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An Error Analysis of the CN Weighed DG θ Method of the Convection Equation

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Abstract: In this research paper, a weighed DG finite element method is proposed for solving convection equations with an easy execution and analysis. The key aim of this method is to design an error estimation for space and time of a discontinuous approximation on general finite element meshes. The efficiency of the parameter θ in the order of convergence of the solutions is also exposed. Some numerical examples were tested that demonstrated the strength and flexibility of the method.

Keywords: CN-DG θ method; convection equation; hyperbolic problems; error analysis

1. Introduction

Convection plays an important role in the application of meteorology, gas dynamics, oceanography, weather forecasting, aeroacoustics, turbulent flows, turbomachinery, oil recovery simulation, modeling of shallow water, the transport of contaminants in porous media, viscoelastic flows, magneto-hydrodynamics, and electro-magnetism. Recently, the discontinuous Galerkin (DG) method has been commonly used in industrial software packages for solving a large set of computational fluid problems. It is evident that it performs better than the standard continuous Galerkin method because of its flexibility in estimating, uneven solutions, mass conservation, unstructured meshes, potential for error control, and mesh adaptation. Among all FEMs, the DG method is a significant tool for solving convection equations. The calculation of the convection problem is simple, but it gives important numerical properties for more complicated problems. The discontinuous Galerkin finite element method (DGFEM) was introduced by Reed and Hill [1] in 1973 for solving the neutron transport equation. They compared their method with the continuous Galerkin finite element method by using numerical experiments and showed worthwhile stability properties of DGFEM. The first exploration of this method was published by Lesaint and Raviart [2] in 1974. Later on, this method was analyzed by using a stronger stability by Johnson, Navert, and J. Pitkäranta in [3] and Johnson and J. Pitkäranta in [4]. Jonson [5] gave an analysis of error control for stiff ODEs of the DG method in 1988, and then, Estep [6] in 1995 extended this analysis to general non-autonomous ODEs. Lastly, Böttcher and Rannacher [7] introduced a new adaptive error control method for ODEs by applying the DG method in 1996. On the other hand, Richter in [8] explained the optimal rate of convergence for any structured two-dimensional non-Cartesian grid, and Lin, Yan, and Zhou in [9] exposed the first-order convergence of DG methods by using piece-wise constant approximations. After that, Lin in [10] reviewed this method and proposed a new error estimation for finite element approximations of hyperbolic equations. Süli [11] in 1996, Süli and Houston [12] in 1997, and Houston and Süli [13] in 2001 studied the a posteriori error analysis of hyperbolic problems. More recently, Houston, Schwab, and Süli in [14] widely studied the upwind finite element approximation for optimal control systems, while Xiong and Li in [15,16] established the a posteriori error estimation for the optimal control problems of the first-order linear hyperbolic and evolution equations, respectively. Then, Brezzi, Marini, and Süli in [17] adapted the upwind method by changing the standard upwind flux with a reliability term and a jump stabilization term. Again, Xiong and Li



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in [18] clarified the convergence properties of the optimal control problem governed by convection-diffusion equations. They extended the a posteriori error estimates and the a priori error estimates for both states, adjoints, and the control variable approximation. Furthermore, Xiong C., Luo F., and Ma X. et al. in [19] derived the a priori error analysis for the streamline diffusion discontinuous Galerkin finite element approximation of the optimal distributed control problem governed by the first-order linear hyperbolic equation. In this article, they analyzed the behavior of the error for h tending to zero and the degree p tending to infinity. Finally, Burman and Stamm in [20] verified that the optimal convergence holds for quadratic and higher polynomial degrees when the jump of the tangential part of the gradient is penalized. Mu and Ye in [21] introduced a simple method for solving first-order hyperbolic equations of polygons/polyhedra of an arbitrary shape, and Wang in [22] presented the optimal L^2 error estimate of a linearized Crank–Nicolson Galerkin FEM for a generalized nonlinear Schrödinger equation without any time step size restriction. The least squares finite element solutions for linear hyperbolic problems were introduced in [23–25]. A significant literature exists on developing DG methods [26–35]. The objective of this article is to explore error estimates for both space and time of the convection equation by using the Crank–Nicolson discontinuous Galerkin (CN-DG) finite element method. The key aim of this method is to design the error estimation for both space and time on general finite element meshes. The reliability and effectiveness of the weighed parameter θ on the order of convergence of the solutions are also exposed. In the numerical experiments, we used the influence of the parameter θ in Figures 1, 2, and 3 on the convergence of the solution. Most of the methods cited here established their approximation in a steady state without time discretization. Those methods (cited here) were considered time discretizations; their analysis was quite complicated and time consuming. It is also noteworthy to point out that it is a different and simpler approach to introduce such a parameter to seek an error analysis in time from all the DG schemes for convection equations. To the best of our knowledge, no proposals have been found in the literature that are relevant to such a type of parameter and penalty term in the error approximation of the convection equation. Therefore, we believe that the present way provides a simpler and more elegant error approximation of the convection equation for both space and time discretization.

The framework of the paper is as follows. The statement of the problem, other notations, and finite element meshes are introduced in Section 2. Moreover, we propose the main results, as well as the corresponding time-discrete scheme and give their detailed proofs in Section 3. Numerical examples are presented in Section 4 to demonstrate our theoretical results. Conclusions are drawn in Section 5.

2. Preliminaries

The purpose of this section is to state the problem in detail, introducing the finite element meshes and space and establishing some others preliminary results.

