

Article

Generalized Affine Connections Associated with the Space of Centered Planes

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Abstract: Our purpose is to study a space Π of centered m -planes in n -projective space. Generalized fiberings (with semi-gluing) are investigated. Planar and normal affine connections associated with the space Π are set in the generalized fiberings. Fields of these affine connection objects define torsion and curvature tensors. The canonical cases of planar and normal generalized affine connections are considered.

Keywords: projective space; space of centered planes; planar generalized affine connection; normal generalized affine connection



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1. Introduction

The theory of connections occupies an important place in differential geometry. The concept of connection was introduced by T. Levi-Civita [1] as a parallel displacement of tangent vectors of a manifold. Weyl introduced the concept of a space of the affine connection [2]. This concept was continued in the works of Cartan [3] and Ehresmann [4]. Different ways to determine the connection were inputted by Nomizu [5], Vagner [6], and Laptev [7]. Veblen and Whitehead [8] introduced a connection in a composite manifold. In [9], Laptev gave an invariant definition of connections as a certain law defining the mapping of infinitely close fibres. He also provided a well-known theoretical-group method on the basis of the Cartan calculation.

Rashevskii [10], Nomizu [11], and Norden [12] studied affine connections. Generalized affine connections were considered in book [13], where a relation was shown between a generalized affine connection and a linear connection (see also [14,15]). In [16], all invariant affine connections on three-dimensional symmetric homogeneous spaces admitting a normal connection were described; canonical connections and natural torsion-free connections were also considered.

The theory of affine connections is widely used in physics [17–21] and by studying geodesics [22] (see, e.g., [23–28]).

This paper is the result of the author's research [29,30], where the concepts of generalized bundles are used (see, e.g., [31]). Principal fibering with gluing to the base was introduced in the paper [32] and generalized principal fibering (a principal fibering with semi-gluing to the base) was introduced in the paper [33]. Connections in the fiberings associated with the space of centered planes were studied in [34,35]. In the present paper, generalized affine connections associated with this space are considered.

The notion of vector, or specifically, principal bundles over a smooth manifold is one of the central notions in modern mathematics and its applications to mathematical and theoretical physics. In particular, all known types of physical interactions (gravitational, electromagnetic, etc.) are described in terms of connections and other geometric structures on vector/principal bundles on underlying manifolds. The properties of physical fields can be formulated in terms of geometric invariants of connections such as curvature and characteristic classes of corresponding vector/principal bundles [36].

There are affine connections which arise in different contexts. These connections are used in geometry for illustration of parallel transports, and they can be applied to mechanics. Therefore, I hope this article will be of interest to geometers and physicists.

The important role of the affine connections and the previous author’s work in the study of manifolds of planes was the motivation for writing this paper.

This paper will be arranged in the following way. Section 2 will describe the analytical apparatus and the space of centered planes as the object of research. In Sections 3 and 4, a review of planar and normal generalized affine connections will be given, respectively. A brief summary of the paper is given in Section 5.

2. Analytical Apparatus and Object of Research

It is a well-known fact that projective space P_n can be represented as a quotient space L_{n+1}/\sim of a linear space L_{n+1} with respect to equivalence (collinearity) \sim of non-zero vectors, that is, $P_n = (L_{n+1} \setminus \{0\})/\sim$. Projective frame in the space P_n is a system formed by points $A_{I'}$, $I' = 0, \dots, n$, and a unit point E (see [37]). In linear space L_{n+1} , linearly independent vectors $e_{I'}$ correspond to the points $A_{I'}$, and a vector $e = \sum_{I'=0}^n e_{I'}$ corresponds to the point E . Moreover, these vectors are determined in the space L_{n+1} with accuracy up to a common factor. The unit point is specified together with the basic points, although you do not have to mention it every time.

In the present paper, we will use the method of a moving frame (see, e.g., [38]) $\{A, A_I\}$, $I, \dots = 1, \dots, n$, the derivation formulae of the vertices of which are

$$dA = \theta A + \omega^I A_I, \quad dA_I = \theta A_I + \omega^J A_J + \omega_I A,$$

where the form θ acts as a proportionality factor, and the structure forms $\omega^I, \omega^J, \omega_I$ of the projective group $GP(n)$, effectively acting on the space P_n , and satisfying the Cartan equations (see [39], cf. [40])

$$D\omega^I = \omega^J \wedge \omega^I_J, \quad D\omega^I_J = \omega^K \wedge \omega^I_{JK} + \delta^I_J \omega_K \wedge \omega^K + \omega_J \wedge \omega^I, \quad D\omega_I = \omega^J \wedge \omega_{IJ}.$$

Throughout this paper, d is the symbol of ordinary differentiation in the space P_n , and D is the symbol of exterior differentiation. This apparatus is successfully used by geometers in Kaliningrad (see, e.g., [41]).

In the projective space P_n , a space Π (see [34,42,43]) of all centered m -dimensional planes is considered (cf. [44–47]). Vertices A and A_a , $a, \dots = 1, \dots, m$, of the moving frame are placed on the centered plane, with vertex A fixed as a centre. The forms $\omega^a, \omega^\alpha, \omega^a_\alpha$ ($\alpha, \dots = m + 1, \dots, n$) are the basic forms of the space Π .

