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# An Operational Characterization of Soft Topologies by Crisp Topologies

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**Abstract:** This paper contributes to the expanding literature on soft topology. We first prove that soft topologies can be characterized by crisp topologies. This takes advantage of two connected constructions that produce soft topologies from crisp topologies and vice versa. Both constructions are explicit and amenable to mathematical manipulations. Various consequences demonstrate that our theory has far-reaching implications for the development of soft topology and its extensions.

**Keywords:** soft topology; topology; base for a topology; second-countable space; separability



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## 1. Introduction

Soft topology is born from the interaction between soft set theory [1] and set-theoretic topology [2,3]. This field investigates a construction on the set of all soft sets that replicates the axiomatic behavior of the topological spaces from the perspective of soft unions and intersections.

Soft set theory is concerned with descriptions of a universal set of alternatives as expressed by a set of apposite characteristics. Since its establishment in 1999, soft set theory has become a flourishing research field (see for example [4,5]). It has been hybridized with other models (e.g., fuzzy soft sets [6], and their extensions by intuitionistic fuzzy soft sets [7] and hesitant fuzzy soft sets [8]; probabilistic and dual probabilistic soft sets [9]). The soft set model has been expanded (e.g., incomplete soft sets [10],  $N$ -soft sets [11]) and applied to many fields like medicine [12] or financial forecasting [13].

By the inspiration of the structure of a topological space, Çağman et al. [14] and Shabir and Naz [15] independently launched the field of soft topology. This novel structure does not consist of a standard topology on a set. The principles of soft topology about unions and intersections do not apply to subsets (i.e., to collections of elements from the ground set) as in topology. Instead, they apply to soft sets (i.e., to individual elements from the ground set). Although directly motivated by the axioms of a topology, soft topology builds on different essentials.

There are many papers with substantial contributions to the development of soft topology. Let us summarize some milestones of this theory.

A large part of the literature on soft topology consists of definitions of concepts and investigation of basic relationships among them. Concepts like soft second-countability are indisputable. However, this is not always the case: for various reasons, the extension of well-established notions in set-theoretic topology to soft topology is sometimes controversial. Let us see some cases.

Consider first the concept of ‘separability’. It had been exported to the field of soft metric spaces [16]. Then, in the general setting of soft topological spaces, in [17], two related axioms of soft separability are proposed; then, their performance and connection to soft second-countability are investigated. The requirements for a reliable concept of ‘soft separability’ are discussed in [17] too. Soft separation axioms are not exported straight away either [18].

Concepts in the vein of topological compactness have been proposed in various articles. Aygünoğlu and Aygün [19] initiate the analysis of soft compactness. Recently, Al-shami et al. [20] have produced seven generalized types of soft semi-compact spaces.

Another branch of the literature investigates procedures that generate soft topologies on the set of alternatives. For example, Al-shami et al. [21] have recently suggested the useful idea of sum of soft topological spaces. In addition, soft topologies can also be produced from elements pertaining to standard topology theory. These elements may be a set of topologies [17,22], or, alternatively, a collection of bases for topologies [17].

The success of the soft topological approach paved the way to subsequent variants of the idea. Fuzzy soft topological spaces have enjoyed a comparable success [23–25]. They were built on fuzzy soft sets [6]. A novel  $N$ -soft topology building on  $N$ -soft sets [11] is the subject of [26] that also produces an application to multi-criteria group decision-making. Hesitant fuzzy soft topology was introduced by [27]. It was built on hesitant fuzzy soft sets [8]. Soft metric spaces [16] have been deeply investigated.

This article has a clear-cut objective. For the first time, proofs are provided stating that soft topologies can be fully characterized in terms of standard topologies on a crisp set. Respective explicit constructions ensure a steady two-way transition from one setting to the other. This achievement has several remarkable upshots. Concepts from soft topology that are controversial or unwieldy can be reduced to the corresponding concepts in topology. Results from topological spaces may be exported to soft topological spaces. In fact, we give examples of these fundamental advances too, namely: (1) Soft points, resp., soft bases, in a soft topological space are identified with points (i.e., single-valued subsets), resp., bases, of the topological space associated with it; (2) Soft second countability of a soft topological space is equivalent to second countability of the topology associated with it; (3) The classical result that second-countable topological spaces are separable can be exported to a property of soft topological spaces.

This research article is organized as follows. Section 2 states preliminary definitions. Then, Section 3 defines two novel mappings between topological and soft topological spaces. Both constructions are explicit. Their fundamental properties and mutual links are explored too. Section 4 proves our main theorem, i.e., the equivalence between soft topologies and (crisp) topologies on a suitable product set. Section 5 exploits this equivalence in order to discuss the cogency of certain soft topological properties. The aim of Section 6 is to conclude the paper and suggest some future lines of research.

## 2. Preliminaries

Along this article,  $X$  denotes a nonempty set, and  $E$  represents a set of attributes. For each set  $U$ , we denote by  $\mathcal{P}(U)$  the set formed by all the subsets of  $U$ , i.e., the set of parts of the set  $U$ .

In this section, we briefly review some basic elements of soft set theory and soft topology. Sometimes, we need to distinguish clearly between topologies and soft topologies, which are not topologies on a set. In this case, we use the term “crisp topology” to mean a topology on a set. The standard notions of set-theoretic topology can be consulted in Munkres [2] and Willard [3].