2.1. Formulation of the Problem

In this paper, a convection equation defined in a polyhedral bounded domain Ω in $\mathbb{R}^d$ ($d = 2, 3$) is considered with the Lipschitz continuous boundary.

$$\partial_t u + \beta \cdot \nabla u + bu = f; \quad u(x, 0) = u_0(x), \quad x \in \partial\Omega; \quad (x, t) \in \Omega \times \mathbb{R}^+. \quad (1)$$

where the convection vector $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_d)^T$ is a constant vector or a vector-valued function. Again, f is assumed to be a sufficiently smooth real-valued function, which could be different in different problems, and b is presumed as a constant. The initial u_0 is assumed to be bounded.

The subsets of $\Gamma = \partial\Omega$ are defined as follows:

$$\Gamma_- = \{x \in \Gamma : \beta(x)n(x) < 0\},$$

$$\Gamma_+ = \{x \in \Gamma : \beta(x)n(x) \geq 0\}.$$

where Γ_- and Γ_+ are known as the inflow boundary and outflow boundary, respectively, and $\mathbf{n}(x)$ is the unit outward normal vector to Γ at $x \in \Gamma$. Here, Γ_- and Γ_+ are not necessarily connected subsets of Ω . For simplicity, Ω is non-characteristic in the sense that $\Gamma_- \cup \Gamma_+$. In addition, it is assumed that there exists a vector $\xi \in \mathbb{R}^d$ such that,

$$b(x) + \frac{1}{2} \nabla \cdot \beta(x) + \beta(x) \cdot \xi \geq 0 \quad a.e. \quad x \in \Omega. \quad (2)$$

For the simplicity of (2), it is assumed $\xi = 0$ in the above hypothesis, and a positive constant b_0 is defined by:

$$b(x) + \frac{1}{2} \nabla \cdot \beta(x) \geq b_0 > 0 \quad a.e. \quad x \in \Omega. \quad (3)$$

2.2. Finite Element Mesh and Space

The approximation in the space of (1) is calculated by using discontinuous finite elements to construct finite element meshes over a polyhedron Ω . A family of partitions $(\mathcal{T}_h)_{h>0}$ is an affine shape-regular or one-irregular mesh sequence of the domain Ω . The mesh $\mathcal{T}_h$ is composed of a combination of triangles T with the diameter h_T . The mesh size is considered as $h = \max_{T \in \mathcal{T}_h} h_T$. For each triangulation $(\mathcal{T}_h)_{h>0}$, V_h is introduced to denote the corresponding discontinuous finite element space of the piecewise polynomial $\mathbb{P}_k(K)$ with the degree $k \geq 1$.

$$V_h = \{v(t) \in L^2(\Omega) : v|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}.$$

2.3. CN-DG Approximation

In this paper, error estimations for both space and time were established for the convection Equation (1) by using the CN-DG finite element method. The purpose of this work is to find an approximation $u_h^n \in V_h$ such that:

$$\begin{aligned} & \left(\frac{\delta_t^+ u^n}{\Delta t}, v_h \right) + (\beta \cdot n \{u_h^{n+\frac{1}{2}}\}, [v_h])_{\partial K_+} - (u_h^{n+\frac{1}{2}}, \beta \cdot \nabla v_h)_K + (\theta \beta \cdot n [u_h^{n+\frac{1}{2}}], [v_h])_{\partial K_+} \\ & + (b u_h^{n+\frac{1}{2}}, v_h)_K = (f, v_h), \forall v_h \in V_h. \end{aligned} \quad (4)$$

where $u^n = u(X, t_n)$, for a sequence of functions $\{u^n\}_{n=0}^N$, is defined, $u^{n+\frac{1}{2}} = \frac{1}{2}(u^n + u^{n+1})$, and $\delta_t^+ u^n = (u^{n+1} - u^n)$, and as usual, $[u_h^{n+\frac{1}{2}}], [v_h]$ denote the jump of $u_h^{n+\frac{1}{2}}, v_h$, respectively, across the edges of an element over which it is calculated. $\{u_h^{n+\frac{1}{2}}\}$ denotes the average of $u_h^{n+\frac{1}{2}}$. In this article, θ was used as a penalty parameter; the term $(\theta \beta \cdot n [u_h^{n+\frac{1}{2}}], [v_h])_{\partial K_+}$ of Equation (4) is called the penalty term. The penalty term is independent of time, convection terms, and source terms. The penalty parameter is only dependent on the mesh geometry of the sub-domain triangulation $\mathcal{T}_h$.

3. Main Results

The aspiration of this section is to introduce the stability of the CN-DG scheme, the error estimate, and the corresponding time-discrete error estimate. For the stability and error analysis, Gronwall inequalities are mostly required. For this reason, the following two lemmas are introduced.

Lemma 1. (Gronwall inequality) Let $y(t)$, $g(t)$, $h(t)$, and $f(t)$ be nonnegative functions, and assume that there is a positive constant M independent of t such that $\int_0^t g(t) dt \leq M$ and:

$$y(t) + \int_0^t h(s) ds \leq y(0) + \int_0^t g(s) y(s) + f(s) ds, \quad \forall 0 \leq t \leq t_N,$$

then

$$y(t) + \int_0^t h(s)ds \leq \exp(M)(y(0) + \int_0^t f(s)ds), \quad \forall 0 \leq t \leq t_N.$$

Lemma 2. (Discrete Gronwall inequality) Let C and a_n, b_n , and d_n , for integer $1 \leq m \leq \frac{t_N}{k}$, be nonnegative numbers such that,

$$a_m + k \sum_{n=0}^m b_n \leq k \sum_{n=0}^m a_n d_n + C, \quad \forall 1 \leq m \leq \frac{t_N}{k}.$$

Assume that $kd_n < \frac{1}{2}$ and $\forall 1 \leq m \leq \frac{t_N}{k}$, then:

$$a_m + k \sum_{n=1}^m b_n \leq C \exp(2k \sum_{n=1}^{m-1} a_n), \quad \forall 1 \leq m \leq \frac{t_N}{k}.$$

3.1. Stability of the CN-DG Scheme

To prove the stability of the scheme, the following DG-norm is defined by:

$$\|u_h^n\|_{DG}^2 = \sum_K \|u_h^n\|_{L^2(K)}^2 + \Delta t \|b_0(x) u_h^n\|_{L^2(K)}^2 + \sum_{e \in \partial K+} \Delta t \theta(\beta \cdot n [u_h^n]^2, 1)_e. \quad (5)$$

Now, the following theorem is proposed for the stability analysis of the error estimate of the problem governed by Equation (1).