Remark 1. *The space Π is a differentiable manifold whose points are m -dimensional centered planes.*

The technique used in the present paper was based on the Laptev–Lumiste method. This, in turn, requires knowledge of calculating external differential forms.

3. Planar Generalized Affine Connection

Now we introduce the following definition.

Definition 1. *A smooth manifold with structure equations*

$$\begin{aligned} D\omega^a &= \omega^b \wedge \omega^a_b + \omega^\alpha \wedge \omega^a_\alpha, & D\omega^\alpha &= \omega^\beta \wedge \omega^\alpha_\beta + \omega^a \wedge \omega^a_\alpha, \\ D\omega^a_\alpha &= \omega^b \wedge (\delta^a_b \omega^\alpha_\beta - \delta^\alpha_\beta \omega^b_a) - \omega^\alpha \wedge \omega_a, & & \\ D\omega^a_b &= \omega^c \wedge \omega^a_c - \omega^c \wedge (\delta^a_c \omega_b + \delta^a_b \omega_c) - \delta^a_b \omega^\alpha \wedge \omega_\alpha + \omega^a_b \wedge \omega^a_\alpha \end{aligned} \tag{1}$$

is called a generalized bundle of planar affine frames [48] and is denoted by $A_{m^2+[m]}(\Pi)$.

Remark 2. The symbol m is enclosed in square brackets, since m forms ω^a are included in both basic and fibre forms. We will call the forms ω^a basic-fibre.

To define an affine connection in the generalized bundle $A_{m^2+[m]}(\Pi)$, we extend to it the Laptev–Lumiste method of defining group connections in principal bundles. We transform the basic-fibre forms ω^a and fibre forms ω_b^a of the fibering $A_{m^2+[m]}(\Pi)$, using linear combinations of the basic forms ω^a , ω^α , ω_α^a [32] as follows:

$$\tilde{\omega}^a = \omega^a - C_b^a \omega^b - C_\alpha^a \omega^\alpha - C_\alpha^{ab} \omega_b^\alpha, \quad \tilde{\omega}_b^a = \omega_b^a - \Gamma_{bc}^a \omega^c - \Gamma_{b\alpha}^a \omega^\alpha - \Gamma_{b\alpha}^{ac} \omega_c^\alpha. \tag{2}$$

Finding the exterior differentials of forms (2) with the help of structure equations (1) and applying the Cartan–Laptev theorem in this generalized case, we obtain

$$\begin{aligned} \Delta C_b^a &= C_{b,c}^a \omega^c + C_{b,\alpha}^a \omega^\alpha + C_{b,\alpha}^{a,c} \omega_c^\alpha, \\ \Delta C_\alpha^a - C_b^a \omega_\alpha^b + C_\alpha^{ab} \omega_b + \omega_\alpha^a &= C_{\alpha,b}^a \omega^b + C_{\alpha,\beta}^a \omega^\beta + C_{\alpha,\beta}^{a,b} \omega_b^\beta, \\ \Delta C_\alpha^{ab} &= C_{\alpha,c}^{ab} \omega^c + C_{\alpha,\beta}^{ab} \omega^\beta + C_{\alpha,\beta}^{ab,c} \omega_c^\beta, \\ \Delta \Gamma_{bc}^a - \delta_b^a \omega_c - \delta_c^a \omega_b &= \Gamma_{bc,d}^a \omega^d + \Gamma_{bc,\alpha}^a \omega^\alpha + \Gamma_{bc,\alpha}^{a,d} \omega_d^\alpha, \\ \Delta \Gamma_{b\alpha}^a - \Gamma_{bc}^a \omega_\alpha^c + \Gamma_{b\alpha}^{ac} \omega_c - \delta_b^a \omega_\alpha &= \Gamma_{b\alpha,c}^a \omega^c + \Gamma_{b\alpha,\beta}^a \omega^\beta + \Gamma_{b\alpha,\beta}^{a,c} \omega_c^\beta, \\ \Delta \Gamma_{b\alpha}^{ac} + \delta_b^c \omega_\alpha^a &= \Gamma_{b\alpha,d}^{ac} \omega^d + \Gamma_{b\alpha,\beta}^{ac} \omega^\beta + \Gamma_{b\alpha,\beta}^{ac,d} \omega_d^\beta, \end{aligned} \tag{3}$$

where $C_{b,c}^a, \dots$ are Pfaffian derivatives [32], and Δ is a tensor differential operator acting according to the law $\Delta C_b^a = dC_b^a + C_b^c \omega_c^a - C_c^a \omega_b^c$ (see, e.g., [49]).

We will use the following terminology (see [50]):

A substructure of a structure S is called *simple* if it is not a union of two substructures of the structure S . A simple substructure is called *the simplest* if it, in turn, does not have a substructure.

Statement 1. The object of a planar generalized affine connection $\overset{P}{\Gamma} = \{C_b^a, C_\alpha^a, C_\alpha^{ab}, \Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$ associated with the space Π of centered planes contains two simplest tensors C_b^a, C_α^{ab} in a simple quasi-tensor of a connection $C = \{C_b^a, C_\alpha^a, C_\alpha^{ab}\}$ and two simplest subquasi-tensors $\Gamma_{bc}^a, \Gamma_{b\alpha}^{ac}$ of a simple quasi-tensor of a planar linear connection $\{\Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$.