A soft set on  $X$  is a pair  $(F, E)$ , where  $E$  consists of all the attributes that are needed to characterize the elements of  $X$ , and  $F$  is a mapping  $F : E \rightarrow \mathcal{P}(X)$ . Thus, according to [1], a soft set over  $X$  is a parameterized family of subsets of  $X$ . From a mathematical viewpoint, a soft set on  $X$  is a multi-function (also called correspondence, point-to-set mapping, or multi-valued mapping)  $F$  from the set of characteristics  $E$  to  $X$ . The set formed by all soft sets on  $X$  with attributes  $E$  shall be expressed as  $SS_E(X)$ . Granted that  $E$  is common knowledge, we may write  $SS(X)$  instead.

When  $a \in E$ , the subset  $F(a) \subseteq X$  will be sometimes denoted as  $(F, E)(a)$  for higher accuracy. This is a subset often called the  $a$ -approximate elements of  $X$ .

When  $(F, E)$  is a soft set on  $X$  with the property that for each  $a \in E$ , the set  $F(a)$  is finite (respectively, countable), we say that it is a finite (resp., countable) soft set on  $X$  [16,28].

In order to operate with soft sets in a more convenient manner, two descriptions are especially helpful.

- (1) A soft set  $(F, E)$  is usually described as  $\{(e, F(e)) : e \in E\}$ .
- (2) In the event that both  $X$  and  $E$  are finite,  $(F, E)$  can be displayed in tabular form too. Figure 1 shows this layout when  $X = \{x_1, x_2, \dots, x_m\}$  and  $E = \{e_1, e_2, \dots, e_n\}$ . By convention, in such a table, we write  $f_{jk} = 1$  when  $x_j \in F(e_k)$ , and  $f_{jk} = 0$  otherwise.

$(F, E)$	$e_1$	$e_2$	$\dots$	$e_n$
$x_1$	$f_{11}$	$f_{12}$	$\dots$	$f_{1n}$
$x_2$	$f_{21}$	$f_{22}$	$\dots$	$f_{2n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$x_m$	$f_{m1}$	$f_{m2}$	$\dots$	$f_{mn}$

Figure 1. A soft set  $(F, E)$  presented in its tabular form.

Two extreme examples are the null and absolute soft sets on  $X$ . The null soft set  $\Phi$  satisfies  $\Phi(a) = \emptyset$  for each  $a \in E$ . The absolute soft set  $\tilde{X}$  satisfies  $\tilde{X}(a) = X$  for each  $a \in E$ . Another special example of soft sets are soft points [16,29], which are soft sets  $(F, E)$  such that there exist  $y \in X$  and  $a \in E$  with the property  $F(a) = \{y\}$  and  $F(a') = \emptyset$  when  $a' \in E \setminus \{a\}$ . Such soft point shall be represented as  $(\{y\}_a, E)$ . Notice, however, that, for certain authors, soft points have a different meaning [22]: a soft point is  $(F, E)$  such that there is  $y \in X$  with the property  $F(a) = \{y\}$  for all  $a \in E$ .

Standard set-theoretic operations are exported to soft set theory. Union, intersection, and inclusion are defined on soft sets from  $SS_E(X)$  as follows: when  $(F_1, E), (F_2, E) \in SS_E(X)$ ,

- (1)  $(F_1, E) \sqcup (F_2, E)$  is  $(F_3, E) \in SS_E(X)$  with  $F_3(a) = F_1(a) \cup F_2(a)$  for each  $a \in E$ . This can also be written as  $((F_1, E) \sqcup (F_2, E))(a) = F_1(a) \cup F_2(a)$  when  $a \in E$ .
- (2)  $(F_1, E) \sqcap (F_2, E)$  is  $(F_4, E) \in SS_E(X)$  such that  $F_4(a) = F_1(a) \cap F_2(a)$  for each  $a \in E$ . This can also be written as  $((F_1, E) \sqcap (F_2, E))(a) = F_1(a) \cap F_2(a)$  when  $a \in E$ .
- (3)  $(F_1, E) \sqsubseteq (F_2, E)$  represents the property  $F_1(a) \subseteq F_2(a)$  for each  $a \in E$ .

The complement of  $(F, E) \in SS_E(X)$  is the soft set  $(F, E)^c = (F^c, E) \in SS_E(X)$  such that, for each  $a \in E$ ,  $F^c(a) = X \setminus F(a)$ .

These operators endow  $SS_E(X)$  with the structure of a Boolean lattice (cf., Aygünoğlu and Aygün ([19] Theorem 2.1)).

**Remark 1.** The respective extensions of the concepts of union and intersection of two soft sets to arbitrary collections of soft sets are direct.

In addition, a concept of soft equality derives from soft inclusions. Thus,  $(F_1, E) = (F_2, E)$  means  $(F_1, E) \sqsubseteq (F_2, E)$  and  $(F_2, E) \sqsubseteq (F_1, E)$ . In other words,  $F_1(e) = F_2(e)$  for each  $e \in E$ .

Note that the soft-theoretic union acts on elements of  $SS_E(X)$  and produces another element of  $SS_E(X)$ . In addition, the same goes for the soft-theoretic intersection. They are different from the set-theoretic operators of union and intersection that act on subsets of a given set in order to produce other subsets of the set. This difference carries forward to the concept of a soft topology that we proceed to recall. To be precise, observe a marked contrast in structure.  $\Sigma$  is a topology on  $X$  when  $\Sigma \subseteq \mathcal{P}(X)$  satisfies a known list of properties [2,3]. By contrast, a soft topology on  $X$  (with a fixed set of attributes  $E$ ) is  $\tau \subseteq SS_E(X)$  that satisfies the properties in the following definition:

**Definition 1** ([14,15]). A soft topology  $\tau$  on  $X$  is a set of soft sets on  $X$ ,  $\tau \subseteq SS_E(X)$ , that satisfies:

- (1) Both  $\Phi$  and  $\tilde{X}$  belong to  $\tau$ .