Theorem 1. Assume that $f \in C^2(\Omega \times [0, T])$ and u_h^n is the solution of (1), then there is a constant $C > 0$ independent of Δt such that the following condition holds:

$$\|u_h^{n+1}\|_{DG}^2 \leq C \Delta t \sum_{i=0}^n \|f(t_i)\|_{L^2(\Omega)}^2.$$

Proof. Let $V_h = u_h^n + u_h^{n+1}$. By using Equation (4), it can be found that:

$$\left(\frac{\delta_t^+ u^n}{\Delta t}, u_h^n + u_h^{n+1} \right) = (\|u_h^{n+1}\|^2 - \|u_h^n\|^2) \cdot \frac{1}{\Delta t}. \quad (6)$$

Take the second and third parts of Equation (4), and apply the rules of integration:

$$\begin{aligned} & (\beta \cdot n \{u_h^{n+\frac{1}{2}}\}, [u_h^{n+\frac{1}{2}}])_{\partial K_+} - (u_h^{n+\frac{1}{2}}, \beta \cdot \nabla u_h^{n+\frac{1}{2}})_K \\ &= (\beta \cdot n [u_h^{n+\frac{1}{2}}], \{u_h^{n+\frac{1}{2}}\})_{\partial K_+} - \frac{1}{2} \int_{\partial K} \beta \cdot n (u_h^{n+\frac{1}{2}})^2 ds + \frac{1}{2} \int_{\partial K} \nabla \cdot \beta (u_h^{n+\frac{1}{2}})^2 dx \\ &= \frac{1}{2} (\beta \cdot n (u_h^{n+\frac{1}{2},+} - u_h^{n+\frac{1}{2},-}), (u_h^{n+\frac{1}{2},+} + u_h^{n+\frac{1}{2},-}))_{\partial K_+} \\ &\quad - \frac{1}{2} (\beta \cdot n ((u_h^{n+\frac{1}{2},+})^2 - (u_h^{n+\frac{1}{2},-})^2), 1)_{\partial K_+} + \frac{1}{2} \int_{\partial K} \nabla \cdot \beta (u_h^{n+\frac{1}{2}})^2 dx \\ &= \frac{1}{2} (\beta \cdot n (u_h^{n+\frac{1}{2},+})^2 - \beta \cdot n (u_h^{n+\frac{1}{2},-})^2)_{\partial K_+} - \frac{1}{2} ((\beta \cdot n (u_h^{n+\frac{1}{2},+})^2 \\ &\quad - \beta \cdot n (u_h^{n+\frac{1}{2},-})^2), 1)_{\partial K_+} + \frac{1}{2} \int_{\partial K} \nabla \cdot \beta (u_h^{n+\frac{1}{2}})^2 dx \\ &= \frac{1}{2} \int_{\partial K} \nabla \cdot \beta (u_h^{n+\frac{1}{2}})^2 dx. \end{aligned} \quad (7)$$

Again, consider the fourth and fifth parts of Equation (4):

$$\begin{aligned}
 & (\theta\beta \cdot n[u_h^{n+\frac{1}{2}}], [u_h^{n+\frac{1}{2}}])_{\partial K_+} + (bu_h^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}})_K \\
 &= (\theta\beta \cdot n[u_h^{n+\frac{1}{2}}]^2, 1)_{\partial K_+} + (b(u_h^{n+\frac{1}{2}})^2, 1)_K \\
 &= (\theta\beta \cdot n[u_h^{n+1}]^2, 1)_{\partial K_+} + (\theta\beta \cdot n[u_h^n]^2, 1)_{\partial K_+} + 2(\theta\beta \cdot n[u_h^n], [u_h^{n+1}])_{\partial K_+} \\
 & \quad + (b(u_h^{n+\frac{1}{2}})^2, 1)_K.
 \end{aligned} \tag{8}$$

Now, combining the above Equations (6)–(8) and using Equation (4), it is found that,

$$\begin{aligned}
 & \|u_h^{n+1}\|^2 + \Delta t(\theta\beta \cdot n[u_h^{n+1}]^2, 1)_{\partial K_+} + \Delta t(\frac{\nabla \cdot \beta}{2}(u_h^{n+\frac{1}{2}})^2, 1)_K + \Delta t(b(u_h^{n+\frac{1}{2}})^2, 1)_K \\
 &= \|u_h^n\|^2 - \Delta t(\theta\beta \cdot n[u_h^n]^2, 1)_{\partial K_+} - 2\Delta t(\theta\beta \cdot n[u_h^n], [u_h^{n+1}])_{\partial K_+} + \Delta t(f, u_h^n + u_h^{n+1}).
 \end{aligned} \tag{9}$$

