Abstract: The present paper deals with a class of second-order PDE constrained controlled optimization problems with application in Lagrange–Hamilton dynamics. Concretely, we formulate and prove necessary conditions of optimality for the considered class of control problems driven by multiple integral cost functionals involving second-order partial derivatives. Moreover, an illustrative example is provided to highlight the effectiveness of the results derived in the paper. In the final part of the paper, we present an algorithm to summarize the steps for solving a control problem such as the one investigated here.

Keywords: multi-time controlled second-order Lagrangian; Euler–Lagrange equations; second-order PDE constraints; multiple integral functional

1. Introduction

Calculus of Variations and Optimal Control Theory are two mathematical fields with strong and important connections, with significant applications in applied sciences, engineering, data analysis, and classification. Over time, many researchers have investigated these areas and the connections between them, obtaining remarkable results. In this regard, we mention only the works of Friedman [1], Hestenes [2], Kendall [3], Udriște [4], Petrat and Tumulka [5], Treanță [6], Deckert and Nickel [7], and Olteanu and Treanță [8]. In fact, before, we wanted to specify in particular the research papers that dealt with problems in several time variables. In the last decade, the study of multi-dimensional optimization problems (with important applications in various branches of mathematical sciences, engineering design, portfolio selection, game theory, decision problems in management science, data analysis, web access problems, query optimization in databases, and so forth), have been studied by Mititelu and Treanță [9], Treanță [10–17], Jayswal et al. [18], and Treanță [19–21]. More precisely, some classes of variational problems driven by multiple and path-independent curvilinear integral cost functionals with isoperimetric and mixed constraints involving PDEs of m-flow type and partial differential inequalities have been introduced and studied. In addition, the isoperimetric constrained optimization problems have been of particular interest to many researchers due to their importance in the applied sciences. We mention, for example, the research works of Schmitendorf [22,23], Forster and Long [24], and Treanță [13]. More specifically, by using the Pontryagin’s principle, Schmitendorf [22] studied the necessary conditions of optimality associated with a class of control problems involving isoperimetric and inequality constraints at the terminal time. Later, Forster and Long [24] (see also Schmitendorf [23]), by considering an alternative transformation technique, established the associated necessary optimality conditions for the same optimization problem. Quite recently, Treanță [13] has focused on the optimization of some simple, multiple or curvilinear integral functionals (governed by second-order Lagrangians) subject to ODE, PDE or isoperimetric constraints. In addition, Pascalis et al. [25] used genetic algorithms in order to theoretically design a range of phononic media that can act to prevent or ensure antiplane elastic wave propagation over a specific range of low frequencies, with each case corresponding to a specific pre-stress level.
In the present paper, inspired by the previous research in this field, we investigate a new class of PDE constrained optimization problems driven by multiple integral cost functionals which involve second-order partial derivatives. More precisely, the proper motivation for studying such issues is:

(i) Considering the bi-temporal optimal problem with pointwise state constraints:

\[
\min_{x,v} \frac{1}{2} \int_{t_0,t_1} \left( x(t) - \sin(2\pi t^2) \right)^2 dt_1 dt_2 + \frac{\alpha}{2} \int_{t_0,t_1} v^2(t) dt_1 dt_2
\]
subject to

\[-\Delta x(t) = v(t), \ t \in (t_0,t_1); \ x(t) = 0, \ t \in \partial(t_0,t_1).\]

This optimization problem has also been studied in Udrişte and Matei [26] by applying a simplified multi-time maximum principle.

(ii) (Neumann boundary control): Find a control function \( v \in L^2(\Gamma) \) that minimizes the cost functional

\[
J(v(\cdot)) = \frac{1}{2} \int_{t_0}^{t_1} (z(t) - z_d)^2 dt_1 \cdots dt_m + \frac{\beta}{2} \int_{\Gamma} v^2(t) ds,
\]

where \((z,v)\) satisfies \(-\Delta z + z = f\) in \(\Omega\) and \(\frac{\partial z}{\partial n} = v\) on \(\Gamma\), the function \(f\) is a given source term, the function \(v\) is a control variable, and \(\Omega\) is a bounded domain in \(\mathbb{R}^m\) with a boundary \(\Gamma\) of class \(C^2\). Since the term \(\frac{\beta}{2} \int_{\Gamma} v^2(t) ds\) (see \(\beta > 0\)) is proportional to the consumed energy, the minimizing of \(J\) is a compromise between the energy consumption and finding \(v\) so that the distribution \(z\) is close to the desired profile \(z_d\). We say that the control function is a boundary control because the control acts on \(\Gamma\).