- (2) The union of any collection of soft sets from  $\tau$  is a member of  $\tau$ .
- (3) The intersection of any finite collection of soft sets from  $\tau$  is a member of  $\tau$ .

The members of  $\tau$  are called soft open sets.

**Remark 2.** For higher accuracy, when needed, we shall say that  $\tau$  in Definition 1 is a soft topology for  $SS_E(X)$ . The standard denomination “ $\tau$  is a soft topology on  $X$ ” eschews the role of  $E$  that will be crucially important in our main Theorem.

**Example 1.** The following soft topologies for  $SS_E(X)$  have been defined in the literature:

- (i) The indiscrete soft topology is  $\tau_{id} = \{\Phi, \tilde{X}\}$ , and the discrete soft topology is  $\tau_d = SS(X)$  [15].
- (ii) The cofinite soft topology was introduced by [28]

$$\tau_c = \left\{ (F, E) \in SS_E(X) \text{ whose complement is a finite soft set} \right\}. \tag{1}$$

In addition to specific examples of soft topologies, two general procedures exist that generate soft topologies from crisp topologies. The first one is quite direct:

**Definition 2** ([17,22]). Let  $\Sigma = \{\Sigma_e\}_{e \in E}$  be a collection of crisp topologies on  $X$ . The set

$$\tau(\Sigma) = \left\{ \{(a, F(a)) : a \in E\} \in SS_E(X) \text{ such that } F(a) \in \Sigma_a \text{ for each } a \in E \right\} \tag{2}$$

produces a soft topology called the soft topology on  $X$  generated by the family of topologies  $\Sigma$ .

When  $\Sigma_a = \Sigma_{a'} = \Sigma$  for each  $a, a' \in E$ , we also write  $\tau(\Sigma) = \tau(\Sigma)$ .

As an application, in the paper [17], it was proved that the cofinite soft topology  $\tau_c$  can be obtained from  $\Sigma_c$ , the cofinite (crisp) topology on  $X$ , by the process defined above. Put briefly,  $\tau_c = \tau(\Sigma_c)$ .

The second procedure that generates soft topologies from crisp topologies relies on the behavior of soft open bases. Let us summarize some preliminary facts.

Quite naturally, soft bases are defined as sets of soft sets which generate soft topologies when we produce all their arbitrary soft unions. Thus,  $\mathcal{B} \subseteq \tau$  is a soft base for the soft topology  $\tau$  if we can represent each  $(F, E) \in \tau$  as the union of a conveniently selected collection of soft sets from  $\mathcal{B}$  [14]. Roy and Samanta produced the more operative concept of a soft open base. Any soft open base produces a soft topology ([30] Theorem 13) in such a way that the soft open base is a soft base of the topology that it generates by this process ([30] Theorem 16).

In ([17] Proposition 1), it was proved that each base for a topology produces a soft open base by an explicit definition. Now a soft topology ensues from this soft open base ([30] Theorem 13). Thus, any base for a crisp topology defines an associated soft topology.

Notice that the two constructions of soft topologies defined above are closely related. Suppose that we fix a base for a crisp topology. We can produce both a crisp topology and a soft open base from it. Then, we can apply Definition 2 to this crisp topology, and we can also produce a soft topology from the soft open base. The conclusion is that the respective soft topologies that we obtain are the same ([17] Theorem 3).

Theoretical consequences follow from these constructions. For example, recall that a soft second-countable (S2C) soft topology is defined by the property that it possesses a base formed by a countable number of elements ([16] Definition 4.32). Then, ([17] Section 4) proves that Definition 2 produces a S2C soft topology, provided that each  $\Sigma_e$  is second-countable and  $E$  is at most countable. Actually, a converse to this result holds true as well ([17] Corollary 2).

### 3. Two Novel Mappings between Topological and Soft Topological Spaces

A good deal of the success of our operational characterization of soft topologies relies upon the behavior of two constructions. The first one yields a soft topological space associated with any crisp topology on a certain Cartesian product of two sets. This is done in Section 3.1. The second works in the opposite direction: with any soft topological space, it produces a crisp topology on the Cartesian product of the universe of discourse  $X$  and the set of attributes. This is achieved in Section 3.2. Then, Section 3.3 shows that, in the particular case of one single attribute, these procedures are linked to a known construction of crisp topologies from soft topologies. Finally, Section 3.4 provides a compact expression of a handy relationship between both constructions.

#### 3.1. A Soft Topological Space Stemming from a Topological Space

To achieve the goal of this section, we need the following preliminary concept:

**Definition 3.** Any  $S \subseteq X \times E$  induces  $(Y_S, E) \in SS_E(X)$  by the expression

$$Y_S(a) = \{x \in X \mid (x, a) \in S\} \text{ for each } a \in E. \tag{3}$$

The next two technical lemmas will prove useful:

**Lemma 1.** Suppose  $P, S \subseteq X \times E$  are such that  $P \cap S \neq \emptyset$ . Then,  $(Y_P, E) \sqcap (Y_S, E) \neq \Phi$ .