Now, by applying Equation (3),

$$\begin{aligned}
 & (\frac{\nabla \cdot \beta}{2}(u_h^{n+\frac{1}{2}})^2, 1)_K + (b(u_h^{n+\frac{1}{2}})^2, 1)_K = (b + \frac{\nabla \cdot \beta}{2})(u_h^{n+\frac{1}{2}})^2, 1)_K \\
 & \geq \frac{1}{8}\|b_0 u_h^{n+1}\|_{L^2(K)}^2 - \frac{1}{4}\|b_0 u_h^n\|_{L^2(K)}^2.
 \end{aligned} \tag{10}$$

$$\Delta t(\theta\beta \cdot n[u_h^n], [u_h^{n+1}])_{\partial K_+} \leq \Delta t(\theta\beta \cdot n\| [u_h^n]\|^2)_{L^2(\partial K_+)} + \Delta t(\theta\beta \cdot n\| [u_h^{n+1}]\|^2)_{L^2(\partial K_+)}. \tag{11}$$

$$(f, u_h^n + u_h^{n+1}) \leq C\|f\|^2 + \|u_h^n\|_{L^2(K)}^2 + \frac{1}{2}\|u_h^{n+1}\|_{L^2(K)}^2. \tag{12}$$

Insert Equations (10)–(12) into Equation (9).

$$\begin{aligned}
 & \|u_h^{n+1}\|_{L^2(K)}^2 + \Delta t(\theta\beta \cdot n[u_h^{n+1}]^2, 1)_{\partial K_+} + \frac{\Delta t}{8}\|b_0 u_h^{n+1}\|_{L^2(K)}^2 \\
 & \leq \|u_h^n\|_{L^2(K)}^2 + \Delta t(\theta\beta \cdot n[u_h^n]^2, 1)_{\partial K_+} + \frac{\Delta t}{4}\|b_0 u_h^n\|_{L^2(K)}^2 + 4C\|f\|^2 \Delta t \\
 & \quad + 9\|u_h^n\|_{L^2(K)}^2 \Delta t + \frac{1}{16}\|u_h^{n+1}\|_{L^2(K)}^2 \Delta t.
 \end{aligned} \tag{13}$$

By the Gronwall inequality, it is obtained:

$$\|u_h^{n+1}\|_{DG}^2 \leq C\Delta t \sum_{i=0}^n \|f(t_i)\|_{L^2(\Omega)}^2. \tag{14}$$

□

3.2. Error Estimates

To describe the error estimate for the time-discrete system, the following theorem is proposed:

Theorem 2. For any sequence $\{e_u^n\}_{n=0}^N$ and $e_u^n = u(X, t_n) - u^n$, there is a constant $C > 0$ independent of Δt such that the following condition holds:

$$\|e_u^{n+1}\|^2 \leq C(\Delta t)^2.$$

Proof. Let Equation (1) be represented as a time scheme in the following way:

$$u(x, t_{n+1}) = u(x, t_n) - \Delta t \beta \nabla u^{n+\frac{1}{2}} + R_1 + R_2,$$

where,

$$R_1 = \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} - \frac{\partial u}{\partial t}(X, t_{n+\frac{1}{2}}) = \frac{(\Delta t)^2}{24} \frac{\partial^3 u}{\partial t^3}(X, t_{n+\frac{1}{2}}) + \dots,$$

$$R_2 = \beta \nabla u^{n+\frac{1}{2}} - \beta \nabla u(X, t_{n+\frac{1}{2}}) = \beta \frac{(\Delta t)^2}{4} \frac{\partial^2}{\partial t^2} \nabla u(X, t_{n+\frac{1}{2}}) + \dots$$

Now, denote $e_u^n = u(X, t_n) - u^n$; therefore,

$$e_u^{n+1} = e_u^n - \frac{1}{2} \Delta t \beta (\nabla e_u^{n+1} + \nabla e_u^n) + \Delta t (R_1 + R_2).$$

Multiplying by $e_u^n + e_u^{n+1}$ in the above equation, it is found that,

$$\begin{aligned} \|e_u^{n+1}\|^2 &= \|e_u^n\|^2 - \frac{1}{2} \Delta t \beta \int_{\Omega} \nabla(e_u^n + e_u^{n+1}) \cdot (e_u^n + e_u^{n+1}) + \Delta t \int_{\Omega} (R_1 + R_2)(e_u^n + e_u^{n+1}) \\ &= \|e_u^n\|^2 - \frac{\Delta t}{4} \int_{(\Gamma_- \cup \Gamma_+)} \beta n (e_u^n + e_u^{n+1})^2 ds + C \Delta t (\|R_1\|^2 + \|R_2\|^2) + \Delta t \|e_u^n\|^2 + \Delta t \|e_u^{n+1}\|^2 \\ &\leq \|e_u^n\|^2 - \frac{\Delta t}{4} \int_{\Gamma_-} (e_u^n + e_u^{n+1})^2 \cdot \beta n + C \Delta t (\|R_1\|^2 + \|R_2\|^2) + \Delta t (\|e_u^n\|^2 + \|e_u^{n+1}\|^2) \\ &= \|e_u^n\|^2 + C \Delta t (\|R_1\|^2 + \|R_2\|^2) + \Delta t (\|e_u^n\|^2 + \|e_u^{n+1}\|^2). \end{aligned}$$

By using the Gronwall inequality,

$$\|e_u^{n+1}\|^2 \leq C(\Delta t)^2. \quad (15)$$

□

3.3. Spatial Error Estimate

To establish the main results, the following lemma is needed, which will play an important role in the error approximation.

