statistics*, 2nd ed.; Wadsworth: Belmont, CA, USA, 1983.
8. Rao, C.R.; Rao, M.B. *Matrix Algebra and Its Applications to Statistics and Econometrics*; World Scientific: Singapore, 1998.
9. Seber, G.A.F. *Matrix Handbook for Statisticians*; Wiley: East Orange, NJ, USA, 2007.
10. Baksalary, O.M. On *k*-potent matrices, Electron. *J. Linear Algebra*, **2013**, *26*, 446–470.
11. Benítez, J.; Liu, X.; Zhu T. Nonsingularity and group invertibility of linear combination of two *k*-potent matrices. *Linear Multilinear Algebra* **2010**, *58*, 1023–1035. [CrossRef]
12. Deng, C.; Cvetkovi ć-Ilić, D.; Wei, Y. On ivertibility of combinations of *k*-potent operators. *Linearna Algebra Appl.* **2012**, *437*, 376–387. [CrossRef]
13. Deng C. Invertibility and stability of linear combinations of *m*-potent operators. *Linear Multilinear Algebra* **2014**, *62*, 1365–1376. [CrossRef]
14. Huang, R.; Liu, J.; Zhu, L. A structural characterization of real *k*-potent matrices. *Linear Multilinear Algebra* **2011**, *59*, 433–439. [CrossRef]
15. Huang, Z.; Lin, H. On *k*-idempotent 0–1 matrices. *Linear Algebra Appl.* **2020**, *597*, 187–197. [CrossRef]
16. Ilišević D.; Thome N. When is the Hermitian/skew-Hermitian part of a matrix a potent matrix? *Electron. J. Linear Algebra* **2012**, *24*, 95–112. [CrossRef]
17. Khatri, C.G. Powers of matrices and idempotency. *Linear Algebra Appl.* **1980**, *33*, 57–65. [CrossRef]
18. McCloskey, J.P. Characterizations of *r*-potent matrices. *Math. Proc. Camb. Phil. Soc.* **1984**, *96*, 213–222. [CrossRef]
19. McCloskey, J.P. Properties of normal *r*-potent matrices. *Linear Multilinear Algebra* **1985**, *17*, 325–336. [CrossRef]
20. Wu, Y.; Linder D.F. On the eigenstructures of functional *k*-potent matrices and their integral forms. *WSEAS Trans. Math.* **2010**, *9*, 244–253.
21. Baksalary, J.K.; Baksalary, O.M. Nonsingularity of linear combinations of idempotent matrices. *Linear Algebra Appl.* **2004**, 388, 25–29. [CrossRef]
22. Koliha, J.J.; Rakočević, V. The nullity and rank of linear combinations of idempotent matrices. *Linear Algebra Appl.* **2006**, *418*, 11–14. [CrossRef]
23. Benítez J.; Thome N. $\{k\}-$Group Periodic Matrices. *SIAM. J. Matrix Anal. Appl.* **2006**, *28*, 9–25. [CrossRef]