**Proof.** Notice that, when  $(x, e) \in P \cap S \neq \emptyset$ , (3) implies  $x \in Y_P(e)$  and  $x \in Y_S(e)$ . This concludes the argument because  $x \in ((Y_P, E) \sqcap (Y_S, E))(e) \neq \emptyset$ .  $\square$

**Lemma 2.** Suppose  $S_i \subseteq X \times E$  for each  $i \in I$ . Define  $S = \cup_{i \in I} S_i$ . Then,  $(Y_S, E) = \sqcup_{i \in I} (Y_{S_i}, E)$ .

**Proof.** In order to prove the equality of these soft sets, let us fix an arbitrary  $e \in E$ . We need to prove the set equality  $(Y_S, E)(e) = \left( \sqcup_{i \in I} (Y_{S_i}, E) \right)(e)$ .

Notice that  $x \in \left( \sqcup_{i \in I} (Y_{S_i}, E) \right)(e)$  if and only if  $x \in \cup_{i \in I} Y_{S_i}(e)$ , by definition of soft union. Now,  $x \in \cup_{i \in I} Y_{S_i}(e)$  if and only if there exists  $j \in I$  with  $x \in Y_{S_j}(e)$ . Using (3), we rewrite this equivalence as  $x \in \cup_{i \in I} Y_{S_i}(e)$  if and only if there exists  $j \in I$  with  $(x, e) \in S_j$ . It is now obvious that  $x \in \cup_{i \in I} Y_{S_i}(e)$  if and only if  $(x, e) \in \cup_{i \in I} S_i = S$  if and only if  $x \in (Y_S, E)$ , again by (3). This concludes the argument.  $\square$

We are now ready to define a novel class of soft topologies. We do this in the following result:

**Proposition 1.** Let  $\Sigma$  be a crisp topology on  $X \times E$ . Then,

$$\tau^\Sigma = \left\{ (Y_G, E) \mid G \in \Sigma \right\} \tag{4}$$

is a soft topology on  $X$ .

**Proof.** We check the three axioms for a soft topology.

- (1)  $\emptyset \in \Sigma$  implies  $(Y_\emptyset, E) = \Phi \in \tau^\Sigma$ ; and  $X \times E \in \Sigma$  implies  $(Y_{X \times E}, E) = \tilde{X} \in \tau^\Sigma$ .
- (2) Suppose that  $(F_i, E) \in \tau^\Sigma$  for each  $i \in I$ . By construction (4), with each  $i$ , we can associate  $G_i \in \Sigma$  such that  $(F_i, E) = (Y_{G_i}, E)$ . Define  $G = \cup_{i \in I} G_i$ . Then,  $G \in \Sigma$  because  $\Sigma$  is a topology; therefore,  $(Y_G, E) \in \tau^\Sigma$  by (4). Lemma 2 proves  $\sqcup_{i \in I} (F_i, E) = (Y_G, E) \in \tau^\Sigma$ .
- (3) Suppose that  $(F_1, E), (F_2, E) \in \tau^\Sigma$ . By construction (4), with each  $i = 1, 2$ , we can associate  $G_i \in \Sigma$  such that  $(F_i, E) = (Y_{G_i}, E)$ . Define  $G = G_1 \cap G_2 \in \Sigma$ ; therefore,  $(Y_G, E) \in \tau^\Sigma$ . We claim  $(F_1, E) \sqcap (F_2, E) = (Y_G, E) \in \tau^\Sigma$ .

To prove this soft set equality, let us fix  $e \in E$ . We will show the set equality  $\left( (F_1, E) \sqcap (F_2, E) \right)(e) = (Y_G, E)(e)$ . Notice that  $x \in \left( (F_1, E) \sqcap (F_2, E) \right)(e)$  if and only if  $x \in (F_i, E)(e)$  for each  $i = 1, 2$ , if and only if  $x \in (Y_{G_i}, E)(e)$  for each  $i = 1, 2$ , if and only if  $(x, e) \in G_1 \cap G_2 = G$ . With the help of (3), we can now conclude  $x \in \left( (F_1, E) \sqcap (F_2, E) \right)(e)$  if and only if  $x \in (Y_G, E)(e)$ .  $\square$

### 3.2. A Topological Space Stemming from a Soft Topological Space

This novel construction works in the opposite direction of the process designed in Section 3.1. It requires the following preliminary concept:

**Definition 4.** Any  $(F, E) \in SS_E(X)$  induces a subset of  $X \times E$  by the expression

$$A_{(F,E)} = \{(x, a) \in X \times E \mid x \in F(a)\}. \tag{5}$$

The next technical result will be useful:

**Lemma 3.** Suppose  $(F_i, E) \in SS_E(X)$  for each  $i \in I$ . Define  $(F, E) = \sqcup_{i \in I} (F_i, E)$ . Then,  $A_{(F,E)} = \cup_{i \in I} A_{(F_i,E)}$ .

**Proof.** We prove  $A_{(F,E)} = \cup_{i \in I} A_{(F_i,E)}$  by double inclusion.

Let us first fix an arbitrary  $(x, e) \in \cup_{i \in I} A_{(F_i,E)} \subseteq X \times E$ . There is  $j \in I$  such that  $(x, e) \in A_{(F_j,E)}$ . Using (5), we deduce  $x \in F_j(e) \subseteq \cup_{i \in I} F_i(e)$ . Then, by definition of soft union and the construction of  $(F, E)$ , we assure  $x \in F(e)$ . Now, (5) entails  $(x, e) \in A_{(F,E)}$ .