Lemma 3. Let us consider Π_h to be an interpolation operator, then by the classical interpolation theory,

$$\|\Pi_h u - u\|_{L^2} + h \|\nabla(\Pi_h u - u)\|_{L^2} + h \leq Ch^{r+1} \|u\|_{H^{r+1}}.$$

For any two function $u, v \in L^2(\Omega)$, define the inner product by:

$$(u, v) = \int_{\Omega} u(x) v(x) dx.$$

Let $P : H_0^1(\Omega) \rightarrow V_h$ be a projection operator defined by:

$$(\nabla(u - Pu), \nabla w) = 0, \quad \text{for all } w \in V_h.$$

By the classical FEM theory,

$$\|u - Pu\|_{L^2} + h \|\nabla(u - Pu)\|_{L^2} \leq Ch^s \|u\|_{H^s}, \quad 1 \leq s \leq r + 1.$$

Now, the following theorem is proposed for the discussion about the main error analysis of the problem governed by Equation (1).

Theorem 3. Let u^n and $u_h^n, n = 1, 2, \dots, N$ denote the solutions of (1) and (4), respectively. For any $0 < \tau \leq T$ and $h > 0$, there is a constant C , which is dependent on τ_h , but independent of u , τ^2 , and h^r . Then, the following error representation formula holds:

$$\|u^n - u_h^n\|_{DG} \leq C(\tau^2 + h^r).$$

Proof. $\zeta_u^n = u_h^n - Pu^n$, and $\rho_u^n = u^n - Pu^n$. Therefore, using Equation (4), it is shown that:

$$\begin{aligned} & (\frac{\delta_t^+ \zeta_u^n}{\Delta t}, v_h) + (\beta n[\zeta_u^{n+\frac{1}{2}}], v_h) - (\zeta_u^{n+\frac{1}{2}}, \beta \nabla v_h) + (\theta \beta \cdot n[\zeta_h^{n+\frac{1}{2}}], [v_h])_{\partial K_+} + (b\zeta_h^{n+\frac{1}{2}}, v_h)_K \\ &= (\frac{\delta_t^+ \rho_u^n}{\Delta t}, v_h) + (\beta n[\rho_u^{n+\frac{1}{2}}], v_h) - (\rho_u^{n+\frac{1}{2}}, \beta \nabla v_h) + (\theta \beta \cdot n[\rho_h^{n+\frac{1}{2}}], [v_h])_{\partial K_+} \\ &+ (b\rho_h^{n+\frac{1}{2}}, v_h)_K, \forall v_h \in V_h. \end{aligned} \quad (16)$$

Let $v_h = \zeta_u^n + \zeta_u^{n+1} = 2\zeta_u^{n+\frac{1}{2}}$; from Equation (16), it is found that:

$$(\frac{\delta_t^+ \zeta_u^n}{\Delta t}, v_h) = \frac{1}{\Delta t} (\|\zeta_u^{n+1}\|^2 - \|\zeta_u^n\|^2). \quad (17)$$

$$(\beta n[\zeta_u^{n+\frac{1}{2}}], \zeta_u^{n+\frac{1}{2}}) - (\zeta_u^{n+\frac{1}{2}}, \beta \nabla \zeta_u^{n+\frac{1}{2}}) = (\frac{\nabla \cdot \beta}{2} (\zeta_h^{n+\frac{1}{2}})^2, 1)_K. \quad (18)$$

$$\begin{aligned} & (\theta \beta \cdot n[\zeta_h^{n+\frac{1}{2}}], [\zeta_h^{n+\frac{1}{2}}])_{\partial K_+} + (b\zeta_h^{n+\frac{1}{2}}, \zeta_h^{n+\frac{1}{2}})_K = \frac{1}{2} (\theta \beta \cdot n[\zeta_u^{n+\frac{1}{2}}]^2, 1)_{\partial K_+} + (b(\zeta_h^{n+\frac{1}{2}})^2, 1)_K \\ &= (\theta \beta \cdot n[\zeta_u^{n+1}]^2, 1)_{\partial K_+} + (\theta \beta \cdot n[\zeta_u^n]^2, 1)_{\partial K_+} + (\theta \beta \cdot n[\zeta_u^{n+1}], [\zeta_u^n])_{\partial K_+} + (b(\zeta_h^{n+\frac{1}{2}})^2, 1)_K. \end{aligned} \quad (19)$$

Again,

$$(\frac{\delta_t^+ \rho_u^n}{\Delta t}, v_h) \leq \|\frac{\delta_t \rho_u}{\Delta t}\|_{L^2} (\|\zeta_u^n\| + \|\zeta_u^{n+1}\|) \leq (\|\frac{\partial u}{\partial t}\|_{H^r}^2 Ch^{2r+2} + \epsilon \|\zeta_u^n\|^2 + \epsilon \|\zeta_u^{n+1}\|^2). \quad (20)$$

Furthermore,

$$\begin{aligned} & (\beta n[\rho_u^{n+\frac{1}{2}}], \zeta_u^{n+\frac{1}{2},+}) - (\rho_u^{n+\frac{1}{2}}, \beta \nabla \zeta_u^{n+\frac{1}{2}}) = (\beta n[\zeta_u^{n+\frac{1}{2}}], \rho_u^{n+\frac{1}{2},-}) + (\beta \nabla \rho_u^{n+\frac{1}{2}}, \zeta_u^{n+\frac{1}{2}}) \\ &\leq \epsilon (\beta n[\zeta_u^{n+1}]^2, 1)_{\partial K_+} + \epsilon (\beta n[\zeta_u^n]^2, 1)_{\partial K_+} + Ch^{2r+2} \|u\|_{H^r}^2 + \epsilon (\|\zeta_u^{n+1}\|^2 + \|\zeta_u^n\|^2) \\ &+ Ch^{2r+2} \|u\|_{H^r}^2. \end{aligned} \quad (21)$$