The foregoing two examples of optimal control problems (involving second-order partial derivatives) have been taken from the literature. For more details, other examples, and different points of view, we refer to Raymond [27] and Udrişte and Matei [26].

The mathematical framework developed in this paper is more general than in Hestenes [2], Schmitendorf [22], Udrişte and Tevy [4], or Treanţă [13], both by the presence of controlled multiple integrals and by the inclusion of the new proof associated with the main result and the second-order partial derivatives. Therefore, this paper can be seen as a fundamental work for researchers in the field of applied sciences, mechanics, data analysis, and classification, where second-order PDEs (partial speed-acceleration constraints) are involved. It should also be mentioned that a simplified language is used, specific to applied mathematics, and readers of pure mathematics are referred to the bibliography. The main purpose is to expose ideas stripped of excessive formalizations, which have a great impact on the reader’s understanding of the natural phenomena encountered in engineering and economics. In this sense, mathematical modeling must be only the scientific support for the intelligent presentation of the real world and not the abstract notions specific to pure mathematics.

The paper is organized as follows. In Section 2, we introduce the optimization problem under study and formulate the main result of this paper, that is, Theorem 1. This theorem provides the necessary optimality conditions for the considered second-order PDE-constrained control problem. Moreover, the final part of this section includes an illustrative application and an algorithm. Finally, Section 3 presents the conclusions of the paper.
2. Second-Order PDE Constrained Controlled Optimization Problem

We consider a $C^3$-class function $\mathcal{L}(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), t)$, called *multi-time controlled second-order Lagrangian*, where $t = (t^n) = (t^1, \ldots, t^m) \in \Lambda_{t_0, t_1} \subset \mathbb{R}^m$, $s = (s^i) = (s^1, \ldots, s^n) : \Lambda_{t_0, t_1} \to \mathbb{R}^n$ is the state function of class $C^4$, and $v = (v^q) = (v^1, \ldots, v^k) : \Lambda_{t_0, t_1} \to \mathbb{R}^k$ is the control function (a piecewise continuous function). We also consider $s_a(t) := \frac{\partial s}{\partial t^a}(t)$, $s_{ab}(t) := \frac{\partial^2 s}{\partial t^a \partial t^b}(t)$, $a, b \in \{1, \ldots, m\}$, and $\Lambda_{t_0, t_1} = [t_0, t_1]$ (multi-time interval in $\mathbb{R}^m$) is a hyper-parallelepiped generated by $t_0, t_1 \in \mathbb{R}^+_m$ (diagonally opposite points).

**Second-order PDE constrained controlled optimization problem.** Find $(s^*, v^*)$ that minimizes the following multiple integral cost functional

$$L(s(\cdot), v(\cdot)) = \int_{\Lambda_{t_0, t_1}} \mathcal{L}(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), t) dt^1 \cdots dt^m$$

among all the functions $(s, v)$ that satisfy

$$s(t_0) = s_0, \quad s(t_1) = s_1, \quad s_\gamma(t_0) = \delta s_0, \quad s_\gamma(t_1) = \delta s_1,$$

or

$$s(t)|_{\partial\Lambda_{t_0, t_1}} = \text{given}, \quad s_\gamma(t)|_{\partial\Lambda_{t_0, t_1}} = \text{given}$$

and the controlled second-order PDE constraints (partial speed-acceleration constraints) defined as follows:

$$g^0_\xi(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), t) = 0, \quad a = 1, 2, \ldots, n, \quad \xi = 1, 2, \ldots, l \leq m.$$

In order to investigate the above controlled optimization problem (1), associated with the aforementioned controlled second-order PDE constraints, we introduce the Lagrange multiplier $b = \left( p^0_\xi(t) \right)$ and build a new multi-time controlled second-order Lagrangian (see summation over the repeated indices, Einstein summation)

$$L_1(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), b(t), t) = \mathcal{L}(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), t) + p^0_\xi(t) g^0_\xi(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), t),$$

that changes the initial controlled optimization problem (with second-order PDE constraints) into a partial speed–acceleration unconstrained controlled optimization problem

$$\min_{(s(\cdot), v(\cdot), b(\cdot))} \int_{\Lambda_{t_0, t_1}} L_1(s(t), s_\gamma(t), s_{\alpha\beta}(t), v(t), b(t), t) dt^1 \cdots dt^m$$

$$s(t_q) = s_q, \quad s_\gamma(t_q) = \delta s_q, \quad q = 0, 1.$$

In accordance with Lagrange theory, under suitable constraint qualifications, an extreme point of (1) is found among the extreme points of (2).