Let us take into account differential Equations (3) in the structure equations of the connection forms (2)

$$\begin{aligned} D\tilde{\omega}^a &= \tilde{\omega}^b \wedge \tilde{\omega}_b^a + T_{bc}^a \omega^b \wedge \omega^c + T_{b\alpha}^a \omega^b \wedge \omega^\alpha + T_{b\alpha}^{ac} \omega^b \wedge \omega_c^\alpha \\ &\quad + T_{\alpha\beta}^a \omega^\alpha \wedge \omega^\beta + T_{\alpha\beta}^{ab} \omega^\alpha \wedge \omega_b^\beta + T_{\alpha\beta}^{abc} \omega_b^\alpha \wedge \omega_c^\beta, \\ D\tilde{\omega}_b^a &= \tilde{\omega}_b^c \wedge \tilde{\omega}_c^a + R_{bcd}^a \omega^c \wedge \omega^d + R_{bca}^a \omega^c \wedge \omega^\alpha + R_{bca}^{ad} \omega^c \wedge \omega_d^\alpha \\ &\quad + R_{b\alpha\beta}^a \omega^\alpha \wedge \omega^\beta + R_{b\alpha\beta}^{ac} \omega^\alpha \wedge \omega_c^\beta + R_{b\alpha\beta}^{acd} \omega_b^\alpha \wedge \omega_d^\beta. \end{aligned} \tag{4}$$

The components of a torsion object $\overset{P}{T} = \{T_{bc}^a, T_{b\alpha}^a, T_{b\alpha}^{ac}, T_{\alpha\beta}^a, T_{\alpha\beta}^{ab}, T_{\alpha\beta}^{abc}\}$ of a planar affine connection are expressed by the formulae

$$\begin{aligned} T_{bc}^a &= \Gamma_{[bc]}^a + C_{[b,c]}^a - C_{[b}^c \Gamma_{ec]}^a, & T_{b\alpha}^a &= \Gamma_{b\alpha}^a + C_\alpha^c \Gamma_{cb}^a - C_b^c \Gamma_{c\alpha}^a + C_{b,\alpha}^a - C_{\alpha,b}^a, \\ & & T_{b\alpha}^{ac} &= \Gamma_{b\alpha}^{ac} - \delta_b^c C_\alpha^a - C_b^e \Gamma_{e\alpha}^{ac} + C_\alpha^{ec} \Gamma_{eb}^a + C_{b,\alpha}^{a,c} - C_{\alpha,b}^{ac}, \\ T_{\alpha\beta}^a &= C_{[\alpha,\beta]}^a - C_{[\alpha}^b \Gamma_{b\beta]}^a, & T_{\alpha\beta}^{ab} &= C_\beta^{cb} \Gamma_{c\alpha}^a - C_\alpha^c \Gamma_{c\beta}^{ab} + C_{\alpha,\beta}^{a,b} - C_{\beta,\alpha}^{ab}, \\ & & T_{\alpha\beta}^{abc} &= C_{[\alpha,\beta]}^{[b,c]} - C_{[\alpha}^e \Gamma_{e\beta]}^{[bc]}, \end{aligned} \tag{5}$$

and the components of the curvature object $R = \{R_{bcd}^a, R_{bc\alpha}^a, R_{bc\alpha}^{ad}, R_{b\alpha\beta}^a, R_{b\alpha\beta}^{ac}, R_{b\alpha\beta}^{acd}\}$ have the form

$$\begin{aligned}
 R_{bcd}^a &= \Gamma_{b[c,d]}^a - \Gamma_{b[c]ed}^e \Gamma_{ed}^a, & R_{bc\alpha}^a &= \Gamma_{bc,\alpha}^a - \Gamma_{b\alpha,c}^a + \Gamma_{b\alpha}^d \Gamma_{dc}^a - \Gamma_{bc}^d \Gamma_{d\alpha}^a, \\
 R_{bc\alpha}^{ad} &= \Gamma_{bc,\alpha}^{a,d} - \Gamma_{b\alpha,c}^{ad} + \Gamma_{b\alpha}^{ed} \Gamma_{ec}^a - \Gamma_{bc}^e \Gamma_{e\alpha}^{ad} - \delta_c^d \Gamma_{b\alpha}^a, \\
 R_{b\alpha\beta}^a &= \Gamma_{b[\alpha,\beta]}^a - \Gamma_{b[\alpha}^c \Gamma_{c\beta]}^a, & R_{b\alpha\beta}^{ac} &= \Gamma_{b\alpha,\beta}^{c,b} - \Gamma_{b\beta,\alpha}^{ac} + \Gamma_{b\beta}^{dc} \Gamma_{d\alpha}^a - \Gamma_{b\alpha}^d \Gamma_{d\beta}^{ac}, \\
 R_{b\alpha\beta}^{acd} &= \Gamma_{b[\alpha,\beta]}^{a[c,d]} - \Gamma_{b[\alpha}^c \Gamma_{\beta]}^{ad}].
 \end{aligned}
 \tag{6}$$

Remark 3. Here and in what follows, the square brackets denote alternation by extreme indices and by pairs of indices.

