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Geometry of Bi-Warped Product Submanifolds of Nearly Trans-Sasakian Manifolds

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Abstract: In the present work, we consider two types of bi-warped product submanifolds, $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ and $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$, in nearly trans-Sasakian manifolds and construct inequalities for the squared norm of the second fundamental form. The main results here are a generalization of several previous results. We also design some applications, in view of mathematical physics, and obtain relations between the second fundamental form and the Dirichlet energy. The relationship between the eigenvalues and the second fundamental form is also established.

Keywords: geometric inequalities; nearly trans-sasakian; nearly sasakian; bi-warped products; dirichlet energy



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1. Introduction

Let $\mathbf{M} = \mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{M}_3 \times \cdots \times \mathbf{M}_k$ be the Cartesian product of Riemannian manifolds $\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k$ and let $\pi_i : \mathbf{M} \rightarrow \mathbf{M}_k$ denote the canonical projection maps for $i = 1, \dots, k$. If the positive-valued functions $f_1, \dots, f_k$ are defined such that $f_1, \dots, f_k : \mathbf{M}_1 \rightarrow (0, \infty)$, then the Riemannian metric g is defined as

$$g(X, Y) = g(\pi_{1*}X, \pi_{1*}Y) + \sum_{i=1}^k (f_i \circ \pi) g(\pi_{i*}X, \pi_{i*}Y),$$

where $*$ is the symbol for tangent map, for any X, Y tangent to $\mathbf{M}$, then $\mathbf{M}$ is called a multiple warped product manifold [1,2]. If we choose two fibers of a multiple warped product $\mathbf{M}^1 \times_{f_1} \mathbf{M}^2 \times \cdots \times_{f_k} \mathbf{M}^k$, such that $\mathbf{M} = \mathbf{M}_1 \times_{f_1} \mathbf{M}_2 \times_{f_2} \mathbf{M}_3$, then $\mathbf{M}$ is defined as a bi-warped product submanifold, which satisfies the following:

$$\nabla_{\mathbb{U}_1} \mathbb{Z} = \sum_{i=2}^3 (\mathbb{U}_1 \ln f_i) \mathbb{Z}_i, \quad (1)$$

where $\mathbb{U}_1 \in \Gamma(T\mathbf{M}_1)$, $\mathbb{Z} \in \Gamma(T(\mathbf{M}_2 \times \mathbf{M}_3))$ and $\mathbb{Z}_i$ tangent to $\mathbf{M}_i$, for each $i = 2, 3$. Moreover, ∇ is the Levi-Civita connection and for more details, see [3–5]. An odd-dimensional analog of the nearly Kähler metric is the nearly Sasakian metric. The nearly Kähler cone over a nearly Sasakian Einstein manifold has applications in physics. The Sasakian geometry has been extensively studied, due to its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics—for example, related to p -brane solutions in superstring theory and the Maldacena conjecture [6]. On the other hand, the vanishing of Dirichlet energies is equivalent to the Dirichlet condition with the unique solution of Poisson's equation $\nabla^2 f = -4f\rho$. This implies Neumann or Dirichlet boundary conditions, as classified by the electrostatic problem (see, e.g., [7]). In the present paper, we consider the bi-warped product submanifolds in a nearly trans-Sasakian manifold, inspired by the publication of Taştan's seminal work [5], and obtained some inequalities for the Dirichlet energy and the second fundamental form. The study of bi-warped product

submanifolds with two distinct fibers has been a topic of great interest; see, e.g., Naghi et al. [4], Ali et al. [8], Siraj et al. [9–11], and Awatif et al. [12]. It has been noted that the class of bi-warped product submanifolds is a generalization of several classes, such as CR-warped products, warped product semi-slant submanifolds, and warped product pseudo-slant submanifolds. On the other hand, as a generalization of nearly cosymplectic, nearly Sasakian [13], nearly Kenmotsu [8,14], nearly α -Sasakian, and nearly β -Kenmotsu manifolds, nearly trans-Sasakian manifolds have been studied on a large scale; see [15–19]. Therefore, our objective was to remove the gap in the nearly trans-Sasakian manifold literature, as they are an interesting structure of the almost contact manifolds that have generalized many others structures. The main goal of this paper was to discuss the geometry of bi-warped product submanifolds of the types $\mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ and $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ in a nearly trans-Sasakian manifold. Some inequalities for the length of the second fundamental form are obtained, including the length of warping functions and slant immersions. Various inequalities, which have been derived in [13,14,18–27], can be recovered from our inequalities under some conditions. Therefore, our results may find applications in mathematical physics.

2. Preliminaries

An odd-dimensional C^∞ -manifold $(\widetilde{\mathbf{M}}, g)$ associated with an almost contact structure (ψ, ζ, η) is referred to as an *almost contact metric manifold* if there exist a $(1, 1)$ tensor field ψ , a vector field ζ (called a characteristic or Reeb vector field), and a 1-form η satisfying the following conditions:

$$\psi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \psi(\zeta) = 0, \quad \eta \circ \psi = 0, \quad (2)$$

$$g(\psi U_1, \psi W_2) = g(U_1, W_2) - \eta(U_1)\eta(W_2), \quad \eta(U_1) = g(U_1, \zeta), \quad (3)$$

$\forall U_1, W_2 \in \Gamma(T\widetilde{\mathbf{M}})$. The above structure (ψ, η, ζ) can be reduced to a nearly trans-Sasakian manifold (cf. [18,19]) if the following holds:

$$\begin{aligned} (\widetilde{\nabla}_{U_1}\psi)V_1 + (\widetilde{\nabla}_{V_1}\psi)U_1 = & \alpha \left(2g(U_1, V_1)\zeta - \eta(U_1)V_1 - \eta(V_1)U_1 \right) \\ & - \beta \left(\eta(V_1)\psi U_1 + \eta(U_1)\psi V_1 \right), \end{aligned} \quad (4)$$

for any $U_1, V_1 \in \Gamma(T\widetilde{\mathbf{M}})$, where $\widetilde{\nabla}$ is the Riemannian connection associated with the metric g on $\widetilde{\mathbf{M}}$. According to the structure, we have the following classifications:

- (i) A nearly trans-Sasakian $\widetilde{\mathbf{M}}$ is a nearly cosymplectic if $\alpha = 0$ and $\beta = 0$ in (4).
- (ii) If $\alpha = 1$ and $\beta = 0$ in (4), then $\widetilde{\mathbf{M}}$ is a nearly Sasakian manifold under the condition

$$(\widetilde{\nabla}_{U_1}\psi)V_1 + (\widetilde{\nabla}_{V_1}\psi)U_1 = 2g(U_1, V_1)\zeta - \eta(V_1)U_1 - \eta(U_1)V_1. \quad (5)$$

- (iii) A nearly trans-Sasakian $\widetilde{\mathbf{M}}$ is a nearly Kenmotsu manifold if $\alpha = 0$ and $\beta = 1$ in (4).
- (iv) Similarly, *nearly α -Sasakian* and *nearly β -Kenmotsu* manifolds can be defined from a *nearly trans-Sasakian* manifold by substituting $\beta = 0$ and $\alpha = 0$ in (4), respectively.

The Gauss and Weingarten formulas, which specify the relation between Levi-Civita connections ∇ on a submanifold $\mathbf{M}$ and $\widetilde{\nabla}$ on an ambient manifold $\widetilde{\mathbf{M}}$, are given by (for more detail, see [28]):

$$\widetilde{\nabla}_{U_1}W_2 = \nabla_{U_1}W_2 + \mathbf{B}(U_1, W_2), \quad (6)$$

$$\widetilde{\nabla}_{U_1}\zeta = -A_\zeta U_1 + \nabla_{U_1}^\perp \zeta, \quad (7)$$

for every $U_1, W_2 \in \Gamma(T\mathbf{M})$ and $\zeta \in \Gamma(T^\perp\mathbf{M})$. In addition, $\mathbf{B}$ and A_ζ are the second fundamental form and shape operator, respectively, having the relation $g(\mathbf{B}(U_1, W_2), \zeta) =$

$g(A_{\xi}U_1, W_2)$. If we replace $U_1 = \xi$, $V_1 = \xi$ in (4), we find that $(\tilde{\nabla}_{\xi}\psi)\xi = 0$, which implies that $\psi\tilde{\nabla}_{\xi}\xi = 0$. Applying ψ and using (4), we get $\tilde{\nabla}_{\xi}\xi = 0$. From the Gauss formula, we get $\nabla_{\xi}\xi = 0$ and $h(\xi, \xi) = 0$. For more classification, see [18,19]. We also have:

$$\psi U_1 = T U_1 + F U_1, \quad (8)$$

in which $F U_1$ and $T U_1$ are normal and tangential elements of ψU_1 , respectively. If M is invariant and/or anti-invariant, then $F U_1$ and/or $T U_1$ are zero, respectively. Similarly, we have

$$\psi \xi = t \xi + f \xi, \quad (9)$$

where $t \xi$ (respectively $f \xi$) are tangential (respectively normal) components of $\psi \xi$. The covariant derivative of the endomorphism ψ is defined by

$$(\tilde{\nabla}_{X_0}\psi)X_5 = \tilde{\nabla}_{X_0}\psi X_5 - \psi \tilde{\nabla}_{X_0}X_5, \quad \forall X_0, X_5 \in \Gamma(TM). \quad (10)$$

There is a motivating class of submanifolds, presented as slant submanifolds. For any non-zero vector U_1 tangential to M at point p , such that U_1 is not proportional to ζ_p , $0 \leq \phi(U_1) \leq \pi/2$ refers to the angle between ψU_1 and $T_p M$, which is named the Wirtinger angle. If $\phi(U_1)$ is constant for any $U_1 \in T_p M - \langle \zeta \rangle$ at point $p \in M$, then M is referred to as the slant submanifold [29] and ϕ is the slant angle of M . We consider the following necessary and sufficient for a submanifold M to be a slant submanifold [29,30]:

$$T^2 = \cos^2 \phi (-I + \eta \otimes \zeta), \quad (11)$$

for $0 \leq \phi \leq \frac{\pi}{2}$ and T is an endomorphism defined in (8). The following result is from Equation (11):

$$g(TU_1, TW_2) = \cos^2 \phi \{g(U_1, W_2) - \eta(U_1)\eta(W_2)\}, \quad (12)$$

$$g(FU_1, FW_2) = \sin^2 \phi \{g(U_1, W_2) - \eta(U_1)\eta(W_2)\}, \quad (13)$$

$\forall U_1, W_2 \in \Gamma(TM)$. The following are the definitions of semi-slant and pseudo-slant submanifolds.

Definition 1. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a semi-slant submanifold if there exists a pair of orthogonal distributions, D and D^ϕ , on M such that

- (i) the tangent bundle TM admits the orthogonal direct decomposition $TM = D \oplus D^\phi \oplus \zeta$;
- (ii) the distribution D is invariant under ψ (i.e., $\psi D = D$); and
- (iii) the distribution D^ϕ is slant, with slant angle ϕ .

Definition 2. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions, D and D^ϕ , on M such that

- (i) the tangent bundle TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D^\phi \oplus \zeta$;
- (ii) the distribution $D^\perp$ is totally real under ψ (i.e., $\psi D^\perp \subset T^\perp M$); and
- (iii) the distribution D^ϕ is slant, with slant angle ϕ .