To formulate the necessary optimality conditions associated with the aforementioned controlled optimization problem, we shall introduce the Saunders’s multi-index notation (see Saunders [28], Treanţă [29]).

The main result of this paper is provided by the following theorem. It formulates the necessary optimality conditions for the second-order PDE constrained optimization problem (1).

**Theorem 1.** If $(s^*(\cdot), v^*(\cdot), b^*(\cdot))$ is a solution for (2), then

$$(s^*(\cdot), v^*(\cdot), b^*(\cdot))$$
is a solution of the following system of Euler-Lagrange PDEs

\[
\frac{\partial L_i}{\partial s^i} - D_\gamma \frac{\partial L_i}{\partial s^\gamma} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_i}{\partial s^\alpha \partial s^\beta} = 0, \quad i = 1, n
\]

\[
\frac{\partial L_1}{\partial \theta} - D_\gamma \frac{\partial L_1}{\partial \theta} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_1}{\partial \theta \partial \xi} = 0, \quad \theta = \Gamma_k
\]

\[
\frac{\partial L_1}{\partial p^\alpha} - D_\gamma \frac{\partial L_1}{\partial p^\gamma} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_1}{\partial p^\alpha \partial p^\beta} = 0, \quad a = 1, r, \quad \zeta = \Gamma_l
\]

where \( p^\alpha_{\theta,\gamma} := \frac{\partial p^\alpha}{\partial v^\gamma}, \ p^\alpha_{a,\beta} := \frac{\partial^2 p^\alpha}{\partial v^\gamma \partial v^\beta}, \ u_{a,\beta} := \frac{\partial^2 v^\alpha}{\partial v^\gamma \partial v^\beta}, \ \alpha, \beta \in \{1, 2, ..., m\}. \)

**Proof.** Consider that \((s(t), v(t), b(t))\) is a solution for (2). By considering the variations \(s(t) + \epsilon h(t), v(t) + \epsilon \eta(t), b(t) + \epsilon \xi(t)\), with \(h(t)|_{\partial \Lambda_{\alpha \beta,1}} = 0, \ \eta(t)|_{\partial \Lambda_{\alpha \beta,1}} = 0, \ \eta \in \{1, 2, ..., m\}\) (see \( h_{\eta} := \frac{\partial h}{\partial \eta} \)), \(b(t) + \epsilon \xi(t)\), with \(f(t)|_{\partial \Lambda_{\alpha \beta,1}} = 0\), and \(v(t) + \epsilon \mu(t)\), with \(m(t)|_{\partial \Lambda_{\alpha \beta,1}} = 0\), the controlled multiple integral cost functional becomes a function depending on \(\epsilon\). Consequently, it is a controlled multiple integral with parameter

\[
I(\epsilon) = \int_{\Lambda_{\alpha \beta,1}} L_1(s(t) + \epsilon h(t), s_\gamma(t) + \epsilon h_\gamma(t), s_{a,\beta}(t) + \epsilon h_{a,\beta}(t), v(t) + \epsilon \mu(t),
\]

\[
b(t) + \epsilon \xi(t), t) dt^1 \cdots dt^m.
\]

By using the hypothesis, we obtain

\[
\frac{d}{d\epsilon} I(\epsilon)_{\epsilon=0} = \int_{\Lambda_{\alpha \beta,1}} \left( \frac{\partial L}{\partial s^i} h^i + \frac{1}{\mu(\alpha, \beta)} \frac{\partial L}{\partial s^\alpha \partial s^\beta} h_{a,\beta} + \frac{\partial L}{\partial \theta} \right) dt^1 \cdots dt^m
\]

\[
= BT + \int_{\Lambda_{\alpha \beta,1}} \left( \frac{\partial L_1}{\partial s^i} - D_\gamma \frac{\partial L_1}{\partial s^\gamma} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_1}{\partial s^\alpha \partial s^\beta} \right) h^i dt^1 \cdots dt^m
\]

\[
+ \int_{\Lambda_{\alpha \beta,1}} \left( \frac{\partial L_1}{\partial \theta} - D_\gamma \frac{\partial L_1}{\partial \theta} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_1}{\partial \theta \partial \xi} \right) \xi^i dt^1 \cdots dt^m
\]

\[
+ \int_{\Lambda_{\alpha \beta,1}} \left( \frac{\partial L_1}{\partial p^\alpha} - D_\gamma \frac{\partial L_1}{\partial p^\gamma} + \frac{1}{\mu(\alpha, \beta)} D^2_{\alpha\beta} \frac{\partial L_1}{\partial p^\alpha \partial p^\beta} \right) \xi^i dt^1 \cdots dt^m = 0.
\]