Remark 4. In the generalized case, just as in the usual case, the structure equations (4) of the connection forms (2) include the components of torsion and curvature objects expressed by the Formulae (5) and (6). Curvature R is not expressed in terms of the quasi-tensor components C and their Pfaffian derivatives.

Theorem 1. In the space Π of centered planes, the objects of torsion T and curvature R of the planar generalized affine connection are tensors containing the simplest subtensors $T_1 = \{T_{bc}^a\}$, $T_2 = \{T_{b\alpha}^{ac}\}$, $T_3 = \{T_{\alpha\beta}^{abc}\}$, $R_1 = \{R_{bcd}^a\}$, $R_2 = \{R_{bc\alpha}^{ad}\}$, $R_3 = \{R_{b\alpha\beta}^{acd}\}$ and simple subtensors $T_4 = \{T_{b\alpha}^a, T_{bc}^a, T_{b\alpha}^{ac}\}$, $T_5 = \{T_{\alpha\beta}^{ab}, T_{\alpha\beta}^{abc}, T_{c\beta}^{ab}\}$, $R_4 = \{R_{bc\alpha}^a, R_{bcd}^a, R_{bc\alpha}^{ad}\}$, $R_5 = \{R_{b\alpha\beta}^{ac}, R_{b\alpha\beta}^{acd}, R_{bd\beta}^{ac}\}$.

Proof. Prolongating Equation (3) of the components of the connection object Γ and using Formulae (5), (6) for expressions for the components of torsion and curvature objects, we find differential congruences modulo the basic forms

$$\begin{aligned}
 \Delta T_{bc}^a &\equiv 0, & \Delta T_{b\alpha}^a - 2T_{bc}^a \omega_\alpha^c + T_{b\alpha}^{ac} \omega_c &\equiv 0, & \Delta T_{b\alpha}^{ac} &\equiv 0, \\
 \Delta T_{\alpha\beta}^a + T_{b[\alpha}^a \omega_{\beta]}^b + T_{[\alpha\beta]}^{ab} \omega_b &\equiv 0, & \Delta T_{\alpha\beta}^{ab} - 2T_{\beta\alpha}^{abc} \omega_c - T_{c\beta}^{ab} \omega_\alpha^c &\equiv 0, & \Delta T_{\alpha\beta}^{abc} &\equiv 0, \\
 \Delta R_{bcd}^a &\equiv 0, & \Delta R_{bc\alpha}^a - 2R_{bcd}^a \omega_\alpha^d + R_{bc\alpha}^{ad} \omega_d &\equiv 0, & \Delta R_{bc\alpha}^{ad} &\equiv 0, \\
 \Delta R_{b\alpha\beta}^a + R_{bc[\alpha}^a \omega_{\beta]}^c + R_{b[\alpha\beta]}^{ac} \omega_c &\equiv 0, & \Delta R_{b\alpha\beta}^{ac} - 2R_{b\alpha\beta}^{acd} \omega_d - R_{bd\beta}^{ac} \omega_\alpha^d &\equiv 0, & \Delta R_{b\alpha\beta}^{acd} &\equiv 0,
 \end{aligned}$$

which these components satisfy. The obtained differential congruences prove the validity of this theorem. \square

Statement 2. Differential congruences for torsion T_i and curvature R_i subtensors correspond to each other, $i = 1, \dots, 5$.

Now we consider the canonical case of a planar affine connection when the tensor components C_b^a, C_α^{ab} of the connection quasi-tensor C vanish. In this case, $\tilde{\omega}^a = \omega^a - C_\alpha^a \omega^\alpha$, therefore, the left and right sides of the Formulae (3)₁ and (3)₃ are identically zero, and the Equations (3)₂ take the form

$$\Delta C_\alpha^a + \omega_\alpha^a = C_{\alpha,b}^a \omega^b + C_{\alpha,\beta}^a \omega^\beta + C_{\alpha,\beta}^{a,b} \omega_b^\beta,$$

where zero means the equalities $C_b^a = 0, C_\alpha^{ab} = 0$ in the expressions (5) for the components of the torsion tensor.

Statement 3. In the canonical case, the quasi-tensor C of the planar generalized affine connection is reduced to the quasi-tensor C_α^a , and the connection object is simplified: $\Gamma = \{0, C_\alpha^a, 0, \Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$.

If the tensor components C_b^a, C_α^{ab} of the connection quasi-tensor C are equal to zero, the expressions for the components of the torsion tensor have the form

$$\begin{aligned} T_{bc}^a &= \Gamma_{[bc]}^a, & T_{b\alpha}^a &= \Gamma_{b\alpha}^a + C_\alpha^c \Gamma_{cb}^a - C_{\alpha,b}^a, & T_{b\alpha}^{ac} &= \Gamma_{b\alpha}^{ac} - \delta_b^c C_\alpha^a, \\ T_{\alpha\beta}^a &= C_{[\alpha,\beta]}^a - C_{[\alpha}^b \Gamma_{b\beta]}^a, & T_{\alpha\beta}^{ab} &= C_{\alpha,\beta}^{a,b} - C_\alpha^c \Gamma_{c\beta}^{ab}, & T_{\alpha\beta}^{abc} &= 0. \end{aligned} \tag{7}$$

Statement 4. Thus, the torsion tensor of canonical connection is non-zero but contains zero components $T_{\alpha\beta}^{abc}$.