The mean curvature vector, $\mathcal{H}$, for an orthonormal frame $\{e_1, e_2, \dots, e_n\}$ of the tangent space TM on M^n is defined by

$$\mathcal{H} = \frac{1}{n} \text{trace}(B) = \frac{1}{n} \sum_{i=1}^n B(e_i, e_i), \quad (14)$$

where $n = \dim M$. In addition, we set

$$\|\mathbf{B}\|^2 = \sum_{\alpha, \beta=1}^n g(\mathbf{B}(e_\alpha, e_\beta), \mathbf{B}(e_\alpha, e_\beta)), \quad (15)$$

where $\|\mathbf{B}\|^2$ is the length of the second fundamental form.

For a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$, a bi-warped product submanifold $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ in $\widetilde{\mathbf{M}}$, where $\mathbf{M}_T, \mathbf{M}_\perp, \mathbf{M}_\phi$ refer to holomorphic, totally real, and proper slant submanifolds of $\widetilde{\mathbf{M}}$, respectively. Suppose that $T\mathbf{M} = \mathbf{D} \oplus \mathbf{D}^\perp \oplus \mathbf{D}^\phi$ and $T^\perp \mathbf{M} = \psi \mathbf{D}^\perp \oplus \mathbf{F} \mathbf{D}^\phi \oplus \mu$, where μ is classified as a ψ -invariant normal sub-bundle of $T^\perp \mathbf{M}$. The next practical consequence will be used later in this paper.

Proposition 1. *Let $\mathbf{M} = \mathbf{M}_1 \times_{f_1} \mathbf{M}_2 \times_{f_2} \mathbf{M}_3$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. Then, $\mathbf{M}$ is only a single warped product if the structure vector field ζ is tangent to the fiber; that is, either $\zeta \in \Gamma(\mathbf{D}_2)$ or $\zeta \in \Gamma(\mathbf{D}_3)$.*

Proof. If we take $\zeta \in \Gamma(\mathbf{D}_2)$, for any $\mathbb{U} \in \Gamma(\mathbf{D}_2)$, we then get

Making use of (1), we obtain

$$\mathbb{U}(\ln f_1)\zeta = \beta \mathbb{U}.$$

By taking the inner product with ζ and using the fact that $\zeta \in \Gamma(\mathbf{D}_2)$, we have

$$\mathbb{U}(\ln f_1) = 0,$$

where f_1 is a constant. Similarly, we can easily obtain that

$$\mathbb{U}(\ln f_2) = 0,$$

which implies that f_2 is constant. $\square$

Proposition 2. *Let $\mathbf{M} = \mathbf{M}_1 \times_{f_1} \mathbf{M}_2 \times_{f_2} \mathbf{M}_3$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. Then,*

$$\zeta(\ln f_a) = \beta, \quad \forall a = 1, 2, \quad (16)$$

for each structure vector field ζ is tangent to $\mathbf{M}_1$.

Proof. For any $\mathbb{X}_0 \in \Gamma(\mathbf{D}_1)$, we expand the following:

$$\tilde{\nabla}_{\mathbb{X}_0} \zeta = \beta \mathbb{X}_0.$$

Using (1) and (8), we obtain

$$\zeta(\ln f_1)\mathbb{X}_0 = \beta \mathbb{X}_0.$$

Taking the inner product with $\mathbb{X}_0$, we obtain

$$\zeta(\ln f_1) = \beta.$$

Similarly, we can find that

$$\zeta(\ln f_2) = \beta.$$

Therefore, the statement is proved. $\square$

Remark 1. It can be noticed that, if the structure vector field ζ is tangent to any fiber, then the warped product will be trivial due to Proposition 1. On the other hand, Proposition 2 assures that the warped product is always non-trivial if the structure vector field ζ is tangent to any base manifold of a nearly trans-Sasakian manifold.

Lemma 1. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. Therefore, we have

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = 0, \quad (17)$$

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) = 0, \quad (18)$$

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \psi \mathbb{X}_3) = -\left(\psi \mathbb{X}_0(\ln f_1) + \alpha \eta(\mathbb{X}_0)\right)g(\mathbb{X}_2, \mathbb{X}_3), \quad (19)$$

$$g(\mathbf{B}(\psi \mathbb{X}_0, \mathbb{X}_2), \psi \mathbb{X}_3) = \left(\mathbb{X}_0(\ln f_1) - \eta(\mathbb{X}_0)\right)g(\mathbb{X}_2, \mathbb{X}_3), \quad (20)$$

$$g(\mathbf{B}(\zeta, \mathbb{X}_2), \psi \mathbb{X}_3) = -\alpha g(\mathbb{X}_2, \mathbb{X}_3), \quad (21)$$

for any $\mathbb{X}_0, \mathbb{X}_1 \in \Gamma(\mathbf{D})$, $\mathbb{X}_2, \mathbb{X}_3 \in \Gamma(\mathbf{D}^\perp)$ and $\mathbb{X}_4 \in \Gamma(\mathbf{D}^\phi)$.

Proof. For all $\mathbb{X}_0, \mathbb{X}_1 \in \Gamma(\mathbf{D})$ and $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$, we have

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_1, \psi \mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_1, \mathbb{X}_2) - g(\tilde{\nabla}_{\mathbb{X}_0} \psi \mathbb{X}_1, \mathbb{X}_2).$$

Equation (1) gives the following:

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_1, \mathbb{X}_2) + \mathbb{X}_0(\ln f_1)g(\psi \mathbb{X}_1, \mathbb{X}_2).$$

The normality of vector fields implies that

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_1, \mathbb{X}_2). \quad (22)$$

Replacing $\mathbb{X}_0$ with $\mathbb{X}_1$ in (22) gives

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_1} \psi) \mathbb{X}_0, \mathbb{X}_2). \quad (23)$$

Combining Equations (22) and (23) results in

$$2g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \psi \mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_1 + (\tilde{\nabla}_{\mathbb{X}_1} \psi) \mathbb{X}_0, \mathbb{X}_2).$$

Then, using Equation (4), we arrive at the first part (17). The second part, (18), can be obtained through a similar process as the first part. For the next part, we have:

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \psi \mathbb{X}_3) = g(\tilde{\nabla}_{\mathbb{X}_2} \mathbb{X}_0, \psi \mathbb{X}_3) = g((\tilde{\nabla}_{\mathbb{X}_2} \psi) \mathbb{X}_0, \mathbb{X}_3) - g(\tilde{\nabla}_{\mathbb{X}_2} \psi \mathbb{X}_0, \mathbb{X}_3)$$

for all $\mathbb{X}_0 \in \Gamma(\mathbf{D})$ and $\mathbb{X}_2, \mathbb{X}_3 \in \Gamma(\mathbf{D}^\perp)$. From (4), (6), and (1), we obtain

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \psi W) &= -g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_2, \mathbb{X}_3) + \alpha \left\{ g(\zeta, \mathbb{X}_3) g(\mathbb{X}_0, \mathbb{X}_2) \right. \\ &\quad \left. - \eta(\mathbb{X}_2) g(\mathbb{X}_0, \mathbb{X}_3) - \eta(\mathbb{X}_0) g(\mathbb{X}_2, \mathbb{X}_3) \right\} \\ &\quad - \beta \left\{ \eta(\mathbb{X}_0) g(\psi \mathbb{X}_2, \mathbb{X}_3) + \eta(\mathbb{X}_2) g(\psi \mathbb{X}_0, \mathbb{X}_3) \right\} - \psi \mathbb{X}_0 (\ln f_1) g(\mathbb{X}_2, \mathbb{X}_3) \\ &= -g(\tilde{\nabla}_{\mathbb{X}_0} \psi \mathbb{X}_2, \mathbb{X}_3) + g(\psi \tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_2, \mathbb{X}_3) - \psi \mathbb{X}_0 (\ln f_1) g(\mathbb{X}_2, \mathbb{X}_3) \\ &\quad - \alpha \eta(\mathbb{X}_0) g(\mathbb{X}_2, \mathbb{X}_3). \end{aligned}$$

This implies the following:

$$\begin{aligned} 2g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \psi \mathbb{X}_3) &= g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_3), \psi \mathbb{X}_2) - \psi \mathbb{X}_0 (\ln f_1) g(\mathbb{X}_2, \mathbb{X}_3) \\ &\quad - \alpha \eta(\mathbb{X}_0) g(\mathbb{X}_2, \mathbb{X}_3). \end{aligned} \quad (24)$$

Replacing $\mathbb{X}_2$ with $\mathbb{X}_3$ in (24) results in

$$\begin{aligned} 2g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_3), \psi \mathbb{X}_2) &= g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \psi \mathbb{X}_3) - \psi \mathbb{X}_0 (\ln f_1) g(\mathbb{X}_2, \mathbb{X}_3) \\ &\quad - \alpha \eta(\mathbb{X}_0) g(\mathbb{X}_2, \mathbb{X}_3), \end{aligned} \quad (25)$$

which implies Equation (19) from (24) and (25). Now, replacing $\mathbb{X}_0$ with $\psi \mathbb{X}_0$ in (19) and using (2), we obtain (20). If we substitute $\mathbb{X}_0 = \zeta$ in (19), we finally reach (21). This completes the proof of the lemma. $\square$

Lemma 2. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. Then, we have

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4) = \frac{1}{2} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi \mathbb{X}_2) = 0, \quad (26)$$

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= \frac{1}{3} \left\{ (\mathbb{X}_0 \ln f_2) - \beta \eta(\mathbb{X}_0) \right\} g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) \\ &\quad - \left\{ \psi \mathbb{X}_0 (\ln f_2) + \alpha \eta(\mathbb{X}_0) \right\} g(\mathbb{X}_4, \mathbb{X}_5), \end{aligned} \quad (27)$$

$$g(\mathbf{B}(\psi \mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) = \frac{1}{3} (\psi \mathbb{X}_0 \ln f_2) g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) + \left\{ \mathbb{X}_0 (\ln f_2) - \eta(\mathbb{X}_0) \right\} g(\mathbb{X}_4, \mathbb{X}_5), \quad (28)$$

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbf{T}\mathbb{X}_5) &= \frac{1}{3} \left\{ (\mathbb{X}_0 \ln f_2) - \beta \eta(\mathbb{X}_0) \right\} \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5) \\ &\quad - \left\{ \psi \mathbb{X}_0 (\ln f_2) + \alpha \eta(\mathbb{X}_0) \right\} g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5), \end{aligned} \quad (29)$$

$$\begin{aligned} g(\mathbf{B}(\psi \mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbf{T}\mathbb{X}_5) &= \frac{1}{3} (\psi \mathbb{X}_0 \ln f_2) \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5) \\ &\quad + \left\{ \mathbb{X}_0 (\ln f_2) - \eta(\mathbb{X}_0) \right\} g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5), \end{aligned} \quad (30)$$

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= -\frac{1}{3} \left\{ (\mathbb{X}_0 \ln f_2) - \beta \eta(\mathbb{X}_0) \right\} \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5) \\ &\quad - \left\{ \psi \mathbb{X}_0 (\ln f_2) + \alpha \eta(\mathbb{X}_0) \right\} g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5), \end{aligned} \quad (31)$$

$$\begin{aligned} g(\mathbf{B}(\psi \mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= -\frac{1}{3} (\psi \mathbb{X}_0 \ln f_2) \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5) \\ &\quad + \left\{ \mathbb{X}_0 (\ln f_2) - \eta(\mathbb{X}_0) \right\} g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5), \end{aligned} \quad (32)$$

$$g(\mathbf{B}(\mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \mathbf{F}\mathbb{X}_5) = -\frac{1}{3} \left\{ \mathbb{X}_0(\ln f_2) - \beta\eta(\mathbb{X}_0) \right\} \cos^2 \phi g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) - \left\{ \psi\mathbb{X}_0(\ln f_2) + \alpha\eta(\mathbb{X}_0) \right\} \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5), \quad (33)$$

$$g(\mathbf{B}(\psi\mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \mathbf{F}\mathbb{X}_5) = -\frac{1}{3} (\ln f_2) \cos^2 \phi g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) + \left\{ \mathbb{X}_0(\ln f_2) - \eta(\mathbb{X}_0) \right\} \cos^2 \phi g(\mathbb{X}_4, \mathbb{X}_5), \quad (34)$$

$$g(\mathbf{B}(\zeta, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) = \frac{1}{3} (\zeta \ln f_2) g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) - \alpha g(\mathbb{X}_4, \mathbb{X}_5) \quad (35)$$

for all $\mathbb{X}_0 \in \Gamma(\mathbf{D})$, $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$ and $\mathbb{X}_4, \mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi)$.