Combining Equations (17) to (21) and using the DG-norm defined in (5), we get,

$$\|\zeta_u^{n+1}\|_{DG}^2 \leq \|\zeta_u^n\|_{DG}^2 + \Delta t Ch^{2r} \|u\|_{H^r} + Ch^{2r+2} \|\frac{\partial u}{\partial t}\|_{H^r} + \Delta t (\epsilon \|\zeta_u^{n+1}\|_{DG}^2 + \epsilon \|\zeta_u^n\|_{DG}^2).$$

By using the Gronwall inequality above, $\|\zeta_u^{n+1}\|_{DG}^2 \leq C(\tau^2 + h^r)$. By using the triangle inequality, $\|u^n - u_h^n\|_{DG} \leq C(\tau^2 + h^r)$. $\square$

4. Numerical Experiments

In this section, a number of numerical experiments are presented to confirm our theoretical analysis derived in Section 3. The mesh generation and all computations were done with FreeFem++ [36]. The error profiles and corresponding convergence rates are given in Tables 1–3. Consider the problem (4) on the circular domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ for all the numerical tests. The boundary condition and the proper function

f and constant b were chosen for all the experiments such that u is the exact solution of (1). For all the experiments, the step size was considered as $h = \frac{1.0}{\text{mesh}}$, and the time step was $dt = h$. The algorithm (4) was executed on the uniform triangular mesh sequence to authenticate the theoretical results. The discrete space V_h was constructed by using piecewise polynomials of a uniform degree. The convergence history of the L^2 -norm of the error $\|u^n - u_h^n\|_{L^2}$ are displayed against the mesh-size. Figures 1–3 explain the behavior of the error $\|u_h^n - u^n\|_{L^2}$ at time $t = 1$ with respect to θ , while Figures 4–6 display the behavior of the error $\|u^n - u_h^n\|_{L^2}$ at time $t = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ at a fixed h (mesh at 40) and θ . Furthermore, Figures 7–9 present the numerical results of u_h^n for the values $0 \leq x \leq 1$ and $0 \leq y \leq 1$ at time $t = 0, 5, 7, 10$.

Table 1. Numerical errors and convergence rates for Experiment 1.

<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>	<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>
5	0.283202		40	0.0251048	1.52436
10	0.180874	0.646841	80	0.00583999	2.10393
20	0.0722165	1.32459	160	0.0013437	2.11976

Table 2. Numerical errors and convergence rates for Experiment 2.

<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>	<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>
5	0.830739		40	0.00211324	1.83117
10	0.097215	1.35544	80	0.000505464	2.06378
20	0.00751948	2.53455	160	0.000125952	2.00474

Table 3. Numerical errors and convergence rates for Experiment 3.

<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>	<i>Mesh</i>	$\ u^n - u_h^n\ _{L^2}$	<i>Order</i>
5	0.753748		40	0.0360737	2.0403
10	0.48435	0.638032	80	0.00950283	1.92452
20	0.148383	1.70673	160	0.00235028	2.01553

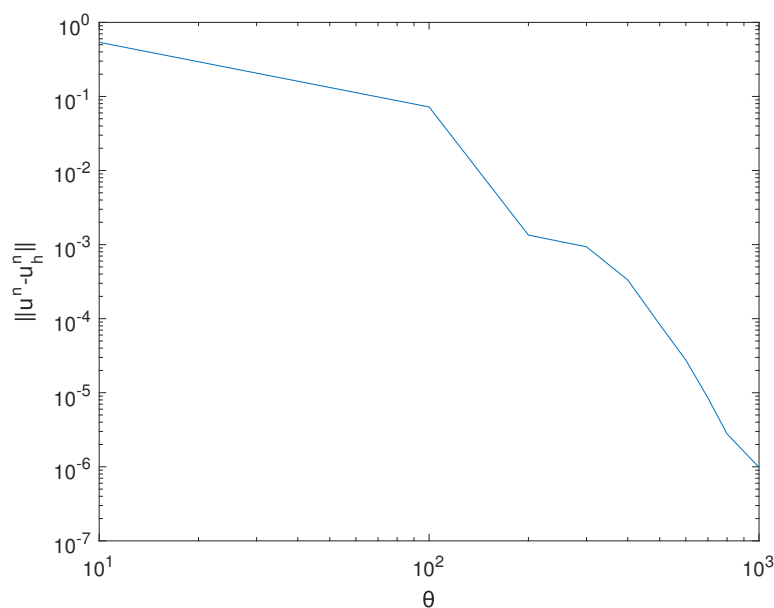


Figure 1. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 and $t = 1$ with respect to θ for Experiment 1.

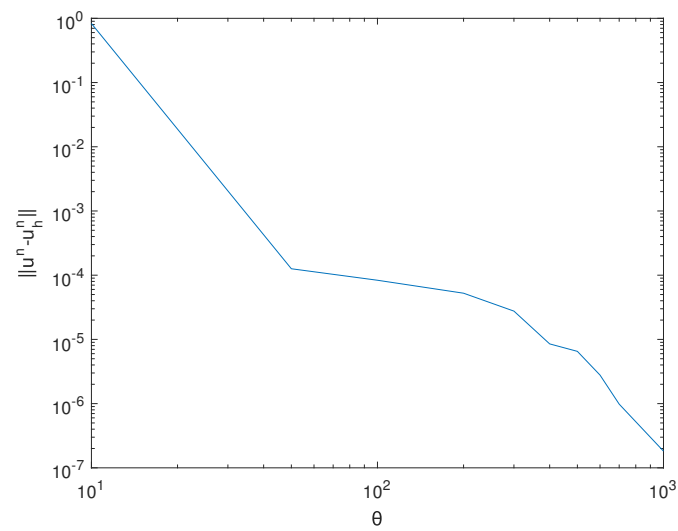


Figure 2. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 and $t = 1$ with respect to θ for Experiment 2.