Conversely, suppose  $(x, e) \in A_{(F,E)}$ . Using (5) and the definition of soft union, we deduce  $x \in F(e) = \cup_{i \in I} F_i(e)$ . Select  $j \in I$  such that  $x \in F_j(e)$ ; then,  $(x, e) \in A_{(F_j,E)} \subseteq \cup_{i \in I} A_{(F_i,E)}$ . This concludes the argument.  $\square$

We can now define a crisp topology associated with any soft topology as follows:

**Proposition 2.** If  $\tau$  is a soft topology on  $X$ ,

$$\Sigma(\tau) = \left\{ A_{(F,E)} \mid (F, E) \in \tau \right\} \tag{6}$$

is a crisp topology on  $X \times E$ .

**Proof.** We check the three standard axioms for a crisp topology [2,3].

- (1)  $\emptyset \in \Sigma(\tau)$  because  $\Phi \in \tau^\Sigma$  and  $A_\Phi = \emptyset$ ; and  $X \times E \in \Sigma(\tau)$  because  $\tilde{X} \in \tau$  and  $A_{\tilde{X}} = X \times E$ .
- (2) Suppose that  $G_i \in \Sigma(\tau)$  for each  $i \in I$ . By construction (6), with each  $i$ , we can associate  $(F_i, E) \in \tau$  such that  $G_i = A_{(F_i,E)}$ . Define  $(F, E) = \sqcup_{i \in I} (F_i, E) \in \tau$ ; then,  $A_{(F,E)} \in \Sigma(\tau)$  by (6). Lemma 3 proves  $\cup_{i \in I} G_i = A_{(F,E)}$ ; therefore, the conclusion  $\cup_{i \in I} G_i \in \Sigma(\tau)$ .
- (3) Suppose that  $G_1, G_2 \in \Sigma$ . By construction (6), for  $i = 1, 2$ , we can associate  $(F_i, E) \in \tau$  such that  $G_i = A_{(F_i,E)}$ . Define  $(F, E) = (F_1, E) \sqcap (F_2, E) \in \tau$ ; then,  $A_{(F,E)} \in \Sigma(\tau)$ . We prove  $G_1 \cap G_2 = A_{(F,E)}$  by double inclusion.

Let us first fix an arbitrary  $(x, e) \in G_1 \cap G_2$ . For  $i = 1, 2$ ,  $(x, e) \in A_{(F_i,E)}$ . Using (5), we deduce  $x \in F_1(e) \cap F_2(e)$ . Then, by definition of soft intersection and the construction of  $(F, E)$ , we assure  $x \in F(e)$ . Now, (5) entails  $(x, e) \in A_{(F,E)}$ .

Conversely, suppose  $(x, e) \in A_{(F,E)}$ . Using (5), we deduce  $x \in F(e) = F_1(e) \cap F_2(e)$ . For  $i = 1, 2$ ,  $x \in F_i(e)$  yields  $(x, e) \in A_{(F_i,E)} = G_i$ . Therefore,  $(x, e) \in G_1 \cap G_2$ .  $\square$



### 3.3. Relationship with the Literature

The founding Shabir and Naz [15] proved that, for each soft topology  $\tau$  on  $X$ , if we fix a parameter  $e \in E$ , then the following expression defines a crisp topology on  $X$  called the  $e$ -parameter topology:

$$\tau(e) = \{F(e) \mid (F, E) \in \tau\}. \tag{7}$$

Terepeta ([22] Theorem 1) noticed that, when  $E$  is a singleton, this topology has a special behavior. It satisfies properties that do not hold in general (i.e., when  $E$  is unrestricted). The poor performance of  $\tau(e)$  leads Terepeta to conclude that “soft topologies on a set  $X$  with  $E$  as the set of parameters are not equivalent to the general topologies on the set  $X$ ” (unless  $E$  is restricted to be a singleton). Our next section overcomes this hindrance to the development of soft set theory, by the recourse to crisp topologies on  $X \times E$ . This is our main result in this paper. However, before moving on to that issue, the next result insists on the fact that the  $e$ -parameter topology with a single-valued attribute set is a distinguished case. The reason is that it can be strongly linked to our construction in Section 3.2:

**Proposition 3.** *Let us fix  $\tau$ , a soft topology on  $X$ . Suppose  $E = \{e\}$ . Then, the mapping*

$$\begin{array}{ccc} \varepsilon : (X, \tau(a)) & \longrightarrow & (X \times E, \Sigma(\tau)) \\ x & & (x, a) \end{array}$$

*is a homeomorphism of topological spaces.*

We recall that, when  $(Y, \Sigma)$  and  $(Y', \Sigma')$  are topological spaces, a mapping  $f : (Y, \Sigma) \longrightarrow (Y', \Sigma')$  is continuous if  $f^{-1}(G') \in \Sigma$  for each  $G' \in \Sigma'$  ([3] Definition 7.1 and Theorem 7.2). In addition,  $f$  is a homeomorphism when it is a bijective (one-to-one and onto) and continuous mapping, whose inverse is also continuous ([3] Definition 7.8). In this case,  $Y$  and  $Y'$  are homeomorphic topological spaces. Willard ([3] Theorem 7.9) provides several simple characterizations of homeomorphisms, and then he explains that “homeomorphic topological spaces are, for the purposes of a topologist, the same”.

Thus, the message of Proposition 3 is now clear: for topological purposes, the  $e$ -parameter topological space is “the same as” the topological space defined in Section 3.2 when  $E$  is a singleton.