Proof. For all $\mathbb{X}_0 \in \Gamma(\mathbf{D})$, $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$ and $\mathbb{X}_4 \in \Gamma(\mathbf{D}^\phi)$, from which the following can be derived:

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4) &= g(\tilde{\nabla}_{\mathbb{X}_2} \mathbb{X}_0, \psi\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_2} \mathbb{X}_0, \mathbf{T}\mathbb{X}_4) \\ &= g((\tilde{\nabla}_{\mathbb{X}_2} \psi)\mathbb{X}_0, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_2} \psi\mathbb{X}_0, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_2} \mathbb{X}_0, \mathbf{T}\mathbb{X}_4) \\ &= -g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_2, \mathbb{X}_4) - \eta(\mathbb{X}_0)g(\psi\mathbb{X}_2, \mathbb{X}_4) - \eta(\mathbb{X}_2)g(\psi\mathbb{X}_0, \mathbb{X}_4) \\ &\quad - \psi\mathbb{X}_0 \ln f_1 g(\mathbb{X}_2, \mathbb{X}_4) - \mathbb{X}_0 \ln f_1 g(\mathbb{X}_2, \mathbf{T}\mathbb{X}_4). \end{aligned}$$

By the use of (4), (10), and the normality of vector fields, we have

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4) &= -g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_2, \mathbb{X}_4) \\ &= -g(\tilde{\nabla}_{\mathbb{X}_0} \psi\mathbb{X}_2, \mathbb{X}_4) + g(\psi\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_2, \mathbb{X}_4) \\ &= g(A_{\psi\mathbb{X}_2} \mathbb{X}_0, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_2, \mathbf{T}\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_2, \mathbf{F}\mathbb{X}_4) \\ &= g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2) - \mathbb{X}_0 \ln f_1 g(\mathbb{X}_2, \mathbf{T}\mathbb{X}_4) - g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4). \end{aligned}$$

Therefore, using (8)–(10) results in

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4) = \frac{1}{2} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2), \quad (36)$$

and we reached (i), as desired. Additionally, we have

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2) &= g(\tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \psi\mathbb{X}_2) \\ &= g((\tilde{\nabla}_{\mathbb{X}_4} \psi)\mathbb{X}_0, \mathbb{X}_2) - g(\tilde{\nabla}_{\mathbb{X}_4} \psi\mathbb{X}_0, \mathbb{X}_2) \\ &= g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_4, \mathbb{X}_2) - (\psi\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_2). \end{aligned}$$

By the use of (4) and (8), we arrive at

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2) &= -g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_4, \mathbb{X}_2) \\ &= g((\tilde{\nabla}_{\mathbb{X}_4} \psi)\mathbb{X}_0, \mathbb{X}_2) + \eta(\mathbb{X}_0)g(\psi\mathbb{X}_4, \mathbb{X}_2) + \eta(\mathbb{X}_4)g(\psi\mathbb{X}_0, \mathbb{X}_2) \\ &= g(\tilde{\nabla}_{\mathbb{X}_0} \psi\mathbb{X}_4, \mathbb{X}_2) + g(\psi\tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \mathbb{X}_2) \\ &= g(\tilde{\nabla}_{\mathbb{X}_0} \mathbf{T}\mathbb{X}_4, \mathbb{X}_2) + g(\tilde{\nabla}_{\mathbb{X}_0} \mathbf{F}\mathbb{X}_4, \mathbb{X}_2) - g(\tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \psi\mathbb{X}_2). \end{aligned}$$

Therefore, it follows from (6), (7), and (1) that

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2) = (\mathbb{X}_0 \ln f_2)g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_2) + g(A_{\mathbf{F}\mathbb{X}_4} \mathbb{X}_0, \mathbb{X}_2) - g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2).$$

Once more, by the use of (10)–(11), we construct

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \psi\mathbb{X}_2) = \frac{1}{2} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_4). \quad (37)$$

From (36) and (37), we obtain the second part of (i). For the third part, we calculate the following:

$$\begin{aligned}
g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= g(\tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \psi \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \mathbf{T}\mathbb{X}_5) \\
&= -g(\psi \tilde{\nabla}_{\mathbb{X}_4} \mathbb{X}_0, \mathbb{X}_5) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&= g((\tilde{\nabla}_{\mathbb{X}_4} \psi) \mathbb{X}_0, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_4} \psi \mathbb{X}_0, \mathbb{X}_5) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&= -g((\tilde{\nabla}_{\mathbb{X}_0} \psi) \mathbb{X}_4, \mathbb{X}_5) + \alpha \left\{ 2g(\mathbb{X}_0, \mathbb{X}_4)g(\zeta, \mathbb{X}_4) \right. \\
&\quad \left. - \eta(\mathbb{X}_4)g(\mathbb{X}_0, \mathbb{X}_5) - \eta(\mathbb{X}_0)g(\mathbb{X}_4, \mathbb{X}_5) \right\} \\
&\quad - \beta \left\{ \eta(\mathbb{X}_0)g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) - \eta(\mathbb{X}_4)g(\psi \mathbb{X}_0, \mathbb{X}_5) \right\} \\
&\quad - (\psi \mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_5) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&= g(\psi(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_4, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_0} \psi \mathbb{X}_4, \mathbb{X}_5) - \beta \eta(\mathbb{X}_0)g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) \\
&\quad - (\psi \mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_5) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) - \alpha \eta(\mathbb{X}_0)g(\mathbb{X}_4, \mathbb{X}_5) \\
&= -g(\tilde{\nabla}_{\mathbb{X}_0} \mathbf{T}\mathbb{X}_4, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_0} \mathbf{F}\mathbb{X}_4, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_4, \mathbf{T}\mathbb{X}_5) - \beta \eta(\mathbb{X}_0)g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) \\
&\quad - g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_4, \mathbf{F}\mathbb{X}_5) - (\psi \mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_5) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&\quad - \alpha \eta(\mathbb{X}_0)g(\mathbb{X}_4, \mathbb{X}_5).
\end{aligned}$$

By the use of (6)–(7) and (8), we have

$$\begin{aligned}
2g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \mathbf{F}\mathbb{X}_4) - (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&\quad - (\psi \mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_5) - \beta \eta(\mathbb{X}_0)g(\mathbf{T}\mathbb{X}_4, \mathbb{X}_5) \\
&\quad - \alpha \eta(\mathbb{X}_0)g(\mathbb{X}_4, \mathbb{X}_5).
\end{aligned} \tag{38}$$

Replacing $\mathbb{X}_4$ with $\mathbb{X}_5$ in (38), we have

$$\begin{aligned}
2g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) &= g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_4), \mathbf{F}\mathbb{X}_5) + (\mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbf{T}\mathbb{X}_5) \\
&\quad - (\psi \mathbb{X}_0 \ln f_2)g(\mathbb{X}_4, \mathbb{X}_5) - \beta \eta(\mathbb{X}_0)g(\mathbf{T}\mathbb{X}_5, \mathbb{X}_4) \\
&\quad - \alpha \eta(\mathbb{X}_0)g(\mathbb{X}_4, \mathbb{X}_5).
\end{aligned} \tag{39}$$

Equations (38) and (39) give the second part. The rest of the terms are derived by interchanging $\mathbb{X}_0$ with $\psi \mathbb{X}_0$, as well as $\mathbb{X}_4$ and $\mathbb{X}_5$ with $\mathbf{T}\mathbb{X}_4$ and $\mathbf{T}\mathbb{X}_5$, respectively. Thus, the proof is completed. $\square$

Remark 2. If $\mathbf{B}(\mathbb{U}_1, \mathbb{X}_2) = 0 \quad \forall \mathbb{U}_1 \in \Gamma(\mathbf{D})$, and $\forall \mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$, $\mathbf{B}(\mathbb{U}_1, \mathbb{X}_3) = 0 \quad \forall \mathbb{U}_1 \in \Gamma(\mathbf{D})$, and $\forall \mathbb{X}_3 \in \Gamma(\mathbf{D}^\phi)$, then the bi-warped product submanifold $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ in a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$ refers to a $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic (respectively, $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic).

Remark 3. It easily proved that a proper bi-warped product submanifold $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$ is trivial, by using the $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic in (19) and (26).

Now, we are in the position to give the proof of our main result. More precisely, we give the following inequality theorem for bi-warped product submanifolds of the type $\mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$.

Theorem 1. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$. Then, the following inequality is satisfied for the second fundamental form $\mathbf{B}$:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 + \alpha^2 - \beta^2 \right\} + n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 - \beta^2 \right\} + \alpha^2 \right\}, \quad (40)$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \dim \mathbf{M}_\phi$. Moreover, $\nabla(\ln f_a)$ is the gradient of $\ln f_a$ for $a = 1, 2$, while $\zeta \in \mathbf{M}_T$, $\mathbf{M}_\phi$, and $\mathbf{M}_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\tilde{\mathbf{M}}$, respectively. If the inequality (40) becomes an equality, then $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

Proof. Assume that $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ is a n -dimensional proper bi-warped product submanifold of the nearly trans-Sasakian manifold $\tilde{\mathbf{M}}^{2m+1}$. In addition, let the local orthonormal vector fields $v_1, \dots, v_n$ of $T\mathbf{M}$ be as follows:

$$\begin{aligned} \mathbf{D} &= \text{span}\{v_1, \dots, v_t, v_{t+1} = \psi v_1, \dots, v_{2t} = \psi v_t\} \\ \mathbf{D}^\perp &= \text{span}\{v_{2t+1} = \hat{v}_1, \dots, v_{2t+l} = \hat{v}_l\} \\ \mathbf{D}^\phi &= \text{span}\{v_{2t+l+1} = v_1^*, \dots, v_{2t+l+k} = v_k^*, v_{2t+l+k+1} = \sec \phi \mathbf{T} v_1^*, \\ &\quad \dots, v_n = \sec \phi \mathbf{T} v_k^*\}. \end{aligned}$$

Therefore, $\dim \mathbf{M}_T = n_1 = 2t + 1$, $\dim \mathbf{M}_\perp = n_2 = l$, and $\dim \mathbf{M}_\phi = n_3 = 2k$. Furthermore, the orthonormal frame fields $v_1, \dots, v_{2m+1-n-l-2k}$ of the normal sub-bundle $T^\perp \mathbf{M}$ are as follows:

$$\begin{aligned} \psi \mathbf{D}^\perp &= \{v_1 = \psi \hat{v}_1, \dots, v_l = \psi \hat{v}_l\} \\ \mathbf{F} \mathbf{D}^\phi &= \text{span}\{v_{l+1} = \tilde{v}_1 = \csc \phi \mathbf{F} v_1^*, \dots, v_{l+k} = \tilde{v}_k = \csc \phi \mathbf{F} v_k^*, \\ &\quad v_{l+k+1} = \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_1^*, \dots, v_{l+2k} = \tilde{v}_{2k} = \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_k^*\} \\ \mu &= \text{span}\{v_{l+2k+1}, \dots, v_{2m+1-n-l-2k}\}. \end{aligned}$$