If the torsion tensor vanishes, then from the expressions (7) we obtain

$$\begin{aligned} \Gamma_{[bc]}^a &= 0, & \Gamma_{b\alpha}^a &= C_{\alpha,b}^a - C_\alpha^c \Gamma_{cb}^a, & \Gamma_{b\alpha}^{ac} &= \delta_b^c C_\alpha^a, \\ C_{[\alpha,\beta]}^a &= C_{[\alpha}^b \Gamma_{b\beta]}^a, & C_{\alpha,\beta}^{a,b} &= C_\alpha^c \Gamma_{c\beta}^{ab}, & T_{\alpha\beta}^{abc} &= 0. \end{aligned}$$

These equalities imply the following theorem.

Theorem 2. The canonical planar generalized torsion-free affine connection $\overset{PO}{\Gamma}$ has the properties:

- (1) The simplest quasi-tensor of affine connection Γ_{bc}^a is symmetric;
- (2) A planar linear connection $\{\Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$ is reduced to an affine subconnection Γ_{bc}^a using the planar subfield of the quasi-tensor of the connection C_α^a of the space Π of centered planes, that is, a subobject $\Gamma_{b\alpha}^a$ is covered by a subobject Γ_{bc}^a , complementing it with a linearly connection object $\{\Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$ by a quasi-tensor C_α^a and its planar Pfaffian derivatives $C_{\alpha,b}^a$;
- (3) The simplest quasi-tensor $\Gamma_{b\alpha}^{ac}$ is formed by the components of the quasi-tensor C_α^a ;
- (4) Alternating normal Pfaffian derivatives $C_{[\alpha,\beta]}^a$ of the connection quasi-tensor C_α^a are formed by alternations of the convolutions of the quasi-tensor C_α^b and the subobject $\Gamma_{b\beta}^a$ of the planar linear connection $\{\Gamma_{bc}^a, \Gamma_{b\alpha}^a, \Gamma_{b\alpha}^{ac}\}$;
- (5) The Pfaffian derivatives $C_{\alpha,\beta}^{a,b}$ of the connection quasi-tensor C_α^a are formed by convolutions of the components of the quasi-tensor C_α^a itself and the components of the simplest quasi-tensor $\Gamma_{c\beta}^{ab}$.

4. Normal Generalized Affine Connection

Definition 2. A smooth manifold with structure equations

$$\begin{aligned} D\omega^a &= \omega^b \wedge \omega_b^a + \omega^\alpha \wedge \omega_\alpha^a, & D\omega^\alpha &= \omega^\beta \wedge \omega_\beta^\alpha + \omega^a \wedge \omega_a^\alpha, \\ D\omega_a^\alpha &= \omega_b^\beta \wedge (\delta_a^\beta \omega_\beta^\alpha - \delta_\beta^a \omega_a^\alpha) - \omega^\alpha \wedge \omega_a, & & \\ D\omega_\beta^\alpha &= \omega_\beta^\gamma \wedge \omega_\gamma^\alpha - \omega^\gamma \wedge (\delta_\gamma^\alpha \omega_\beta + \delta_\beta^\alpha \omega_\gamma) - \delta_\beta^\alpha \omega^a \wedge \omega_a - \omega_a^\alpha \wedge \omega_\beta^a \end{aligned} \tag{8}$$

is called the generalized bundle of normal affine frames [48] and is denoted by $A_{h^2+[h]}(\Pi)$, where $h = n - m$.

Remark 5. The symbol h is enclosed in square brackets, since $n - m$ forms ω^α are included in both basic forms and fibre forms. We will call the forms ω^α basic-fibre.

To define an affine connection in the generalized bundle $A_{h^2+[h]}(\Pi)$ we extend to it the Laptev–Lumiste method. We transform the basic-fibre forms ω^α and fibre forms ω_β^α of the fibering $A_{h^2+[h]}(\Pi)$ using linear combinations of the basic forms $\omega^\alpha, \omega_a^\alpha, \omega^a$

$$\tilde{\omega}^\alpha = \omega^\alpha - L_a^\alpha \omega^a - L_\beta^\alpha \omega^\beta - L_\beta^{\alpha a} \omega_a^\beta, \quad \tilde{\omega}_\beta^\alpha = \omega_\beta^\alpha - \Gamma_{\beta a}^\alpha \omega^a - \Gamma_{\beta\gamma}^\alpha \omega^\gamma - \Gamma_{\beta\gamma}^{\alpha a} \omega_a^\gamma. \tag{9}$$