**Proof. of Proposition 3.** We must show that both  $\varepsilon$  and its inverse mapping, which is obviously defined by  $\varepsilon^{-1}(x, a) = x$  for each  $(x, a) \in X \times E$ , are continuous.

To prove that  $\varepsilon$  is continuous, fix  $G \in \Sigma(\tau)$ . By construction (6), there exists  $(F, E) \in \tau$  such that  $G = A_{(F,E)} = \{(x, a) \in X \times E \mid x \in F(a)\}$ . Now, obviously  $\varepsilon^{-1}(G) = F(a)$ , and  $F(a) \in \tau(a)$  by (7). Therefore,  $\varepsilon^{-1}(G) \in \tau(a)$ .

To prove that  $\varepsilon^{-1}$  is continuous, fix  $G \in \tau(a)$ . By construction (7), there is  $(F, E) \in \tau$  for which  $G = F(a)$ . Now,  $(\varepsilon^{-1})^{-1}(G) = \varepsilon(G) = \{(x, a) \in X \times E \mid x \in F(a) = G\}$ , thus  $(\varepsilon^{-1})^{-1}(G) = A_{(F,E)}$  by construction (5). This set belongs to  $\Sigma(\tau)$  by (6).  $\square$

### 3.4. Relationship between the Two Constructions

Our next lemma will be helpful to streamline the proof of Theorem 1 in Section 4. However, it is also remarkable as a succinct expression of a neat relationship between the constructions in Sections 3.1 and 3.2:

**Lemma 4.** *The following statements hold:*

1. For each  $(F, E) \in SS_E(X)$ :  $(F, E) = (Y_{A_{(F,E)}}, E)$ .
2. For each  $S \subseteq X \times E$ :  $S = A_{(Y_S, E)}$ .

**Proof.** To prove the first claim, let us fix  $e \in E$ . Then,  $x \in F(e)$  if and only if  $(x, e) \in A_{(F,E)}$  if and only if  $x \in Y_{A_{(F,E)}}(e)$ . Equation (5) justifies the first equivalence, and (3) justifies the second.

To prove the second claim, observe  $(x, e) \in S$  if and only if  $x \in (Y_S, E)$  if and only if  $(x, e) \in A_{(Y_S,E)}$ . Equation (3) justifies the first equivalence, and (5) justifies the second.  $\square$

#### 4. Operational Characterization of Soft Topologies by Crisp Topologies

We are ready to state and prove our main result in this paper. Thus far, the progress of soft set theory has been severely handicapped by the lack of a universal connection with crisp topology. Theorem 1 below shows that this inconvenience is easily overcome with the assistance of crisp topologies on  $X \times E$ . For notation in the statement of the next result, recall the convention explained in Remark 2.

**Theorem 1.** *The mapping*

$$T : \{ \text{Crisp topologies on } X \times E \} \longrightarrow \{ \text{Soft topologies for } SS_E(X) \}$$

$$\Sigma \qquad \qquad \qquad \tau^\Sigma$$

is a bijection. Its inverse mapping is

$$T^{-1} : \{ \text{Soft topologies for } SS_E(X) \} \longrightarrow \{ \text{Crisp topologies on } X \times E \}$$

$$\tau \qquad \qquad \qquad \Sigma(\tau)$$

**Proof.** The thesis follows from the next two facts:

- (a)  $(T \circ T^{-1})(\tau) = \tau$  for each soft topology  $\tau$  for  $SS_E(X)$ , and
- (b)  $(T^{-1} \circ T)(\Sigma) = \Sigma$  for each crisp topology  $\Sigma$  on  $X \times E$ .

Notice that (a) means that  $T \circ T^{-1}$  is the identity function on the set of soft topologies for  $SS_E(X)$ , whereas (b) means that  $T^{-1} \circ T$  is the identity function on the set of crisp topologies on  $X \times E$ . A well-known characterization of bijections in essential set theory ([2] §2, Exercise 5.(e)) ensures that these two properties prove our thesis.

To demonstrate that (a) and (b) are true, observe that we can rewrite both statements in the following convenient form:

$$\tau^{\Sigma(\tau)} = \tau \text{ for each soft topology } \tau \text{ for } SS_E(X), \text{ and} \tag{8}$$

$$\Sigma(\tau^\Sigma) = \Sigma \text{ for each crisp topology } \Sigma \text{ on } X \times E. \tag{9}$$

Then, (a) is equivalent to (8), and (b) is equivalent to (9).

We are ready to prove (8), the equivalent form of a). We argue by double inclusion.

Let us first fix  $(F, E) \in \tau$ . If we define  $G = A_{(F,E)}$ , then  $G \in \Sigma(\tau)$  by (6), and  $(F, E) = (Y_G, E)$  by statement 1 in Lemma 4. We appeal to (4) to ensure  $(F, E) \in \tau^{\Sigma(\tau)}$ .

For the converse inclusion, we select an arbitrary  $(F, E) \in \tau^{\Sigma(\tau)}$ . Then, there must be  $G \in \Sigma(\tau)$  such that  $(F, E) = (Y_G, E)$ , by (4). Because  $G \in \Sigma(\tau)$ , using (6), we can claim that there is  $(F', E) \in \tau$  such that  $G = A_{(F',E)}$ . This, in turn, entails  $(F, E) = (Y_G, E) = (Y_{A_{(F',E)}}, E) = (F', E)$ , by virtue of statement 1 in Lemma 4. We conclude  $(F, E) = (F', E) \in \tau$ .

In a like manner, we argue by double inclusion, in order to prove (9), the equivalent form of (b).