According the definition $\mathbf{B}$, we have

$$\|\mathbf{B}\|^2 = \sum_{a,b=1}^n g(\mathbf{B}(v_a, v_b), \mathbf{B}(v_a, v_b)).$$

The above equation can be broken into the components of submanifolds of $\mathbf{M}_T$, $\mathbf{M}_\phi$, and $\mathbf{M}_\perp$ as follows:

$$\begin{aligned} \|\mathbf{B}\|^2 &= \sum_{r=1}^l \sum_{a,b=1}^n g^2(\mathbf{B}(v_a, v_b), \psi \hat{v}_r) + \sum_{r=l+1}^{l+2k} \sum_{a,b=1}^n g^2(\mathbf{B}(v_a, v_b), v_r) \\ &\quad + \sum_{r=l+2k+1}^{2m+1-n-l-2k} \sum_{a,b=1}^n g^2(\mathbf{B}(v_a, v_b), v_r), \end{aligned} \quad (41)$$

where clearly $\dim \mu = 2m + 1 - n - l - 2k$. Expanding the above equation, according to the orthonormal bases of $\mathbf{D}^\phi$, $\mathbf{D}$, and $\mathbf{D}^\perp$ (except for the last term), we arrive at

$$\begin{aligned} \|\mathbf{B}\|^2 &\geq \sum_{r=1}^l \sum_{a,b=1}^{2t+1} g^2(\mathbf{B}(v_a, v_b), \psi \hat{v}_r) + \sum_{r=1}^l \sum_{a,b=1}^l g(\mathbf{B}(\hat{v}_a, \hat{v}_b), \psi \hat{v}_r) \\ &+ \sum_{r=1}^l \sum_{a,b=1}^{2k} g(\mathbf{B}(v_a^*, v_b^*), \psi \hat{v}_r) + \sum_{r=1}^{2k} \sum_{a,b=1}^{2t+1} g^2(\mathbf{B}(v_a, v_b), \tilde{v}_r) \\ &+ \sum_{r=1}^{2k} \sum_{a,b=1}^l g^2(\mathbf{B}(\hat{v}_a, \hat{v}_b), \tilde{v}_r) + \sum_{r=1}^{2k} \sum_{a,b=1}^{2k} g^2(\mathbf{B}(v_a^*, v_b^*), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^l \sum_{a=1}^{2t+1} \sum_{b=1}^l g^2(\mathbf{B}(v_a, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^l \sum_{a=1}^{2t+1} \sum_{b=1}^{2k} g^2(\mathbf{B}(v_a, v_b^*), \psi \hat{v}_r) \\ &+ 2 \sum_{r=1}^l \sum_{a=1}^{2k} \sum_{b=1}^l g^2(\mathbf{B}(v_a^*, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^{2k} \sum_{a=1}^l \sum_{b=1}^{2k} g^2(\mathbf{B}(\hat{v}_a, v_b^*), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^{2k} \sum_{a=1}^{2t+1} \sum_{b=1}^{2k} g^2(\mathbf{B}(v_a, v_b^*), \tilde{v}_r) + 2 \sum_{r=1}^{2k} \sum_{a=1}^{2t+1} \sum_{b=1}^l g^2(\mathbf{B}(v_a, \hat{v}_b), \tilde{v}_r). \end{aligned}$$

Utilizing Equations (17), (18), and (26) in the preceding equation, we obtain

$$\begin{aligned} \|\mathbf{B}\|^2 &\geq \sum_{r=1}^l \sum_{a,b=1}^l g(\mathbf{B}(\hat{v}_a, \hat{v}_b), \psi \hat{v}_r) + \sum_{r=1}^l \sum_{a,b=1}^{2k} g(\mathbf{B}(v_a^*, v_b^*), \psi \hat{v}_r) \\ &+ \sum_{r=1}^{2k} \sum_{a,b=1}^{2k} g^2(\mathbf{B}(v_a^*, v_b^*), \tilde{v}_r) + \sum_{r=1}^{2k} \sum_{a,b=1}^l g^2(\mathbf{B}(\hat{v}_a, \hat{v}_b), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^{2k} \sum_{a=1}^l \sum_{b=1}^{2k} g^2(\mathbf{B}(\hat{v}_a, v_b^*), \tilde{v}_r) + 2 \sum_{r=1}^l \sum_{a=1}^{2k} \sum_{b=1}^l g^2(\mathbf{B}(v_a^*, \hat{v}_b), \psi \hat{v}_r) \\ &+ 2 \sum_{r=1}^l \sum_{a=1}^{2t+1} \sum_{b=1}^l g^2(\mathbf{B}(v_a, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^{2k} \sum_{a=1}^{2t+1} \sum_{b=1}^{2k} g^2(\mathbf{B}(v_a, v_b^*), \tilde{v}_r). \quad (42) \end{aligned}$$

Leaving out all terms except for the last two relations in the above equation, we find

$$\begin{aligned} \|\mathbf{B}\|^2 &\geq 2 \sum_{r=1}^l \sum_{a=1}^{2t} \sum_{b=1}^l g^2(\mathbf{B}(v_a, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^l \sum_{b=1}^l g^2(\mathbf{B}(\zeta, \hat{v}_b), \psi \hat{v}_r) \\ &+ 2 \sum_{r=1}^{2k} \sum_{a=1}^{2t} \sum_{b=1}^{2k} g^2(\mathbf{B}(v_a, v_b^*), \tilde{v}_r) + 2 \sum_{r=1}^{2k} \sum_{b=1}^{2k} g^2(\mathbf{B}(\zeta, v_b^*), \tilde{v}_r). \end{aligned}$$

From the frame fields of tangent and normal sub-bundles of $\mathbf{M}$, after ignoring the μ -components part in (41), we derive

$$\begin{aligned} \|\mathbf{B}\|^2 &\geq 2 \sum_{r=1}^l \sum_{a=1}^t \sum_{b=1}^l g^2(\mathbf{B}(v_a, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^l \sum_{a=1}^t \sum_{b=1}^l g^2(\mathbf{B}(\psi v_a, \hat{v}_b), \psi \hat{v}_r) \\ &+ 2 \sum_{r=1}^l \sum_{b=1}^l g^2(\mathbf{B}(\zeta, \hat{v}_b), \psi \hat{v}_r) + 2 \sum_{r=1}^{2k} \sum_{b=1}^{2k} g^2(\mathbf{B}(\zeta, v_b^*), \tilde{v}_r) \\ &+ 2 \csc^2 \phi \sum_{r,b=1}^k \sum_{a=1}^t \left\{ g^2(\mathbf{B}(v_a, v_b^*), \mathbf{F}v_r^*) + g^2(\mathbf{B}(\psi v_a, v_b^*), \mathbf{F}v_r^*) \right\} \\ &+ 2 \csc^2 \phi \sec^2 \phi \sum_{r,b=1}^k \sum_{a=1}^t \left\{ g^2(\mathbf{B}(v_a, \mathbf{T}v_b^*), \mathbf{F}v_r^*) + g^2(\mathbf{B}(\psi v_a, \mathbf{T}v_b^*), \mathbf{F}v_r^*) \right\} \end{aligned}$$

$$\begin{aligned}
& + 2 \csc^2 \phi \sec^2 \phi \sum_{r,b=1}^k \sum_{a=1}^t \left\{ g^2(\mathbf{B}(\hat{v}_a, v_b^*), \mathbf{F} \mathbf{T} v_r^*) + g^2(\mathbf{B}(\psi v_a, v_b^*), \mathbf{F} \mathbf{T} v_r^*) \right\} \\
& + 2 \csc^2 \phi \sec^4 \phi \sum_{r,b=1}^k \sum_{a=1}^t \left\{ g^2(\mathbf{B}(\hat{v}_a, \mathbf{T} v_b^*), \mathbf{F} \mathbf{T} v_r^*) + g^2(\mathbf{B}(\psi v_a, \mathbf{T} v_b), \mathbf{F} \mathbf{T} v_r^*) \right\}.
\end{aligned}$$

Utilizing Equations (19), (20), and (21) in the first terms, we substitute (26)–(35) to obtain

$$\begin{aligned}
\|\mathbf{B}\|^2 & \geq 2l \sum_{a=1}^t \left\{ \left(-\psi v_a \ln f_1 + \alpha \eta(v_a) \right)^2 + \left(v_a \ln f_1 - \eta(v_a) \right)^2 \right\} \\
& + 2k \csc^2 \phi \sum_{a=1}^t \left\{ \left(-\psi v_a \ln f_2 - \alpha \eta(v_a) \right)^2 + \left(v_a \ln f_2 - \eta(v_a) \right)^2 \right\} \\
& + \frac{2k}{9} \csc^2 \phi \sec^2 \phi \cos^4 \phi \sum_{a=1}^t \left\{ \left(v_a \ln f_2 - \beta \eta(v_a) \right)^2 + \left(-\psi v_a \ln f_2 \right)^2 \right\} \\
& + \frac{2k}{9} \csc^2 \phi \sec^2 \phi \cos^4 \phi \sum_{a=1}^t \left\{ \left(v_a \ln f_2 - \beta \eta(v_a) \right)^2 + \left(-\psi v_a \ln f_2 \right)^2 \right\} \\
& + 2k \csc^2 \phi \sec^4 \phi \cos^4 \phi \sum_{a=1}^t \left\{ \left(-\psi v_a \ln f_2 - \alpha \eta(v_a) \right)^2 + \left(v_a \ln f_2 - \eta(v_a) \right)^2 \right\} \\
& + 2l\alpha^2 + 4k\alpha^2.
\end{aligned}$$

From the orthonormal frame $\eta(v_a) = 0$ for $1 \leq i \leq 2t$, and with some rearrangement in the remaining terms, we arrive at

$$\|\mathbf{B}\|^2 \geq 2l \sum_{a=1}^{2t} (v_a \ln f_1)^2 + 2l\alpha^2 + 4k \csc^2 \phi \sum_{a=1}^{2t} (v_a \ln f_1)^2 + \frac{4k}{9} \cot^2 \phi \sum_{a=1}^{2t} (v_a \ln f_1)^2 + 4k\alpha^2.$$

Using trigonometric identities and adding and subtracting some terms, we obtain

$$\begin{aligned}
\|\mathbf{B}\|^2 & \geq 2l \|\nabla \ln f_1\|^2 - 2l(v_{2t+1} \ln f_1)^2 + 2l\alpha^2 + \frac{4k}{9} (10 \csc^2 \phi - 1) \|\nabla \ln f_2\|^2 \\
& - \frac{4k}{9} (10 \csc^2 \phi - 1) (v_{2t+1} \ln f_2)^2 + 4k\alpha^2.
\end{aligned}$$

For a nearly trans-Sasakian manifold, two conditions are satisfied: $\zeta \ln f_1 = \beta$ and $\zeta \ln f_2 = \beta$ for the structure vector field ζ tangent to the base $\mathbf{M}_T$. The following inequality is obtained by substituting into the proceeding equation:

$$\begin{aligned}
\|\mathbf{B}\|^2 & \geq 2l \|\nabla \ln f_1\|^2 - 2l\beta^2 + 2l\alpha^2 \\
& + \frac{4k}{9} (10 \csc^2 \phi - 1) \|\nabla \ln f_2\|^2 - \frac{4k}{9} (10 \csc^2 \phi - 1) \beta^2 + 4k\alpha^2,
\end{aligned}$$

from which we reach the final result, (40), which we wanted to prove.