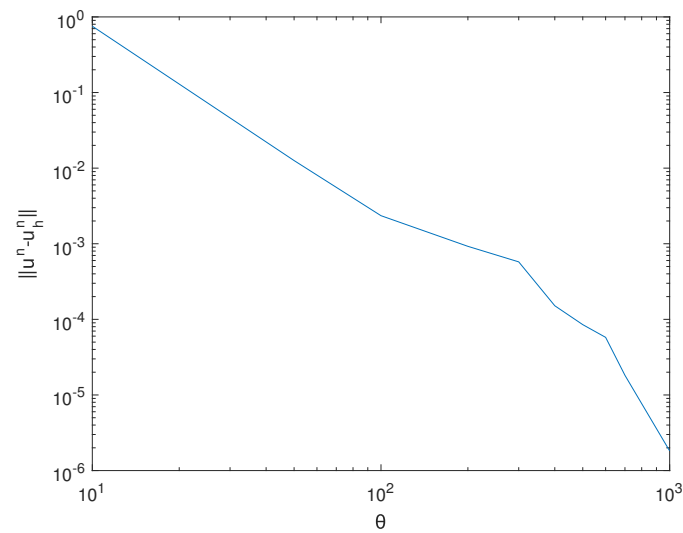


Figure 3. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 and $t = 1$ with respect to θ for Experiment 3.

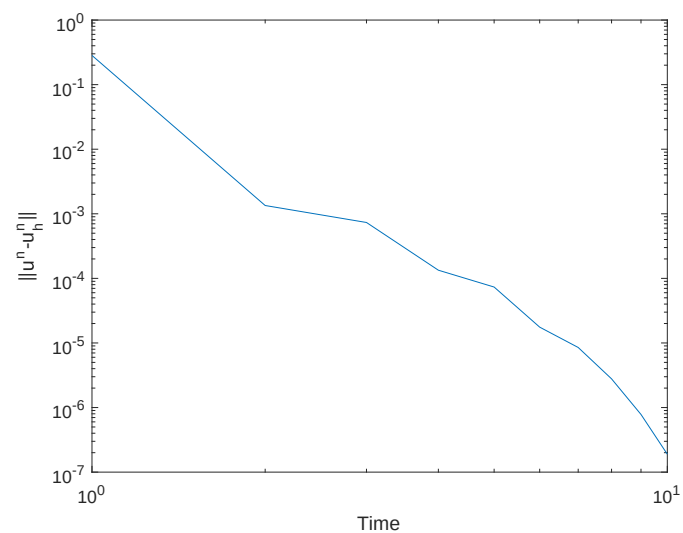


Figure 4. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 with respect to time for Experiment 1.

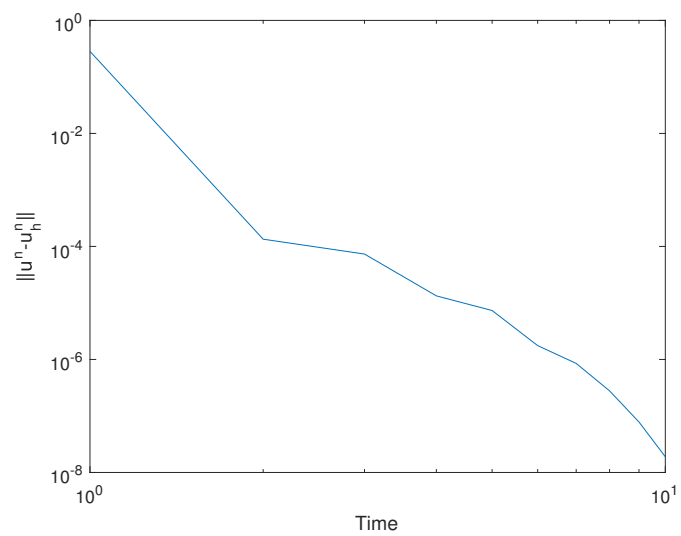


Figure 5. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 with respect to time for Experiment 2.

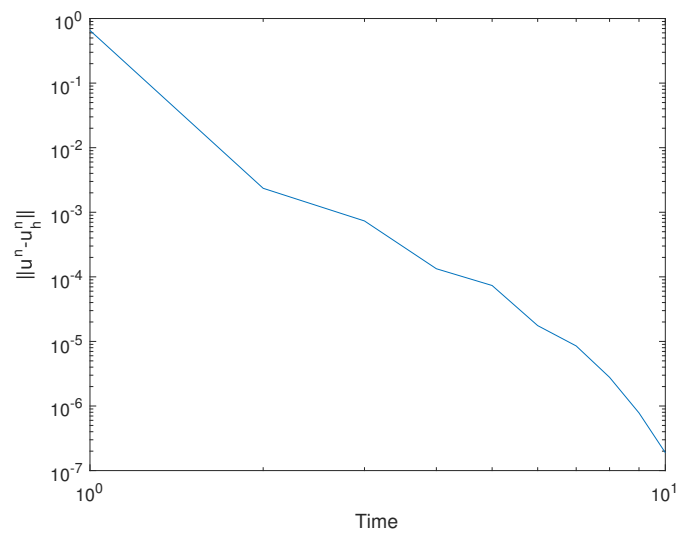


Figure 6. The behavior of the error $\|u^n - u_h^n\|_{L^2}$ at mesh 40 with respect to time for Experiment 3.