In addition, by using the formula of integration by parts, it results that

\[
\frac{\partial L_1}{\partial s^i} h^i = -h^i D_\gamma \frac{\partial L_1}{\partial s^\gamma} + D_\gamma \left( \frac{\partial L_1}{\partial s^\gamma} h^i \right),
\]

\[
\frac{1}{\mu(\alpha, \beta)} \frac{\partial L_1}{\partial s^\alpha \partial s^\beta} h_{a,\beta} = \frac{1}{\mu(\alpha, \beta)} \left[ h^i D^2_{\alpha\beta} \frac{\partial L_1}{\partial s^i \partial s^\beta} - D_\beta \left( h^i D^2_{\alpha\beta} \frac{\partial L_1}{\partial s^i \partial s^\beta} + D_\beta \left( \frac{\partial L_1}{\partial s^\alpha \partial s^\beta} h^i \right) \right) \right]
\]

and taking into account the divergence formula, the boundary terms \(BT\) (given below) vanish (see \( h|_{\partial \Lambda_{\alpha \beta,1}} = m|_{\partial \Lambda_{\alpha \beta,1}} = f|_{\partial \Lambda_{\alpha \beta,1}} = 0, \ h_{\eta}|_{\partial \Lambda_{\alpha \beta,1}} = 0, \ n^i(t) = \) the unit normal vector on \(\partial \Lambda_{\alpha \beta,1}\), and \(\delta_{\gamma \xi} = \) the Kronecker’s symbol).
Find the extremals of $L$ and the extremals are characterized by the following PDEs

Let the double integral cost functional be given by

$$
\int_{\Lambda_{0,1}} D_\gamma \left( \frac{\partial L_1}{\partial s_\gamma} \right) dt^1 \cdots dt^m = \int_{\partial \Lambda_{0,1}} \delta_{\nu \xi} \frac{\partial L_1}{\partial v^\nu} v^\xi d\sigma,
$$

$$
\int_{\Lambda_{0,1}} D_\alpha \left( \frac{h^\nu D_\beta \frac{\partial L_1}{\partial s^\nu_\beta}}{\partial s^\nu_\beta} \right) dt^1 \cdots dt^m = \int_{\partial \Lambda_{0,1}} \delta_{\nu \xi} \frac{\partial L_1}{\partial v^\nu} v^\xi d\sigma,
$$

$$
\int_{\Lambda_{0,1}} D_\beta \left( \frac{\partial L_1}{\partial s_\beta} h^\nu \right) dt^1 \cdots dt^m = \int_{\partial \Lambda_{0,1}} \delta_{\nu \xi} \frac{\partial L_1}{\partial v^\nu} v^\xi d\sigma.
$$

Now, since the "small" variations $h, f, m$ were taken arbitrarily, by using a fundamental lemma of variational calculus, the relation (3) leads to the system of partial differential equations formulated in theorem, and the proof is complete. 

\* Remark 1. \* The system of Euler–Lagrange PDEs given in Theorem 1 can be reformulated as follows

$$
\frac{\partial L_1}{\partial s_i} - D_h \frac{\partial L_1}{\partial s_i} + \frac{1}{\mu(\alpha, \beta)} D^2 \frac{\partial L_1}{\partial s^\nu_\beta} = 0, \quad i = 1, n
$$

$$
\frac{\partial L_1}{\partial s_\beta} - D_h \frac{\partial L_1}{\partial s_\beta} + \frac{1}{\mu(\alpha, \beta)} D^2 \frac{\partial L_1}{\partial s^\nu_\beta} = 0, \quad \beta = 1, k
$$

$$
g^r(s(t), s_\alpha(t), s_{\alpha \beta}(t), v(t), t) = 0, \quad a = 1, 2, \cdots, r \leq n, \ z = 1, 2, \cdots, l \leq m.
$$

\* Example 1. \* Let the double integral cost functional be given by

$$
L(s(\cdot), v(\cdot)) = \int_{[0,1]^2} \left( s^2(t) + v^2(t) \right) dt^1 dt^2.
$$

Find the extremals of $L(s(\cdot), v(\cdot))$ subject to $s_{\alpha}(t) + s_{\beta}(t) = 0$ and the boundary conditions $s(0, 0) = s(1, 1) = 0$.