If the inequality (40) becomes an equality, the missing third part in (41) gives:

$$\mathbf{B}(\mathbf{T}\mathbf{M}, \mathbf{T}\mathbf{M}) \perp \mu. \quad (43)$$

Using terms (i) and (ii), which were not considered in (17), and (42), we can derive $\mathbf{B}(\mathbf{D}, \mathbf{D}) \perp \psi \mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}, \mathbf{D}) \perp \mathbf{F}\mathbf{D}^\phi$. It is obtained that $\mathbf{B}(\mathbf{D}, \mathbf{D}) = 0$. The missing first and fourth terms in (42) lead to the following:

$$\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \perp \psi \mathbf{D}^\perp \quad \& \quad \mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \perp \mathbf{F}\mathbf{D}^\phi. \quad (44)$$

It can easily be seen that $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, by using the missing terms in (43) and (44). In addition, the second and third terms that were left in (42) give $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$. This leads to $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) = 0$. In addition, term numbers five and six that were missed in (42) imply that $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$. These give us $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$. We obtain $\mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) \subset \psi \mathbf{D}^\perp$ by considering the second term in (26), along with (43). From the first term in (26) along with (43), we obtain $\mathbf{B}(\mathbf{D}, \mathbf{D}^\phi) \subset \mathbf{F}\mathbf{D}^\phi$. As $\mathbf{M}_T$ is totally geodesic in $\tilde{\mathbf{M}}$, by the use of this point along with $\mathbf{B}(\mathbf{D}, \mathbf{D}) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$, we find that $\mathbf{M}_T$ is totally geodesic in $\tilde{\mathbf{M}}$. As $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical in $\mathbf{M}$ and we have $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp$, and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$, it can be concluded that $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ together are totally umbilical in $\tilde{\mathbf{M}}$. Furthermore, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$ imply that $\mathbf{M}$ is neither $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic nor $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic at $\tilde{\mathbf{M}}$, in light of Remark 2. This completes the proof of the main theorem. $\square$

Some Geometric Consequences

If we consider $\dim \mathbf{M}_\phi = 2k = 0$, then Theorem 1 gives the following:

Theorem 2. (Theorem 4.1 of [18]) Let $\mathbf{M} = \mathbf{M}_T \times_f \mathbf{M}_\perp$ be a CR-warped product submanifold in a nearly trans-Sasakian manifold. Then, we have:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 + \alpha^2 - \beta^2 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$.

If $\mathbf{M}_\perp$ vanishes, then Theorem 1 implies the following:

Theorem 3. (Theorem 4.1 of [19]) Let $\mathbf{M} = \mathbf{M}_T \times_f \mathbf{M}_\phi$ be a warped product semi-slant submanifold of a nearly trans-Sasakian $\tilde{\mathbf{M}}$, then inequality (40) implies the following inequality:

$$\|\mathbf{B}\|^2 \geq n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 - \beta^2 \right\} + \alpha^2 \right\},$$

where $n_3 = \dim \mathbf{M}_\phi$.

Therefore, Theorem 1 is an extension of Theorem 4.1 of [19]. Similarly, for $\alpha = 0$, we have:

Theorem 4. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly β -Kenmotsu manifold $\tilde{\mathbf{M}}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2l \left\{ \|\nabla(\ln f_1)\|^2 - \beta^2 \right\} + 2k \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 - \beta^2 \right\} \right\},$$

where $l = \dim \mathbf{M}_\perp$, $k = \frac{1}{2} \dim \mathbf{M}_\phi$. In addition, $\nabla(\ln f_a)$ is the gradient of $\ln f_a$, while $\zeta \in \mathbf{M}_T$, $\mathbf{M}_\phi$, and $\mathbf{M}_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\tilde{\mathbf{M}}$, respectively. If inequality (40) becomes an equality, then $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic

totally in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

Inserting $\alpha = 0$ and $\beta = 1$ into Theorem 1, we have

Theorem 5. (Theorem 1 of [8]) Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\mathbf{M}}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 - 1 \right\} + n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 - 1 \right\} \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. In addition, $\nabla(\ln f_a)$ is the gradient of $\ln f_a$, while $\zeta \in \mathbf{M}_T$, $\mathbf{M}_\phi$, and $\mathbf{M}_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\tilde{\mathbf{M}}$, respectively. If inequality (40) becomes an equality, then $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

If we put $\alpha = 0$, $\beta = 0$ into Theorem 1, the following is obtained for the bi-warped product submanifold of a nearly cosymplectic manifold:

Theorem 6. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly cosymplectic manifold $\tilde{\mathbf{M}}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 \right\} + n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \|\nabla(\ln f_2)\|^2 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. Moreover, $\nabla(\ln f_a)$ is the gradient of $\ln f_a$, while $\zeta \in \mathbf{M}_T$, $\mathbf{M}_\phi$, and $\mathbf{M}_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\tilde{\mathbf{M}}$, respectively. If inequality (40) becomes an equality, then $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is totally geodesic in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

Again, we get the following by using $\alpha = 0$, $\beta = 0$, and $\dim \mathbf{M}_\phi = 0$.

Theorem 7. (Theorem 3.2 of [31]) Let $\mathbf{M} = \mathbf{M}_T \times_f \mathbf{M}_\perp$ be a CR-warped product submanifold of a nearly cosymplectic manifold. Then, we have:

$$\|\mathbf{B}\|^2 \geq 2n_2 \|\nabla(\ln f_1)\|^2,$$

where $n_2 = \dim \mathbf{M}_\perp$.

Inserting $\alpha = 0$, $\beta = 0$ and $\dim \mathbf{M}_\perp = 0$ into Theorem 1, we obtain the following:

Theorem 8. (Theorem 3.5 of [32]) Let $\mathbf{M} = \mathbf{M}_T \times_f \mathbf{M}_\phi$ be a warped product semi-slant submanifold of a nearly cosymplectic $\tilde{\mathbf{M}}$. Then, inequality (40) implies the following inequality:

$$\|\mathbf{B}\|^2 \geq 4k \left\{ \left(\frac{1}{9} \cot^2 \phi + 1 \csc^2 \phi \right) \|\nabla(\ln f_2)\|^2 \right\},$$

where $2k = \dim \mathbf{M}_\phi$.

By inserting $\beta = 0$ and $\alpha = 1$, $\beta = 0$ into Theorem 1, the following results for the bi-warped product submanifolds of a nearly α -Sasakian manifold and a nearly Sasakian manifold, respectively, can be obtained:

Theorem 9. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly α -Sasakian manifold $\tilde{\mathbf{M}}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 + \alpha^2 \right\} + n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 \right\} + \alpha^2 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. If the above inequality (40) becomes an equality, then $\mathbf{M}_\perp, \mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

Theorem 10. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly Sasakian manifold $\tilde{\mathbf{M}}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|\nabla(\ln f_1)\|^2 + 1 \right\} + n_3 \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 \right\} + 1 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. If inequality (40) becomes an equality, then $\mathbf{M}_\perp, \mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $\tilde{\mathbf{M}}$. The $\mathbf{D} \oplus \mathbf{D}^\perp$ -mixed totally geodesic and $\mathbf{D} \oplus \mathbf{D}^\phi$ -mixed totally geodesic do not exist in $\mathbf{M}$ of $\tilde{\mathbf{M}}$.

Remark 4. It can be easily seen that Theorem 4.1 of [18], Theorem 4.1 of [19], Theorem 3.2 of [31], Theorem 3.5 of [32], and Theorem 1 of [8] are special cases of our main Theorem 1.

3. Inequality for Bi-Warped Product Submanifold of the Type $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_T$

In this section, we consider the bi-warped product submanifold of type $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_T$ in a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, with respect to the tangent spaces of $\mathbf{M}_T, \mathbf{M}_\perp$ and $\mathbf{M}_\phi$, which are integral manifolds of $\mathbf{D}, \mathbf{D}^\perp$, and $\mathbf{D}^\phi$, respectively.

Lemma 3. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ be a bi-warped product submanifold of a trans-Sasakian manifold $\tilde{\mathbf{M}}$. Then, we construct the following:

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) = \left\{ (\mathbf{T}\mathbb{X}_4 \ln f_1) + \alpha \eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \mathbb{X}_1) \quad (45)$$

$$2g(\mathbf{B}(\mathbb{X}_2, \mathbb{X}_3), \mathbf{F}\mathbb{X}_4) = g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \psi\mathbb{X}_3) + g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_3), \psi\mathbb{X}_2) + 2 \left\{ \mathbf{T}\mathbb{X}_4 \ln f_2 + \alpha \eta(\mathbb{X}_4) \right\} g(\mathbb{X}_2, \mathbb{X}_3) \quad (46)$$

$$g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5),$$

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \mathbf{F}\mathbb{X}_4) = 0.$$

Proof. For any $\mathbb{X}_0, \mathbb{X}_1 \in \Gamma(\mathbf{D})$ and $\mathbb{X}_4, \mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi \oplus \zeta)$, we have

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) &= g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_1, \mathbf{F}\mathbb{X}_4) = g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_1, \psi\mathbb{X}_4) - (\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_1, \mathbf{T}\mathbb{X}_4) \\ &= g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_1, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_0} \psi\mathbb{X}_1, \mathbb{X}_4) + g(\mathbb{X}_1, \tilde{\nabla}_{\mathbb{X}_0} \mathbf{T}\mathbb{X}_4). \end{aligned}$$

Using (4), (8), (10), and (1), we get

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) &= -g((\tilde{\nabla}_{\mathbb{X}_1})\mathbb{X}_0, \mathbb{X}_4) + \left\{ 2g(\mathbb{X}_0, \mathbb{X}_1)g(\zeta, \mathbb{X}_4) - \eta(\mathbb{X}_0)g(\mathbb{X}_1, \mathbb{X}_4) - \eta(\mathbb{X}_1)g(\mathbb{X}_0, \mathbb{X}_4) \right\} \\ &\quad - \beta \left\{ \eta(\mathbb{X}_0)g(\psi\mathbb{X}_1, \mathbb{X}_4) + \eta(\mathbb{X}_1)(\psi, \mathbb{X}_0, \mathbb{X}_4) \right\} + \mathbf{T}\mathbb{X}_4(\ln f_1)g(\mathbb{X}_0, \mathbb{X}_1) \\ &= -g(\tilde{\nabla}_{\mathbb{X}_1}\psi\mathbb{X}_0, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_1}\mathbb{X}_0, \mathbf{F}\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_1}\mathbb{X}_0, \mathbf{T}\mathbb{X}_4) + 2\alpha\eta(\mathbb{X}_4)g(\mathbb{X}_0, \mathbb{X}_1) \\ &\quad + \mathbf{T}\mathbb{X}_4(\ln f_1)g(\mathbb{X}_0, \mathbb{X}_1) \\ &= (\mathbb{X}_4 \ln f_1)g(\psi\mathbb{X}_0, \mathbb{X}_1) - g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) + 2\alpha\eta(\mathbb{X}_4)g(\mathbb{X}_0, \mathbb{X}_1) \\ &\quad + 2\mathbf{T}\mathbb{X}_4(\ln f_1)g(\mathbb{X}_0, \mathbb{X}_1) - (\mathbb{X}_4 \ln f_1)g(\psi\mathbb{X}_0, \mathbb{X}_1). \end{aligned}$$