Let us first fix  $G \in \Sigma$ . Then, (4) justifies  $(Y_G, E) \in \tau^\Sigma$ , which in turn guarantees  $A_{(Y_G,E)} \in \Sigma(\tau^\Sigma)$  by (6). We conclude because  $G = A_{(Y_G,E)}$  by virtue of statement 2 in Lemma 4.

Conversely, let us fix  $G \in \Sigma(\tau^\Sigma)$ . There must be  $(F, E) \in \tau^\Sigma$  such that  $G = A_{(F,E)}$  by (6). Because  $(F, E) \in \tau^\Sigma$ , using (4), we can claim that there is  $G' \in \Sigma$  such that



$(F, E) = (Y_{G'}, E)$ . We conclude the result because  $G = A_{(F,E)} = A_{(Y_{G'},E)} = G'$  by virtue of statement 2 in Lemma 4; thus,  $G = G' \in \Sigma$ .  $\square$

Figure 2 summarizes the elements that are referred to in Theorem 1, and their relationships.

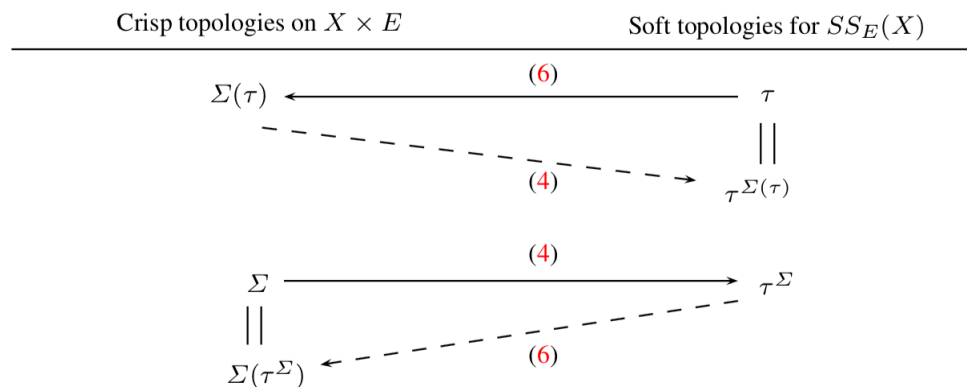


Figure 2. A graphical display of Theorem 1.

### 5. Application: Axioms and Properties of Soft Topological Spaces

This section proves that the theoretical implications of our construction in Section 4 are far-reaching. We prove results of various different genres:

1. Characterization of soft topological concepts (like soft bases) by their crisp counterparts.
2. Characterization of properties of soft topological spaces (like soft second-countability) by their crisp counterparts.
3. The achievement of new results by importing known theorems from crisp topology.

As to the first goal, Proposition 4 and Corollary 1 below show that concepts in soft topology can be characterized by related concepts in crisp topology.

**Proposition 4.** *Let us fix  $\Sigma$ , a crisp topology on  $X \times E$ . Then, the following statements are equivalent:*

1.  $\{B_i\}_{i \in I}$  is a base of  $\Sigma$ .
2.  $\{(Y_{B_i}, E)\}_{i \in I}$  is a soft base for the soft topology  $\tau^\Sigma$ .

**Proof.** Suppose that  $\{B_i\}_{i \in I}$  is a base of  $\Sigma$ , and select  $(F, E) \in \tau^\Sigma$ . There is  $G \in \Sigma$  such that  $(F, E) = (Y_G, E)$  by (4). There exists  $J \subseteq I$  with  $G = \cup_{j \in J} B_j$  because  $\{B_i\}_{i \in I}$  is a base of  $\Sigma$ . Lemma 2 assures that  $(Y_G, E) = \sqcup_{j \in J} (Y_{B_j}, E)$ ; therefore,  $(F, E) = \sqcup_{j \in J} (Y_{B_j}, E)$ . This proves that the desired conclusion that  $\{(Y_{B_i}, E)\}_{i \in I}$  is a soft base for  $\tau^\Sigma$ .

Now, suppose that  $\{(Y_{B_i}, E)\}_{i \in I}$  is a soft base for  $\tau^\Sigma$ , and select  $G \in \Sigma$ . We know  $G \in \Sigma(\tau^\Sigma)$  by (9). Thus, there must exist  $(F, E) \in \tau^\Sigma$  such that  $G = A_{(F,E)}$  by (6). The definition of a soft base entails the existence of  $J \subseteq I$  such that  $(F, E) = \sqcup_{j \in J} (Y_{B_j}, E)$ . Lemma 3 assures that  $G = A_{(F,E)} = \cup_{j \in J} A_{(Y_{B_j}, E)}$ . By statement 2 in Lemma 4,  $B_j = A_{(Y_{B_j}, E)}$  for each  $j \in J$ ; therefore,  $G = \cup_{j \in J} B_j$ . This proves the desired conclusion that  $\{B_i\}_{i \in I}$  is a base of  $\Sigma$ .  $\square$

**Corollary 1.** *Let us fix  $\tau$ , a soft topology for  $SS_E(X)$ . The following statements are equivalent:*

1.  $\{(F_i, E)\}_{i \in I}$  is a soft base for  $\tau$ .
2.  $\{A_{(F_i,E)}\}_{i \in I}$  is a base of the crisp topology  $\Sigma(\tau)$ .