Solution. The associated auxiliary controlled Lagrangian is given by

$$
L_1 = s^2(t) + v^2(t) + b(t)(s_{\alpha}(t) + s_{\beta}(t))
$$

and the extremals are characterized by the following PDEs

$$
2s - \frac{\partial b}{\partial t^1} - \frac{\partial b}{\partial t^2} = 0,
$$

$$
2u = 0,
$$

$$
s_{\alpha}(t) + s_{\beta}(t) = 0,
$$

implying that $(s^*, v^*) = (s^*, 0)$ is the optimal solution of the considered PDE constrained optimization problem, corresponding to the Lagrange multiplier $b$ satisfying $p_{(t)\gamma}(t) + 2p_{(t)\gamma}(t) + p_{(t)\gamma}(t) = 0$.

The intention of the following algorithm (see Algorithm 1) is to summarize the steps for solving a control problem such as the one investigated here. More precisely, for a multiple integral cost functional and a set of mixed (boundary conditions and second-order PDE) restrictions, the goal is to find $(s^*, v^*)$ (which fulfils the set of mixed constraints) such that $L(s^*, v^*) \leq L(s, v)$ for all feasible points $(s, v)$. In this direction, we begin with a feasible point $(s, v)$. If $(s, v)$ satisfies the necessary conditions of optimality formulated in Theorem 1, then the “Generating Stage” (see below) is completed and we go to the next step; else, the algorithm stops. For $(s^*, v^*)$ obtained in “Generating Stage”, if $L(s^*, v^*) \leq L(s, v)$
is true, for all feasible points \((s, v)\), then \((s^*, v^*)\) is an optimal solution for the considered problem; else, the algorithm stops.

**Algorithm 1:** The steps for solving the control problem

**DATA:**
- controlled multiple integral cost functional
  \[
  \min_{(s, v)} L(s, v) = \int_{\Lambda_0} L(s(t), s_\gamma(t), s_\alpha(t), v(t), t) dt_1 \cdots dt_m;
  \]
- set of boundary condition and second-order PDE constraints
  \[
  g^a_\zeta(s(t), s_\gamma(t), s_\alpha(t), v(t), t) = 0, \quad a = 1, 2, \cdots, r \leq n, \quad \zeta = 1, 2, \cdots, l \leq m,
  \]
  and
  \[
  s(t_q) = s_q, \quad s_\gamma(t_q) = \tilde{s}_\gamma q, \quad q = 0, 1,
  \]
  or
  \[
  s(t)|_{\partial \Lambda_0} = \text{given}, \quad s_\gamma(t)|_{\partial \Lambda_0} = \text{given};
  \]

**RESULT:**
\[
S = \{(s^*, v^*)|L(s^*, v^*) \leq L(s, v), \quad \text{with } (s^*, v^*) \text{ satisfying the set of boundary conditions and second-order PDE constraints})\};
\]

**BEGIN**
- Generating Stage: let \((s, v)\) be a feasible point
  if the necessary optimality conditions (see Theorem 1) are not compatible with respect to \((s, v)\) then STOP
  else GO to the next step
- Deciding Stage: let \((s^*, v^*)\) be obtained in Generating Stage
  if \(L(s, v) \geq L(s^*, v^*)\) holds for all feasible points \((s, v)\) then \((s^*, v^*)\) is an optimal solution
  else STOP

**3. Conclusions**

In this paper, we have introduced and investigated a class of second-order PDE constrained controlled optimization problems with application in Lagrange–Hamilton dynamics. More specifically, necessary conditions of optimality were established for the considered class of variational problems driven by multiple integral cost functionals involving second-order partial derivatives. Moreover, the theoretical results are accompanied by an illustrative example. Finally, we have presented an algorithm to synthesize the steps for solving a controlled optimization problem such as the one investigated in the paper.

A new direction of research, on the considered class of control problems introduced in this paper, is given, for example, by the study of well-posedness and well-posedness in the generalized sense.

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