This implies the first part (45) of the lemma. For the next part, we have, from (8),

$$g(\mathbf{B}(\mathbb{X}_2, \mathbb{X}_3), \mathbf{F}\mathbb{X}_4) = g(\tilde{\nabla}_{\mathbb{X}_2}\mathbb{X}_3, \psi\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_2}\mathbb{X}_3, \mathbf{T}\mathbb{X}_4).$$

Using (10) and (1), we obtain

$$g(\mathbf{B}(\mathbb{X}_2, \mathbb{X}_3), \mathbf{F}\mathbb{X}_4) = g((\tilde{\nabla}_{\mathbb{X}_2}\psi)\mathbb{X}_3, \mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_2}\psi\mathbb{X}_3, \mathbb{X}_4) + \mathbf{T}\mathbb{X}_4 \ln f_2 g(\mathbb{X}_2, \mathbb{X}_3).$$

By use of (4), (8), and (10), we get

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_2, \mathbb{X}_3), \mathbf{F}\mathbb{X}_4) &= -g((\tilde{\nabla}_{\mathbb{X}_3}\psi)\mathbb{X}_2, \mathbb{X}_4) + \alpha \left\{ 2\eta(\mathbb{X}_4)g(\mathbb{X}_2, \mathbb{X}_3) \right. \\ &\quad \left. - \eta(\mathbb{X}_2)g(\mathbb{X}_3, \mathbb{X}_4) - \eta(\mathbb{X}_3)g(\mathbb{X}_2, \mathbb{X}_4) \right\} \\ &\quad - \beta \left\{ \eta(\mathbb{X}_2)g(\psi\mathbb{X}_3, \mathbb{X}_4) - \eta(\mathbb{X}_3)g(\psi\mathbb{X}_2, \mathbb{X}_4) \right\} + \mathbf{T}\mathbb{X}_4 \ln f_2 g(\mathbb{X}_2, \mathbb{X}_3) \\ &\quad + g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \psi\mathbb{X}_3) \\ &= g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \psi\mathbb{X}_3) + g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_3), \psi\mathbb{X}_2) \\ &\quad + \mathbf{T}\mathbb{X}_4 \ln f_2 g(\mathbb{X}_2, \mathbb{X}_3) + 2\alpha\eta(\mathbb{X}_4)g(\mathbb{X}_2, \mathbb{X}_3) \\ &\quad - g(\tilde{\nabla}_{\mathbb{X}_3}\mathbb{X}_2, \mathbf{F}\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_3}\mathbb{X}_2, \mathbf{T}\mathbb{X}_4) \\ &= g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \psi\mathbb{X}_3) + g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_3), \psi\mathbb{X}_2) \\ &\quad + 2\mathbf{T}\mathbb{X}_4 \ln f_2 g(\mathbb{X}_2, \mathbb{X}_3) + 2\alpha\eta(\mathbb{X}_4)g(\mathbb{X}_2, \mathbb{X}_3). \end{aligned}$$

This is equivalent to (46). For any $\mathbb{X}_4, \mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi)$, $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$,

$$g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\tilde{\nabla}_{\mathbb{X}_4}\mathbb{X}_5, \psi\mathbb{X}_2) = g((\tilde{\nabla}_{\mathbb{X}_4}\psi)\mathbb{X}_5, \mathbb{X}_2) - g(\tilde{\nabla}_{\mathbb{X}_4}\psi\mathbb{X}_5, \mathbb{X}_2).$$

Utilizing (4), we get the following:

$$g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\tilde{\nabla}_{\mathbb{X}_4}\mathbb{X}_2, \psi\mathbb{X}_5) = g(\nabla_{\mathbb{X}_4}\mathbb{X}_2, \mathbf{F}\mathbb{X}_5) + g(\tilde{\nabla}_{\mathbb{X}_4}\mathbb{X}_2, \mathbf{F}\mathbb{X}_5).$$

In view of (6) and (7), we get

$$g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\mathbf{B}(\mathbb{X}_4, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5).$$

For the last part, we have

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \mathbf{F}\mathbb{X}_4) &= g(\tilde{\nabla}_{\mathbb{X}_5}\mathbb{X}_0, \psi\mathbb{X}_4) - g(\tilde{\nabla}_{\mathbb{X}_5}\mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \\ &= -g(\tilde{\nabla}_{\mathbb{X}_5}\psi\mathbb{X}_0, \mathbb{X}_4) + g((\tilde{\nabla}_{\mathbb{X}_5}\psi)\mathbb{X}_0, \mathbb{X}_4) - \mathbb{X}_5(\ln f_1)g(\mathbb{X}_0, \mathbf{T}\mathbb{X}_4), \end{aligned}$$

for any $\mathbb{X}_0 \in \Gamma(\mathbf{D})$ and $\mathbb{X}_4, \mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi)$. Using (4), (7), and the normality of vector fields, we arrive at

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \mathbf{F}\mathbb{X}_4) = 0.$$

Thus, the required results are obtained. $\square$

The following equalities can be derived by exchanging $\mathbb{X}_0$ with $\psi\mathbb{X}_0$ and $\mathbb{X}_1$ with $\psi\mathbb{X}_1$ in Lemma 3:

$$g(\mathbf{B}(\psi\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbb{X}_4) = \left\{ \mathbf{T}\mathbb{X}_4(\ln f_1) + \alpha\eta(\mathbb{X}_4) \right\} g(\psi\mathbb{X}_0, \mathbb{X}_1), \quad (47)$$

$$g(\mathbf{B}(\mathbb{X}_0, \psi\mathbb{X}_1), \mathbf{F}\mathbb{X}_4) = \left\{ \mathbf{T}\mathbb{X}_4(\ln f_1) + \alpha\eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \psi\mathbb{X}_1), \quad (48)$$

and

$$g(\mathbf{B}(\psi\mathbb{X}_0, \psi\mathbb{X}_1), \mathbf{F}\mathbb{X}_4) = \left\{ (\mathbf{T}\mathbb{X}_4 \ln f_1) + \alpha\eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \mathbb{X}_1). \quad (49)$$

Similarly, if we interchange $\mathbb{X}_4$ with $\mathbf{T}\mathbb{X}_4$ in Lemma 3, (47)–(49), and (16), we can derive

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbf{T}\mathbb{X}_4) = -\cos^2 \phi \left\{ (\mathbb{X}_4 \ln f_1) - \eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \mathbb{X}_1), \quad (50)$$

$$g(\mathbf{B}(\psi\mathbb{X}_0, \mathbb{X}_1), \mathbf{F}\mathbf{T}\mathbb{X}_4) = -\cos^2 \phi \left\{ (\mathbb{X}_4 \ln f_1) - \eta(\mathbb{X}_4) \right\} g(\psi\mathbb{X}_0, \mathbb{X}_1) \quad (51)$$

$$g(\mathbf{B}(\mathbb{X}_0, \psi\mathbb{X}_1), \mathbf{F}\mathbf{T}\mathbb{X}_4) = -\cos^2 \phi \left\{ (\mathbb{X}_4 \ln f_1) - \eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \psi\mathbb{X}_1), \quad (52)$$

and

$$g(\mathbf{B}(\psi\mathbb{X}_0, \psi\mathbb{X}_1), \mathbf{F}\mathbf{T}\mathbb{X}_4) = -\cos^2 \phi \left\{ (\mathbb{X}_4 \ln f_1) - \eta(\mathbb{X}_4) \right\} g(\mathbb{X}_0, \mathbb{X}_1). \quad (53)$$

Again, interchanging $\mathbb{X}_4$ with $\mathbf{T}\mathbb{X}_4$ in (46), Lemma 3, and (16), we obtain

$$\begin{aligned} 2g(\mathbf{B}(\mathbb{X}_2, \mathbb{X}_3), \mathbf{F}\mathbf{T}\mathbb{X}_4) &= g(\mathbf{B}(\mathbf{T}\mathbb{X}_4, \mathbb{X}_2), \psi\mathbb{X}_3) + g(\mathbf{B}(\mathbf{T}\mathbb{X}_4, \mathbb{X}_3), \psi\mathbb{X}_2) \\ &\quad - 2\cos^2 \phi \left(-\eta(\mathbb{X}_4) + (\mathbb{X}_4 \ln f_2) \right) g(\mathbb{X}_2, \mathbb{X}_3). \end{aligned} \quad (54)$$

Lemma 4. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then, we have

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5) = 0, \quad (55)$$

for any $\mathbb{X}_0 \in \Gamma(\mathbf{D})$, $\mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi \oplus \zeta)$ and $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$.

Proof. For any $\mathbb{X}_0 \in \Gamma(\mathbf{D})$, $\mathbb{X}_5 \in \Gamma(\mathbf{D}^\phi \oplus \zeta)$, and $\mathbb{X}_2 \in \Gamma(\mathbf{D}^\perp)$, we have

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5) &= g(\tilde{\nabla}_{\mathbb{X}_2} \mathbb{X}_0, \psi\mathbb{X}_5) + g(\tilde{\nabla}_{\mathbb{X}_2} \mathbf{T}\mathbb{X}_5, \mathbb{X}_0), \\ &= g((\tilde{\nabla}_{\mathbb{X}_2} \psi)\mathbb{X}_0, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_2} \psi\mathbb{X}_0, \mathbb{X}_5) + \mathbf{T}\mathbb{X}_5(\ln f_2)g(\mathbb{X}_0, \mathbb{X}_2). \end{aligned}$$

Equations (4) and (10) imply $g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5) = 0$. For the next equality, we compute

$$\begin{aligned} g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5) &= g(\tilde{\nabla}_{\mathbb{X}_0} \mathbb{X}_2, \psi\mathbb{X}_5) + g(\tilde{\nabla}_{\mathbb{X}_0} \mathbf{T}\mathbb{X}_5, \mathbb{X}_2), \\ &= g((\tilde{\nabla}_{\mathbb{X}_0} \psi)\mathbb{X}_2, \mathbb{X}_5) - g(\tilde{\nabla}_{\mathbb{X}_0} \psi\mathbb{X}_2, \mathbb{X}_5) + \mathbf{T}\mathbb{X}_5(\ln f_1)g(\mathbb{X}_0, \mathbb{X}_2). \end{aligned}$$

Again making use of (4) and (1), we find the following:

$$g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_5), \psi\mathbb{X}_2) = g(\mathbf{B}(\mathbb{X}_0, \mathbb{X}_2), \mathbf{F}\mathbb{X}_5),$$

which is the first equality. Therefore, the result is proved. $\square$

In this direction, we provide a relationship between the squared norm of the second fundamental form and the warping function for the bi-warped product. Before giving the next relationship, we define an orthonormal frame. Taking ζ tangent to the base manifold $\mathbf{M}_\phi$ of an n -dimensional bi-warped product submanifold $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ in a $(2n + 1)$ -dimensional nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$, we consider the dimensions $\dim(\mathbf{M}_T) = n_1$, $\dim(\mathbf{M}_\perp) = n_2$, and $\dim(\mathbf{M}_\phi) = n_3$. We provide proof of the main theorem as follows.