Finding the exterior differentials of forms (9) by using structure equations (8) and applying the Cartan–Laptev theorem in this generalized case, we get

$$\begin{aligned}
 \Delta L_a^\alpha &= L_{a,b}^\alpha \omega^b + L_{a,\beta}^\alpha \omega^\beta + L_{a,\beta}^{\alpha,b} \omega_b^\beta, \\
 \Delta L_\beta^\alpha + L_\beta^{\alpha a} \omega_a - L_a^\alpha \omega_\beta^a &= L_{\beta,a}^\alpha \omega^a + L_{\beta,\gamma}^\alpha \omega^\gamma + L_{\beta,\gamma}^{\alpha,a} \omega_a^\gamma, \\
 \Delta L_\beta^{\alpha a} &= L_{\beta,b}^{\alpha a} \omega^b + L_{\beta,\gamma}^{\alpha a} \omega^\gamma + L_{\beta,\gamma}^{\alpha a,b} \omega_b^\gamma, \\
 \Delta \Gamma_{\beta a}^\alpha - \delta_\beta^\alpha \omega_a &= \Gamma_{\beta a,b}^\alpha \omega^b + \Gamma_{\beta a,\gamma}^\alpha \omega^\gamma + \Gamma_{\beta a,\gamma}^{\alpha,b} \omega_b^\gamma, \\
 \Delta \Gamma_{\beta\gamma}^\alpha - \Gamma_{\beta a}^\alpha \omega_\gamma^a + \Gamma_{\beta\gamma}^{\alpha a} \omega_a - \delta_\beta^\alpha \omega_\gamma - \delta_\gamma^\alpha \omega_\beta &= \Gamma_{\beta\gamma,a}^\alpha \omega^a + \Gamma_{\beta\gamma,\mu}^\alpha \omega^\mu + \Gamma_{\beta\gamma,\mu}^{\alpha,a} \omega_a^\mu, \\
 \Delta \Gamma_{\beta\gamma}^{\alpha a} - \delta_\gamma^\alpha \omega_\beta^a &= \Gamma_{\beta\gamma,b}^{\alpha a} \omega^b + \Gamma_{\beta\gamma,\mu}^{\alpha a} \omega^\mu + \Gamma_{\beta\gamma,\mu}^{\alpha a,b} \omega_b^\mu.
 \end{aligned}
 \tag{10}$$

Statement 5. The object of normal generalized affine connection $\overset{N}{\Gamma} = \{L_a^\alpha, L_\beta^\alpha, L_\beta^{\alpha a}, \Gamma_{\beta a}^\alpha, \Gamma_{\beta\gamma}^\alpha, \Gamma_{\beta\gamma}^{\alpha a}\}$ associated with the space of centered planes contains two simplest subtensors $L_a^\alpha, L_\beta^{\alpha a}$ of a simple tensor of connection $L = \{L_a^\alpha, L_\beta^\alpha, L_\beta^{\alpha a}\}$ and two simplest subquasi-tensors $\Gamma_{\beta a}^\alpha, \Gamma_{\beta\gamma}^{\alpha a}$ of a simple quasi-tensor of normal linear connection $\{\Gamma_{\beta a}^\alpha, \Gamma_{\beta\gamma}^\alpha, \Gamma_{\beta\gamma}^{\alpha a}\}$.

We substitute the differential Equations (10) into the structure equations of the connection forms (9)

$$\begin{aligned}
 D\tilde{\omega}^\alpha &= \tilde{\omega}^\beta \wedge \tilde{\omega}_\beta^\alpha + T_{ab}^\alpha \omega^a \wedge \omega^b + T_{\beta a}^\alpha \omega^\beta \wedge \omega^a + T_{a\beta}^{\alpha b} \omega^a \wedge \omega_b^\beta \\
 &\quad + T_{\beta\gamma}^\alpha \omega^\beta \wedge \omega^\gamma + T_{\beta\gamma}^{\alpha a} \omega^\beta \wedge \omega_a^\gamma + T_{\beta\gamma}^{\alpha b} \omega_b^\beta \wedge \omega_a^\gamma, \\
 D\tilde{\omega}_\beta^\alpha &= \tilde{\omega}_\beta^\gamma \wedge \tilde{\omega}_\gamma^\alpha + R_{\beta ab}^\alpha \omega^a \wedge \omega^b + R_{\beta\gamma a}^\alpha \omega^\gamma \wedge \omega^a + R_{\beta a\gamma}^{\alpha b} \omega^a \wedge \omega_b^\gamma \\
 &\quad + R_{\beta\gamma\mu}^{\alpha a} \omega^\gamma \wedge \omega^\mu + R_{\beta\gamma\mu}^{\alpha a} \omega^\gamma \wedge \omega_a^\mu + R_{\beta\gamma\mu}^{\alpha a b} \omega_a^\gamma \wedge \omega_b^\mu.
 \end{aligned}
 \tag{11}$$