**Proof.** Proposition 4 justifies that  $\{A_{(F_i,E)}\}_{i \in I}$  is a base of  $\Sigma(\tau)$  if and only if  $\{(Y_{A_{(F_i,E)}}, E)\}_{i \in I}$  is a soft base for the soft topology  $\tau^{\Sigma(\tau)}$ . This soft topology is  $\tau$  by Theorem 1 (alternatively, see Figure 2).

The conclusion follows from statement 1 in Lemma 4.  $\square$

The combination of Proposition 4 and Corollary 1 readily proves the next remarkable consequence. Corollary 2 shows that not only definitions, but also properties can be transferred from soft topologies to crisp topologies (or the other way round) as claimed by our second goal:

**Corollary 2.** *Let us fix  $\tau$ , a soft topology for  $SS_E(X)$ . Then,  $\tau$  is soft second-countable if and only if  $\Sigma(\tau)$  is second-countable.*

Another definition that we shall take advantage of is the idea of ‘singletons’. The next two characterizations hold true by routine verifications:

**Lemma 5.** *The following statements are equivalent:*

1.  $(F, E) \in SS_E(X)$  is the soft point  $(\{y\}_a, E)$ , i.e.,  $(F, E) = (\{y\}_a, E)$ .
2.  $A_{(F,E)}$  is the singleton  $\{(y, a)\} \subseteq X \times E$ .

**Lemma 6.** *The following statements are equivalent:*

1.  $A \subseteq X \times E$  is the singleton  $\{(y, a)\}$ , i.e.,  $A = \{(y, a)\}$ .
2.  $(Y_A, E)$  is the soft point  $(\{y\}_a, E)$ .

We now proceed to prove a result that is interesting in its own right, and will help us prove another implication in soft topology. In order to state it, we recall the following concept.

**Definition 5 ([17]).** *The soft topology  $\tau$  for  $SS_E(X)$  is soft-points countably-dense (abbreviated as SPCD) when a countable set  $\mathcal{F}$  of soft points exists which is soft  $\tau$ -dense in  $X$ . In addition,  $\mathcal{F} \subseteq \mathcal{P}(SS_E(X))$  is soft  $\tau$ -dense in  $X$  when it satisfies: if  $(F, E) \in \tau$ ,  $\Phi \neq (F, E)$ , then there is  $(G, E) \in \mathcal{F}$  for which  $\Phi \neq (F, E) \sqcap (G, E)$ .*

A weaker extension of the spirit of topological separability to the field of soft topology was defined in [17] under the name countable soft-set density. There are technical arguments that qualify these properties as good candidates for separability-like axioms [17]. At any rate, we can now prove:

**Proposition 5.** *Let us fix  $\Sigma$ , a crisp topology on  $X \times E$ . If  $\Sigma$  is separable, then the soft topology  $\tau^\Sigma$  is SPCD.*

**Proof.** Suppose that  $\{(x_i, e_i)\}_{i \in I}$  is a dense subset for  $\Sigma$ , and  $I$  is at most countable. Define the singletons  $P_i = \{(x_i, e_i)\}$  for each  $i \in I$ . We claim that  $\{(Y_{P_i}, E)\}_{i \in I}$  is soft  $\tau^\Sigma$ -dense. Observe that, indeed, each  $(Y_{P_i}, E)$  is a soft point by Lemma 6.

Select an arbitrary  $(F, E) \in \tau^\Sigma$ ,  $(F, E) \neq \Phi$ . There is  $G \in \Sigma$  such that  $(F, E) = (Y_G, E)$  by (4), and necessarily  $G \neq \emptyset$ . Since  $\{(x_i, e_i)\}_{i \in I}$  is dense for  $\Sigma$ , there exists  $i \in I$  with  $(x_i, e_i) \in G$  or  $P_i \cap G \neq \emptyset$ . Lemma 1 assures that  $(Y_{P_i}, E) \sqcap (Y_G, E) \neq \Phi$ . This proves the desired conclusion that  $\{(Y_{P_i}, E)\}_{i \in I}$  is soft  $\tau^\Sigma$ -dense.  $\square$

Finally, Proposition 5 allows us to embed a result from [17] into our new theory. Therefore, the next achievement supports our third goal in this section:

**Proposition 6 ([17] (Proposition 1)).** *Let us fix  $\tau$ , a soft topology for  $SS_E(X)$ . If  $\tau$  is S2C, then it is SPCD.*

**Proof.** Corollary 2 assures that  $\Sigma(\tau)$  is a second-countable crisp topology; therefore, it is separable. Now, Proposition 5 entails that  $\tau^{\Sigma(\tau)}$  is SPCD. Finally, we know  $\tau^{\Sigma(\tau)} = \tau$  by virtue of (8).  $\square$

## 6. Conclusions

Soft topologies can be interpreted as standard or crisp topologies on a set. It is in fact simple to move from one setting to the other and Section 4 explains this two-way transition. Corollary 1 and Lemmas 5 and 6 show that concepts can be transferred in an explicit and natural manner. Corollary 2 illustrates how we can characterize properties of one of these structures by checking an associated property of the other. Proposition 6 demonstrates that the combination of the techniques developed here with properties in crisp topology is capable of producing results about soft topology.

Therefore, a promising line of research consists of the characterization of other properties of soft topological spaces by their counterparts in crisp topology. Our roadmap for research also comprises the import of theorems in topology to soft topology.

It may furthermore be feasible to produce characterizations similar to that in Section 4 in the setting of (hesitant) fuzzy soft topologies or  $N$ -soft topologies.

Finally, since rough set theory considers binary relations and topology, it may now be feasible to relate that theory to crisp topology and soft topology as well.

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