For the second type of bi-warped product submanifold, $\mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$, we prove the following result:

Theorem 11. Assume that $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ is a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. If $\mathbf{D}^\perp - \mathbf{D}^\phi$ is mixed totally geodesic and ζ is tangent to $\mathbf{M}_\phi$, then the length of the second fundamental form $\mathbf{B}$ is defined as

$$\|\mathbf{B}\|^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left(\|\vec{\nabla} \ln f_1\|^2 - \beta^2 \right) + n_2 \cot^2 \phi \left(\|\vec{\nabla} \ln f_2\|^2 - \beta^2 \right), \quad (56)$$

where $n_1 = \dim(\mathbf{M}_T)$ and $n_2 = \dim(\mathbf{M}_\perp)$. The gradients $\vec{\nabla}(\ln f_1)$ and $\vec{\nabla}(\ln f_2)$ of $\ln f_1$ and $\ln f_2$ are along $\mathbf{M}_T$ and $\mathbf{M}_\perp$, respectively. If the inequality (56) becomes an equality, then $\mathbf{M}_T$ and $\mathbf{M}_\perp$ are totally umbilical submanifolds, and $\mathbf{M}_\phi$ is a totally geodesic submanifold in $\widetilde{\mathbf{M}}$. Furthermore, $\mathbf{M}$ is a $\mathbf{D}^\phi$ -totally geodesic submanifold of $\widetilde{\mathbf{M}}$.

Proof. Suppose the orthogonal frames of the corresponding tangent spaces of $\mathbf{D}$, $\mathbf{D}^\perp$, and $\mathbf{D}^\phi$, are as follows:

$$\begin{aligned} \mathbf{D} &= \text{span}\{v_1, \dots, v_p, v_{p+1} = \psi v_1, \dots, v_{n_1} = v_{2p} = \psi v_p\}, \\ \mathbf{D}^\perp &= \text{span}\{v_{n_1+1} = \bar{v}_1, \dots, v_{n_1+n_2} = \bar{v}_{n_2}\}, \\ \mathbf{D}^\phi &= \text{span}\{v_{n_1+n_2+1} = v_1^*, \dots, v_{n_1+n_2+q} = v_q^*, v_{n_1+n_2+q+1} = v_{q+1}^* = \sec \phi \mathbf{T} v_1^* \\ &\quad, \dots, v_{n_1+n_2+2q} = v_{2q}^* = \sec \phi \mathbf{T} v_q^*, v_m = v_{n_3}^* = v_{2q+1}^* = \zeta\}. \end{aligned}$$

Then, the orthonormal frame fields of the normal sub-bundles of $\psi \mathbf{D}^\perp$, $\mathbf{F} \mathbf{D}^\phi$, and μ , respectively, are as follows:

$$\begin{aligned} \psi \mathbf{D}^\perp &= \text{span}\{v_{m+1} = \bar{v}_1 = \psi \bar{v}_1, \dots, v_{m+n_2} = \bar{v}_{n_2} = \psi \bar{v}_{n_2}\}, \\ \mathbf{F} \mathbf{D}^\phi &= \text{span}\{v_{m+n_2+1} = \bar{v}_{n_2+1} = \csc \phi \mathbf{F} v_1^*, \dots, v_{m+n_2+q} = \bar{v}_{n_2+q} = \csc \phi \mathbf{F} v_q^*, \\ &\quad v_{m+n_2+q+1} = \bar{v}_{n_2+q+1} = \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_1^*, \dots, v_{m+n_2+n_3-1} = \bar{v}_{n_2+n_3-1} = \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_q^*\}, \\ \mu &= \text{span}\{v_{m+n_2+n_3} = \bar{v}_{n_2+n_3}, \dots, v_{2n+1} = \bar{v}_{2(n-n_2-n_3+1)-n_1}\}. \end{aligned}$$

From the definition of the second fundamental form, we have

$$\|\mathbf{B}\|^2 = \sum_{a,b=1}^m g(\mathbf{B}(v_a, v_b), \mathbf{B}(v_a, v_b)) = \sum_{r=m+1}^{2n+1} \sum_{a,b=1}^m g(\mathbf{B}(v_a, v_b), v_r)^2.$$

The above expression can be expanded, according the frame vector fields, as

$$\begin{aligned} \|\mathbf{B}\|^2 &= \sum_{r=1}^{n_2} \left(\sum_{a,b=1}^m g(\mathbf{B}(v_a, v_b), \bar{v}_r)^2 \right) + \sum_{r=n_2+1}^{n_2+n_3-1} \\ &\quad + \sum_{r=n_2+n_3}^{2(n-n_2-n_3+1)-n_1} \left(\sum_{a,b=1}^m g(\mathbf{B}(v_a, v_b), \bar{v}_r)^2 \right). \end{aligned} \quad (57)$$

Ignoring the μ -components and implementing the constructed frame fields of $\mathbf{D}$, $\mathbf{D}^\perp$, and $\mathbf{D}^\phi$, we obtain

$$\begin{aligned}
 \|\mathbf{B}\|^2 &\geq \sum_{r=1}^{n_2} \sum_{a,b=1}^{n_1} g(\mathbf{B}(v_a, v_b), \psi \bar{v}_r)^2 + \sum_{r=1}^{n_2} \sum_{a,b=1}^{n_2} g(\mathbf{B}(\bar{v}_a, \bar{v}_b), \psi \bar{v}_r)^2 \\
 &+ \sum_{r=1}^{n_2} \sum_{a,b=1}^{n_3} g(\mathbf{B}(v_a^*, v_b^*), \psi \bar{v}_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_2} \sum_{b=1}^{n_2} g(\mathbf{B}(v_a, \bar{v}_b), \psi \bar{v}_r)^2 \\
 &+ 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(\mathbf{B}(v_a, v_b^*), \psi \bar{v}_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(\mathbf{B}(\bar{v}_a, v_b^*), \psi \bar{v}_r)^2 \\
 &+ \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a,b=1}^{n_1} g(\mathbf{B}(v_a, v_b), \bar{v}_r)^2 + \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a,b=1}^{n_2} g(\mathbf{B}(\bar{v}_a, \bar{v}_b), \bar{v}_r)^2 \\
 &+ \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a,b=1}^{n_3} g(\mathbf{B}(v_a^*, v_b^*), \bar{v}_r)^2 + 2 \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a=1}^{n_2} \sum_{b=1}^{n_2} g(\mathbf{B}(v_a, \bar{v}_b), \bar{v}_r)^2 \\
 &+ 2 \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(\mathbf{B}(v_a^*, v_b^*), \bar{v}_r)^2 + 2 \sum_{r=n_2+1}^{n_2+n_3-1} \sum_{a=1}^{n_2} \sum_{b=1}^{n_3} g(\mathbf{B}(\bar{v}_a, v_b^*), \bar{v}_r)^2. \quad (58)
 \end{aligned}$$

As we assumed that $\mathbf{M}$ is a $\mathbf{D}^\perp - \mathbf{D}^\phi$ mixed totally geodesic, this forces the third, sixth, and twelfth terms to vanish. In view of Lemma 4, the fifth and tenth terms are zero. Similarly, we ignore the first, second, fourth, and ninth terms, due to the lack of connections for warping functions. Now, the rest of the terms (i.e., the seventh and eighth) can be written as

$$\begin{aligned}
 \|\mathbf{B}\|^2 &\geq \sum_{r=1}^q \sum_{a,b=1}^{n_1} g(\mathbf{B}(v_a, v_b), \csc \phi \mathbf{F} v_r^*)^2 + \sum_{r=1}^q \sum_{a,b=1}^{n_1} g(\mathbf{B}(v_a, v_b), \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_r^*)^2 \\
 &+ \sum_{r=1}^q \sum_{a,b=1}^{n_2} g(\mathbf{B}(\bar{v}_a, \bar{v}_b), \csc \phi \mathbf{F} v_r^*)^2 + \sum_{r=1}^q \sum_{a,b=1}^{n_2} g(\mathbf{B}(\bar{v}_a, \bar{v}_b), \csc \phi \sec \phi \mathbf{F} \mathbf{T} v_r^*)^2.
 \end{aligned}$$

Inserting the equations from Lemma 3 with (47)–(54), we derive

$$\begin{aligned}
 \|\mathbf{B}\|^2 &\geq n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=1}^q \left(\mathbf{T} v_r^*(\ln f_1) + \alpha \eta(v_r^*) \right)^2 \\
 &+ n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^q \left(v_r^*(\ln f_1) - \eta(v_r^*) \right)^2 \\
 &+ n_2 \csc^2 \phi \sum_{r=1}^q \left(\mathbf{T} v_r^*(\ln f_1) + \alpha \eta(v_r^*) \right)^2 + n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^q \left(v_r^*(\ln f_2) - \eta(v_r^*) \right)^2.
 \end{aligned}$$

From the orthonormal frame $\eta(e_r^*) = 0$, for $1 \leq r \leq q$, we have

$$\begin{aligned}
 \|\mathbf{B}\|^2 &\geq n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=1}^{2q+1} (\mathbf{T} v_r^*(\ln f_1))^2 - n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=q+1}^{2q} g(v_r^*, \mathbf{T} \vec{v}(\ln f_1))^2 \\
 &- n_1 \csc^2 \phi (1 + \sec^2 \phi) (\mathbf{T} v_{2q+1}^*(\ln f_1))^2 + n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^q (v_r^*(\ln f_1))^2 \\
 &+ n_2 \csc^2 \phi \sum_{r=1}^{2q+1} (\mathbf{T} v_r^*(\ln f_2))^2 - n_2 \csc^2 \phi \sum_{r=q+1}^{2q} g(v_r^*, \mathbf{T} \vec{v}(\ln f_2))^2 \\
 &- n_2 \csc^2 \phi (\mathbf{T} v_{2q+1}^*(\ln f_2))^2 + n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^q (v_r^*(\ln f_2))^2.
 \end{aligned}$$

The third and seventh terms are equal to zero by $e_{2q+1}^* = \zeta$ and $T\zeta = 0$. Thus, the preceding inequality takes the following form:

$$\begin{aligned} \|\mathbf{B}\|^2 &\geq n_1 \csc^2 \phi (1 + \sec^2 \phi) \|\mathbf{T}\vec{\nabla}(\ln f_1)\|^2 - n_1 \csc^2 \phi (1 + \sec^2 \phi) \sec^2 \phi \sum_{r=1}^q g(\mathbf{T}v_r^*, \mathbf{T}\vec{\nabla}(\ln f_1))^2 \\ &\quad + n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^q (v_r^*(\ln f_1))^2 + n_2 \csc^2 \phi \|\mathbf{T}\vec{\nabla}(\ln f_2)\|^2 \\ &\quad - n_2 \csc^2 \phi \sec^2 \phi \sum_{r=1}^q g(\mathbf{T}v_r^*, \mathbf{T}\vec{\nabla}(\ln f_2))^2 \\ &\quad + n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^q (v_r^*(\ln f_2))^2. \end{aligned}$$

Using (11) and the fact that $\xi(\ln f_a) = \beta$, $i = 1, 2$ from Proposition 2, we finally obtain

$$\|\mathbf{B}\|^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left(\|\vec{\nabla}(\ln f_1)\|^2 - \beta^2 \right) + n_2 \cot^2 \phi \left(\|\vec{\nabla}(\ln f_2)\|^2 - \beta^2 \right).$$

For the equality case in (56), considering the third μ -component term in (57), we derive

$$\mathbf{B}(\mathbf{TM}, \mathbf{TM}) \perp \mu. \quad (59)$$

By using the missing first, second, and fourth terms in (58), we find that

$$\mathbf{B}(\mathbf{D}, \mathbf{D}) \perp \psi \mathbf{D}^\perp, \mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \perp \psi \mathbf{D}^\perp, \mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) \perp \psi \mathbf{D}^\perp. \quad (60)$$

Evaluating the ninth term of (58), we get

$$\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp F\mathbf{D}^\phi. \quad (61)$$

Then, from (59) and (60), we obtain

$$\mathbf{B}(\mathbf{D}, \mathbf{D}) \in F\mathbf{D}^\phi, \mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \in F\mathbf{D}^\phi, \mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) \in F\mathbf{D}^\phi. \quad (62)$$