The components of a torsion object $\overset{N}{T} = \{T_{ab}^\alpha, T_{\beta a}^\alpha, T_{a\beta}^{\alpha b}, T_{\beta\gamma}^\alpha, T_{\beta\gamma}^{\alpha a}, T_{\beta\gamma}^{\alpha ab}\}$ of normal affine connection are expressed by the formulae

$$\begin{aligned}
 T_{ab}^\alpha &= L_{[a,b]}^\alpha - L_{[a}^\beta \Gamma_{\beta b]}^\alpha, \quad T_{\beta a}^\alpha = \Gamma_{\beta a}^\alpha + \Gamma_{\gamma\beta}^\alpha L_a^\gamma - L_\beta^\gamma \Gamma_{\gamma a}^\alpha - L_{a,\beta}^\alpha + L_{\beta,a}^\alpha, \\
 T_{a\beta}^{\alpha b} &= L_{a,\beta}^{\alpha b} - L_{\beta,a}^{\alpha b} - \Gamma_{\gamma\beta}^{\alpha b} L_a^\gamma + L_\beta^{\gamma b} \Gamma_{\gamma a}^\alpha + \delta_a^b \delta_\beta^\alpha - \delta_a^b L_{\beta}^{\alpha a}, \\
 T_{\beta\gamma}^\alpha &= L_{[\beta,\gamma]}^\alpha + \Gamma_{[\beta\gamma]}^\alpha - L_{[\beta}^\mu \Gamma_{\mu\gamma]}^\alpha, \\
 T_{\beta\gamma}^{\alpha a} &= L_{\beta,\gamma}^{\alpha a} + \Gamma_{\beta\gamma}^{\alpha a} - L_\beta^\mu \Gamma_{\mu\gamma}^{\alpha a} + L_\gamma^\mu \Gamma_{\mu\beta}^{\alpha a} - L_{\gamma,\beta}^{\alpha a}, \quad T_{\beta\gamma}^{\alpha ab} = L_{[\beta,\gamma]}^{\alpha a b} + \Gamma_{[\beta}^{\alpha a} L_{\gamma]}^{\mu b}],
 \end{aligned}
 \tag{12}$$

and the curvature object $\overset{N}{R} = \{R_{\beta ab}^\alpha, R_{\beta\gamma a}^\alpha, R_{\beta a\gamma}^{\alpha b}, R_{\beta\gamma\mu}^\alpha, R_{\beta\gamma\mu}^{\alpha a}, R_{\beta\gamma\mu}^{\alpha ab}\}$ has components

$$\begin{aligned}
 R_{\beta ab}^\alpha &= \Gamma_{\beta[a,b]}^\alpha + \Gamma_{\gamma[a}^\alpha \Gamma_{\beta b]}^\gamma, \quad R_{\beta\gamma a}^\alpha = \Gamma_{\beta\gamma,a}^\alpha - \Gamma_{\beta a,\gamma}^\alpha - \Gamma_{\mu a}^\alpha \Gamma_{\beta\gamma}^\mu + \Gamma_{\mu\gamma}^\alpha \Gamma_{\beta a}^\mu, \\
 R_{\beta a\gamma}^{\alpha b} &= \Gamma_{\beta a,\gamma}^{\alpha b} - \Gamma_{\beta\gamma,a}^{\alpha b} + \Gamma_{\mu a}^\alpha \Gamma_{\beta\gamma}^{\mu b} - \Gamma_{\mu\gamma}^{\alpha b} \Gamma_{\beta a}^\mu - \delta_\beta^b \Gamma_{\beta\gamma}^{\alpha a}, \\
 R_{\beta\gamma\mu}^\alpha &= \Gamma_{\beta[\gamma,\mu]}^\alpha + \Gamma_{\eta[\gamma}^\alpha \Gamma_{\beta\mu]}^\eta, \quad R_{\beta\gamma\mu}^{\alpha a} = \Gamma_{\beta[\gamma,\mu]}^{\alpha a} - \Gamma_{\beta\gamma,\mu}^{\alpha a} + \Gamma_{\eta\gamma}^\alpha \Gamma_{\beta\mu}^{\eta a} - \Gamma_{\eta\mu}^{\alpha a} \Gamma_{\beta\gamma}^\eta, \\
 R_{\beta\gamma\mu}^{\alpha ab} &= \Gamma_{\beta[\gamma,\mu]}^{\alpha a b} + \Gamma_{\eta[\gamma}^{\alpha a} \Gamma_{\beta\mu]}^{\eta b}].
 \end{aligned}
 \tag{13}$$

Remark 6. In the generalized case, as in the usual case, the structure Equations (11) of the connection forms (9) include the components of torsion and curvature objects expressed by Formulae (12) and (13). The curvature $\overset{N}{R}$ is not expressed in terms of the quasi-tensor L components and their Pfaffian derivatives.

Theorem 3. In the space Π of centered planes, the torsion $\overset{N}{T}$ and curvature $\overset{N}{R}$ objects of normal affine connection are tensors containing the simplest subtensors $\overset{N}{T}_1 = \{T_{ab}^\alpha\}, \overset{N}{T}_2 = \{T_{a\beta}^{\alpha b}\}, \overset{N}{T}_3 = \{T_{\beta\gamma}^{\alpha ab}\}, \overset{N}{R}_1 = \{R_{\beta ab}^\alpha\}, \overset{N}{R}_2 = \{R_{\beta a\gamma}^{\alpha b}\}, \overset{N}{R}_3 = \{R_{\beta\gamma\mu}^{\alpha ab}\}$ and simple subtensors $\overset{N}{T}_4 = \{T_{\beta a}^\alpha, T_{ab}^\alpha, T_{a\beta}^{\alpha b}\}, \overset{N}{T}_5 = \{T_{\beta\gamma}^{\alpha a}, T_{\beta\gamma}^{\alpha ab}, T_{b\gamma}^{\alpha a}\}, \overset{N}{R}_4 = \{R_{\beta\gamma a}^\alpha, R_{\beta ab}^{\alpha b}, R_{\beta a\gamma}^{\alpha b}\}, \overset{N}{R}_5 = \{R_{\beta\gamma\mu}^{\alpha a}, R_{\beta\mu\gamma}^{\alpha ab}, R_{\beta b\mu}^{\alpha a}\}$.