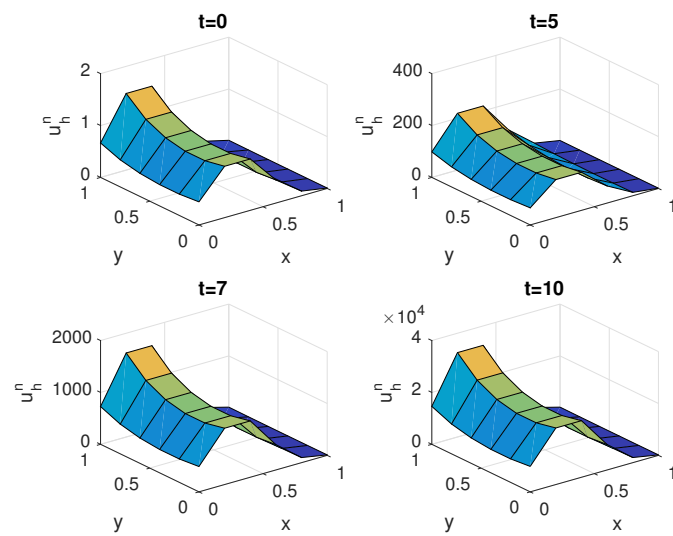


Figure 7. Numerical results of u_h^n for the values $0 \leq x \leq 1$ and $0 \leq y \leq 1$ at time $t = 0, 5, 7, 10$ for Experiment 1.

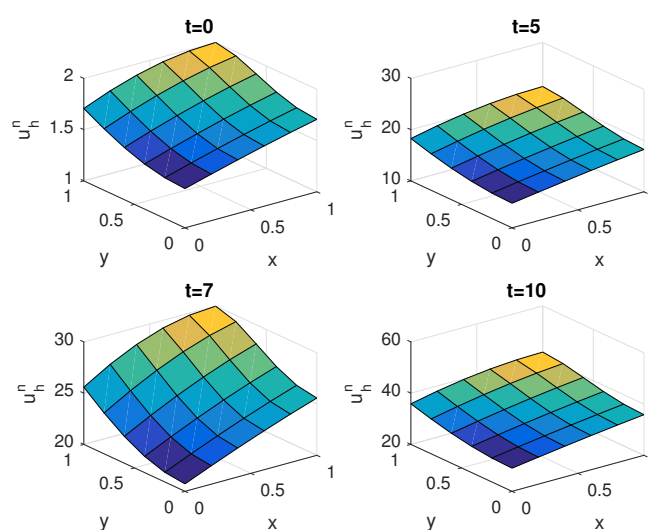


Figure 8. Numerical results of u_h^n for the values $0 \leq x \leq 1$ and $0 \leq y \leq 1$ at time $t = 0, 5, 7, 10$ for Experiment 2.

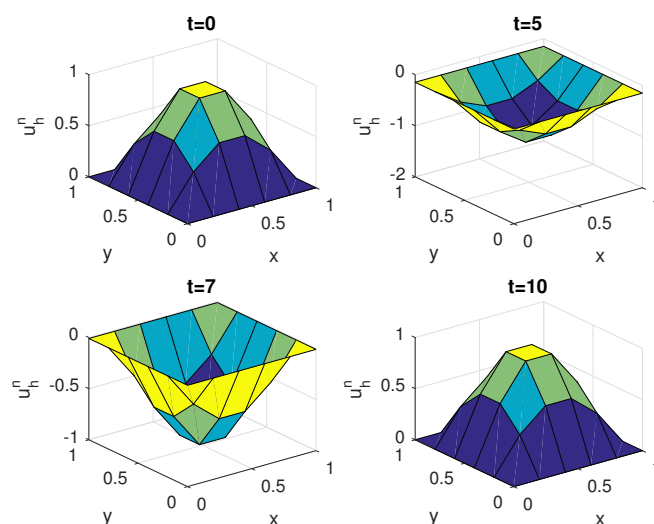


Figure 9. Numerical results of u_h^n for the values $0 \leq x \leq 1$ and $0 \leq y \leq 1$ at time $t = 0, 5, 7, 10$ for Experiment 3.

4.1. Experiment 1

For the first experiment, the convection vector is given by $\beta = (2 - y^2, 2 - x)^T$. The stability parameter $\theta = 200$, and the exact solution is chosen as:

$$u(t, x, y) = e^{t-10(x-0.3)^2+(y-0.3)^2}.$$

4.2. Experiment 2

For the second test, the convection vector was $\beta = (2 - y, 2 - x)^T$. The stability parameter $\theta = 50$, and the exact solution was chosen as:

$$u(t, x, y) = (1 + 2t)(1 + \sin(\frac{\pi}{8}(1 + x)(1 + y^2))).$$

4.3. Experiment 3

In this experiment, consider the convection vector $\beta = (0.5 - y, x - 0.5)^T$. The stability parameter $\theta = 100$, and the exact solution is given as:

$$u(t, x, y) = \cos(2t)\sin(\pi x)\sin(\pi y).$$

Tables 1–3 present the L^2 -norm of the error $\|u^n - u_h^n\|_{L^2}$ and the order of convergence against the meshes at 5, 10, 20, 40, 80, and 160, respectively, at time $t = 1$. From all Tables 1–3, it is observed that the quantity $\|u^n - u_h^n\|_{L^2}$ has the convergence rates as predicted by the theorem, and the computed order of convergence is more than two.

5. Conclusions

In this article, the L^2 error estimation was developed by using the Crank–Nicolson discontinuous Galerkin finite element method. A spatial type of error estimate was designed for the fully discrete system, as well as the time-discrete system. Numerical experiments were presented that emphasized the reliability and effectiveness of the parameter θ on the optimal order of convergence of the solutions. The method used in this paper can also be extended to higher order schemes to obtain the L^2 error estimate.

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