As $\mathbf{M}$ is $\mathbf{D}^\perp - \mathbf{D}^\phi$ mixed totally geodesic, we can conclude the following:

$$\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0. \quad (63)$$

From the vanishing third term of (58), we get

$$\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp. \quad (64)$$

From (59), (61), and (64), we get

$$\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) = 0. \quad (65)$$

Evaluating the fifth, tenth, and eleventh terms of (58), we can derive

$$\mathbf{B}(\mathbf{D}, \mathbf{D}^\phi) \perp \psi \mathbf{D}^\perp, \mathbf{F}(\mathbf{D}, \mathbf{D}^\perp) \perp F\mathbf{D}^\phi, \mathbf{B}(\mathbf{D}, \mathbf{D}^\phi) \perp F\mathbf{D}^\phi. \quad (66)$$

Therefore, with (59), (60), and (66), we obtain

$$\mathbf{B}(\mathbf{D}, \mathbf{D}^\phi) = 0, \mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) = 0. \quad (67)$$

It is well-known that $\mathbf{M}_\phi$ is a totally geodesic submanifold in $\mathbf{M}$ and we can analyze that $\mathbf{M}_\phi$ is totally geodesic in $\widetilde{\mathbf{M}}$ by (63), (65), and (67). Additionally, $\mathbf{M}_T$ and $\mathbf{M}_\perp$ are totally umbilical submanifolds in $\widetilde{\mathbf{M}}$ by (62), as $\mathbf{M}_T$ and $\mathbf{M}_\perp$ are totally umbilical in $\mathbf{M}$. Equations

(60)–(67) show that $\mathbf{M}$ is a $\mathbf{D}^\phi$ -totally geodesic submanifold in $\tilde{\mathbf{M}}$. This completes the proof of the theorem. $\square$

3.1. Geometric Applications

In this section, we find a particular case of our main results. Particularizing $\beta = 0$ and $\alpha = 1$ or $\alpha = 0$, along with $n_1 = \dim \mathbf{M}_T = 0$, in Theorem 11, we get the following:

Theorem 12. (Theorem 3.1 of [13] and Theorem 3.1 of [33]) Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a nearly Sasakian manifold or a nearly cosymplectic manifold. Then, we have

$$\|\mathbf{B}\|^2 \geq n_2 \cot^2 \phi \|\vec{\nabla} \ln f_2\|^2,$$

where $\dim \mathbf{M}_\perp = n_2$.

If we assume that $\alpha = 0$, $\beta = 1$ with $\dim \mathbf{M}_\perp = 0$, we find another theorem:

Theorem 13. Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_T$ be a warped product semi-slant submanifold of nearly Kenmotsu manifold. Then, we have

$$\|\mathbf{B}\|^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left(\|\vec{\nabla} \ln f_1\|^2 - 1 \right).$$

Remark 5. It can be noted that only on a nearly Kenmotsu manifold does the warped product semi-slant submanifold of the type $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_T$ exist; in other structures, it becomes a trivial case (see [22]).

Similarly, if $\alpha = 0$, $\beta = 1$ with $\dim \mathbf{M}_T = 0$, then we have:

Theorem 14. (Theorem 4.2 of [14]) Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a nearly Kenmotsu manifold. Then, we have

$$\|\mathbf{B}\|^2 \geq n_2 \cot^2 \phi \left(\|\vec{\nabla} \ln f_2\|^2 - 1 \right),$$

where $\dim \mathbf{M}_\perp = n_2$.

3.2. Some Applications Related to Mathematical Physics

In this section, we investigate the Dirichlet energy, which satisfies the following for a compact submanifold $\mathbf{M}$ and differentiable function $\theta : \mathbf{M} \rightarrow \mathbb{R}$:

$$E(\theta) = \frac{1}{2} \int_{\mathbf{M}} \|\nabla \theta\|^2 dV, \quad (68)$$

where dV is a volume element. Considering this, we give the following Theorem by combining (40) and (68), where $\mathbf{M}_T$ is compact without boundary.

Theorem 15. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_T$ is compact without boundary. Then, we have

$$\begin{aligned} & 4n_2 E(\ln f_1) + 2n_3 \left(\frac{2k}{9} \cot^2 \phi + 2 \csc^2 \phi \right) E(\ln f_2) \\ & \leq \int_{\mathbf{M}_T \times \{n_2\} \times \{n_3\}} \left\{ \|\mathbf{B}\|^2 + n_3 \beta^2 \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \right. \\ & \quad \left. + 2n_2 (\beta^2 - \alpha^2) - n_3 \alpha^2 \right\} dV, \end{aligned}$$

where $E(\ln f_1)$ and $E(\ln f_2)$ are the Dirichlet energies of the warping functions f_1 and f_2 , respectively. The equality cases are the same as in Theorem 1

Similarly, if $\mathbf{M}_\phi$ is a compact base without boundary, then Theorems 11 and (68) give the following:

Theorem 16. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_\phi$ is compact without boundary. Then, we have

$$2n_1 \csc^2 \phi (1 + \cos^2 \phi) E(\ln f_1) + n_2 \cot^2 \phi E(\ln f_2) \leq \int_{\mathbf{M}_\phi \times \{n_1\} \times \{n_2\}} \left\{ \|\mathbf{B}\|^2 + \beta^2 \left(n_2 \cot^2 \phi + n_1 \csc^2 \phi (1 + \cos^2 \phi) \right) \right\} dV.$$

As immediate applications of Theorem 15, we give corollaries in the following.

Corollary 1. Assume that $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp$ is a CR-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_T$ is compact without boundary. Then, we have

$$E(\ln f_1) \leq \frac{1}{4n_2} \int_{\mathbf{M}_T \times \{n_2\}} \left(\|\mathbf{B}\|^2 + 2n_2 (\beta^2 - \alpha^2) \right) dV.$$

Corollary 2. Assume that $\mathbf{M} = \mathbf{M}_T \times_{f_2} \mathbf{M}_\phi$ is a warped product semi-slant submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_T$ is compact without boundary. Then, we have

$$E(\ln f_2) \leq \left(\frac{9 \sin^2 \phi}{8k(10 - \sin^2 \phi)} \right) \int_{\mathbf{M}_T \times \{n_3\}} \left\{ \|\mathbf{B}\|^2 + 2k\beta^2 \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \theta \right) \right\} dV.$$

On the other hand, substituting $n_1 = 0$ and $n_2 = 0$ into Theorem 16, we can derive the following corollaries.

Corollary 3. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_\perp$ be a warped product pseudo-slant submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_\phi$ is compact without boundary. Then, we have

$$E(\ln f_2) \leq \frac{1}{n_2 \cot^2 \phi} \int_{\mathbf{M}_T \times \{n_2\}} \left\{ \|\mathbf{B}\|^2 + \beta^2 n_2 \cot^2 \phi \right\} dV.$$

Corollary 4. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_2} \mathbf{M}_T$ be a warped product semi-slant submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_\phi$ is compact without boundary. Then, we have

$$2n_1 \csc^2 \phi (1 + \cos^2 \phi) E(\ln f_1) \leq \int_{\mathbf{M}_T \times \{n_1\}} \left\{ \|\mathbf{B}\|^2 + \beta^2 n_1 \csc^2 \phi (1 + \cos^2 \phi) \right\} dV.$$

For the Laplacian, many applications in mathematics as well as in physics can be found. This is possible due to the eigenvalue problem of Δ . The corresponding Laplace eigenvalue equation is defined as follows: A real number λ is called an eigenvalue if there exists a non-vanishing function θ which satisfies the following equation:

$$\Delta \theta = \lambda \theta, \quad \text{on } \mathbf{M},$$

with appropriate boundary conditions. Considering a Riemannian manifold $\mathbf{M}^n$ with no boundary, the first non-zero eigenvalue of Δ , defined as λ_1 , includes variational properties (cf. [34]):

$$\lambda_1 = \inf \left\{ \frac{\int_{\mathbf{M}} \|\nabla \theta\|^2 dV}{\int_{\mathbf{M}} \|\theta\|^2 dV} \mid \theta \in W^{1,2}(\mathbf{M}^n) \setminus \{0\}, \int_{\mathbf{M}} \theta dV = 0 \right\}.$$

Inspired by the above characterization, using the first non-zero eigenvalue of the Laplace operator and the maximum principle for the first non-zero eigenvalue λ_1 , we deduce the following:

Theorem 17. Let $\mathbf{M} = \mathbf{M}_T \times_{f_1} \mathbf{M}_\perp \times_{f_2} \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$ with $\zeta \in \mathbf{M}_T$ and such that $\mathbf{M}_T$ is a compact base without boundary. If λ_1 and μ_1 are the first eigenvalues of eigenfunctions f_1 and f_2 , respectively, then we have following inequality:

$$\begin{aligned} \int_{\mathbf{M}_T \times \{n_2\} \times \{n_3\}} \|\mathbf{B}\|^2 dV &\geq 2n_2 \int_{\mathbf{M}_T \times \{n_2\} \times \{n_3\}} \left\{ \lambda_1 (\ln f_1)^2 + \alpha^2 - \beta^2 \right\} dV \\ &+ n_3 \int_{\mathbf{M}_T \times \{n_2\} \times \{n_3\}} \left\{ \left(\frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \mu_1 (\ln f_2)^2 - \beta^2 \right\} + \alpha^2 \right\} dV. \end{aligned} \quad (69)$$

Proof. Assuming that θ is a non-constant warping function, by use of the minimum principle for the first eigenvalue λ_1 , one can obtain (p. 186, [34]):

$$\lambda_1 \int_{\mathbf{M}^n} (\theta)^2 dV \leq \int_{\mathbf{M}^n} \|\nabla \theta\|^2 dV. \quad (70)$$

The equality holds if and only if $\Delta \theta = \lambda_1 \theta$. Taking the integral in (40) and using (70), we get the required result (69). $\square$

Similarly, using Theorem 11, we have

Theorem 18. Let $\mathbf{M} = \mathbf{M}_\phi \times_{f_1} \mathbf{M}_T \times_{f_2} \mathbf{M}_\perp$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\widetilde{\mathbf{M}}$. If $\mathbf{D}^\perp - \mathbf{D}^\phi$ is mixed totally geodesic and ζ is tangent to the compact base $\mathbf{M}_\phi$, then we get

$$\begin{aligned} n_1 \csc^2 \phi (1 + \cos^2 \phi) \lambda_1 \int_{\mathbf{M}_\phi \times \{n_1\} \times \{n_2\}} (\ln f_1)^2 dV &+ n_2 \cot^2 \phi \mu_1 \int_{\mathbf{M}_\phi \times \{n_1\} \times \{n_2\}} (\ln f_2)^2 dV \\ &\leq \int_{\mathbf{M}_\phi \times \{n_1\} \times \{n_2\}} \left\{ \|\mathbf{B}\|^2 + \beta^2 \left(n_2 \cot^2 \phi + n_1 \csc^2 \phi (1 + \cos^2 \phi) \right) \right\} dV. \end{aligned}$$

4. Conclusions

It is noted that the nearly trans-Sasakian structures generalize some remarkable geometric structures on manifolds, like nearly cosymplectic, nearly Sasakian, nearly Kenmotsu, nearly α -Sasakian, and nearly β -Kenmotsu. The main target of this paper is to discuss the geometry of bi-warped product submanifolds of some special types in nearly trans-Sasakian manifolds. In particular, we derived some basic inequalities, which turn out to be generalization of various known results obtained by several mathematicians in the last decade. The eigenvalues inequalities are established and the Dirichlet energy inequalities are derived that give the new motivation of such studies.

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