Proof. Prolongating Equations (10) of the components of the connection object $\overset{N}{\Gamma}$ and using the Formulae (12) and (13) for the expressions of the components of the torsion and curvature objects, we find the differential congruences

$$\begin{aligned} \Delta T_{ab}^\alpha &\equiv 0, & \Delta T_{\beta a}^\alpha + 2T_{ab}^\alpha \omega_\beta^b - T_{a\beta}^{\alpha b} \omega_b &\equiv 0, & \Delta T_{a\beta}^{\alpha b} &\equiv 0, \\ \Delta T_{\beta\gamma}^\alpha - T_{[\beta a}^\alpha \omega_\gamma^a] + T_{[\beta\gamma]}^{\alpha a} \omega_a &\equiv 0, & \Delta T_{\beta\gamma}^{\alpha a} - 2T_{\gamma\beta}^{\alpha ab} \omega_b - T_{b\gamma}^{\alpha a} \omega_\beta^b &\equiv 0, & \Delta T_{\beta\gamma}^{\alpha ab} &\equiv 0, \\ \Delta R_{\beta ab}^\alpha &\equiv 0, & \Delta R_{\beta\gamma a}^\alpha + 2R_{\beta ab}^\alpha \omega_\gamma^b - R_{\beta a\gamma}^{\alpha b} \omega_b &\equiv 0, & \Delta R_{\beta a\gamma}^{\alpha b} &\equiv 0, & \Delta R_{\beta\gamma\mu}^{\alpha ab} &\equiv 0, \\ \Delta R_{\beta\gamma\mu}^\alpha - R_{\beta[\gamma a}^\alpha \omega_\mu^a] + R_{\beta[\gamma\mu]}^{\alpha a} \omega_a &\equiv 0, & \Delta R_{\beta\gamma\mu}^{\alpha a} - 2R_{\beta\mu\gamma}^{\alpha ab} \omega_b - R_{\beta b\mu}^{\alpha a} \omega_\gamma^b &\equiv 0, \end{aligned}$$

which are satisfied by the components of these objects. The theorem is thereby proved. \square

Statement 6. Differential congruences for torsion $\overset{N}{T}_i$ and curvature $\overset{N}{R}_i$ subtensors correspond to each other, $i = 1, \dots, 5$.

We consider a canonical case of normal affine connection when the connection tensor L vanishes. In this case, $\tilde{\omega}^\alpha = \omega^\alpha$, that is, transformation of basic-fibre forms ω^α is not converted, and the connection object is simplified: $\overset{N0}{\Gamma} = \{0, 0, 0, \Gamma_{\beta a}^\alpha, \Gamma_{\beta\gamma}^\alpha, \Gamma_{\beta\gamma}^{\alpha a}\}$. If we take into account the equalities $L_a^\alpha = 0, L_\beta^\alpha = 0, L_\beta^{\alpha a} = 0$ in expressions (12) for the components of the torsion tensor, then

$$\overset{0}{T}_{ab}^\alpha = 0, \quad \overset{0}{T}_{\beta a}^\alpha = \Gamma_{\beta a}^\alpha, \quad \overset{0}{T}_{a\beta}^{\alpha b} = \delta_a^b \delta_{\beta'}^\alpha, \quad \overset{0}{T}_{\beta\gamma}^\alpha = \Gamma_{[\beta\gamma]}^\alpha, \quad \overset{0}{T}_{\beta\gamma}^{\alpha a} = \Gamma_{\beta\gamma}^{\alpha a}, \quad \overset{0}{T}_{\beta\gamma}^{\alpha ab} = 0.$$

These equalities indicate the validity of the following theorem.

Theorem 4. In the canonical case, the torsion tensor of a normal affine connection contains zero components $T_{ab}^\alpha, T_{\beta\gamma}^{\alpha b}$, the components $T_{\beta a}^\alpha, T_{\beta\gamma}^{\alpha a}$ coincide with the corresponding components of the connection object; components $T_{\beta\gamma}^\alpha$ are alternations of analogous components of a connection object; and components $T_{a\beta}^{\alpha b}$ are equal to the product of the Kronecker symbols.

Corollary 1. The canonical normal generalized affine connection associated with the space of centered planes is always with torsion. Thus, this connection is a canonical connection of the second kind (see [51], cf. [52]).

5. Conclusions

This paper studied planar and normal generalized affine connections, which are associated with the space of centered planes in projective space P_n . It was shown that, by using the Cartan–Laptev–Lumiste theory, the torsion tensor of a canonical planar generalized affine connection is non-zero but contains zero components, and the canonical normal generalized affine connection is always with torsion. Moreover, some properties have been obtained for the canonical planar generalized torsion-free connection.

It is important to emphasize that affine connections are very popular in various kinds of research [53,54], and, therefore, we hope that this paper will be useful for geometers.

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