mathematics

MDPI

Article

Some Properties of Euler's Function and of the Function τ and Their Generalizations in Algebraic Number Fields

Nicuşor Minculete * and Diana Savin

Faculty of Mathematics and Computer Science, Transilvania University, Iuliu Maniu Street 50, 500091 Braşov, Romania; diana.savin@unitbv.ro or dianet72@yahoo.com

* Correspondence: minculete.nicusor@unitbv.ro

Abstract: In this paper, we find some inequalities which involve Euler's function, extended Euler's function, the function τ, and the generalized function τ in algebraic number fields.

Keywords: arithmetic functions; algebraic number fields; Dedekind rings

MSC: primary: 11A25; 11R04; secondary: 11Y70; 13F05

check for updates

Citation: Minculete, N.; Savin, D. Some Properties of Euler's Function and of the Function τ and Their Generalizations in Algebraic Number Fields. *Mathematics* **2021**, *9*, 1710. https://doi.org/10.3390/math9151710

Academic Editor: Adolfo Ballester-Bolinches

Received: 21 June 2021
Accepted: 19 July 2021
Published: 21 July 2021

1. Introduction and Preliminaries

Let the function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$, $\varphi(n) = |\{k \in \mathbb{N}^* | k \le n,\ (k,n) = 1\}|$, $(\forall)\ n \in \mathbb{N}^*$. φ is called *Euler's function* or *Euler's totient function*. We remark that $\varphi(n)$ is the number of invertible elements in the unitary ring $\mathbb{Z}/n\mathbb{Z}$. If $n \in \mathbb{N}, n \ge 2$, the following formula to calculate $\varphi(n)$ is known: if $l \in \mathbb{N}^*$, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_l^{\alpha_l}$, where $p_1, p_2 \ldots p_l \in \mathbb{N}$ are unique distinct prime numbers and $\alpha_i \in \mathbb{N}^*$, $i = \overline{1,l}$ then $\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_l}\right)$. A known property is that φ is a multiplicative function, but it is immediately noticed that φ is not a completely multiplicative function. An important number of monographs in number theory studied these types of functions [1–5].

Let the function $\tau : \mathbb{N}^* \to \mathbb{N}^*$, $\tau(n) = |\{k \in \mathbb{N}^* | k | n\}|$, $(\forall)\ n \in \mathbb{N}^*$. $\tau(1) = 1$. If $n \in \mathbb{N}, n \ge 2,, n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_l^{\alpha_l}$, where $p_1, p_2 \ldots p_l \in \mathbb{N}$ are unique distinct prime numbers and $\alpha_i \in \mathbb{N}^*$, then $\tau(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdot \ldots \cdot (\alpha_l + 1)$. τ is a multiplicative function, but τ is not a completely multiplicative function.

Many analytics properties of these functions can be found in [6–8]. In [9] Rassias introduced the function

$$\phi(n, A, B) = \sum_{A \le k \le B, (n,k)=1} 1,$$

as a generalized totient function. He proved that:

$$\phi(n, A, B) = \sum_{d|n} \mu(d) \cdot \left(\left[\frac{B}{d} \right] - \left[\frac{A}{d} \right] \right),$$

where μ is Möbius' function (see Lemma 5.22 from [9]) and

$$\phi(n, A, B) = \frac{B - A}{n} \cdot \varphi(n) + \delta_{n,A} + O\left(\sum_{d|n} \mu(d)^2 \right),$$

for each $n, A, B \in \mathbb{N}$, $n > 1$, where $\delta_{n,A} = 1$ if $(n, A) = 1$, and 0 otherwise (see Proposition 5.23 from [9]).

Let n be a positive integer, $n \ge 2$, and let K be an algebraic number field, with degree $[K : \mathbb{Q}] = n$. Let $\mathcal{O}_K$ be the ring of integers of the field K, and let Spec $(\mathcal{O}_K)$ be the set of the prime ideals of the ring $\mathcal{O}_K$. It is known that the ring of integers of an arbitrary algebraic number field K is a Dedekind domain. Let $\mathbb{J}$ be the set of ideals of the ring $\mathcal{O}_K$.

It is known that Euler's function was extended to the set $\mathbb{J}$ like this: let I be an ideal from the set $\mathbb{J}$. Taking into account that $\mathcal{O}_K$ is a Dedekind domain and the fact that in any Dedekind domain, there is a factorization theorem for ideals similar to the fundamental theorem of arithmetics in the set of integer numbers, $I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \ldots \cdot P_l^{\alpha_l}$, where P_1, P_2, $\ldots$, P_l are unique different prime ideals in the ring $\mathcal{O}_K$ and $\alpha_i \in \mathbb{N}^*$ $i = \overline{1,l}$, and then

$$\varphi_{ext} : \mathbb{J} \to \mathbb{N}^*,$$

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) \cdot \ldots \cdot \left(1 - \frac{1}{N(P_l)}\right),$$

where $N(I)$ is the norm of the ideal I. We recall the norm of an ideal I is defined as follows $N(I) = [\mathcal{O}_K : I]$. The following properties of the norm function are known:

Proposition 1.

$$N(I_1 \cdot I_2) = N(I_1) \cdot N(I_2),$$

for $(\forall)$ *nonzero ideals* I_1, I_2 *from the set* $\mathbb{J}$.

Proposition 2. *If* I *is an ideal from* $\mathbb{J}$ *with the property* $N(I)$ *as a prime number, then* $I \in Spec(\mathcal{O}_K)$.

Proposition 3. *If* $P \in Spec(\mathcal{O}_K)$ *and* p *is a prime positive integer such that the ideal* P *divides the ideal* $p\mathcal{O}_K$, *then* $N(P) = p^f$, *where* $f \in \mathbb{N}^*$ *is the residual degree of the ideal* P.

Proposition 4. *The norm function* $N : \mathbb{J} \to \mathbb{N}^*$ *is not injective.*

We recall that:

Proposition 5. *If* I_1 *and* I_2 *are nonzero ideals from* $\mathbb{J}$ *such that* $I_1 + I_2 = \mathcal{O}_K$, *then*

$$\varphi_{ext}(I_1 \cdot I_2) = \varphi_{ext}(I_1) \cdot \varphi_{ext}(I_2).$$

These results can be found in [6,10–17].

In the paper [18], the authors extended the function τ to the set $\mathbb{J}$ of the ideals of the ring $\mathcal{O}_K$. We denote this function with τ_{ext} to distinguish it from the function $\tau : \mathbb{N}^* \to \mathbb{N}^*$. Thus, $\tau_{ext} : \mathbb{J} \to \mathbb{N}^*$, $\tau_{ext}(I) =$ the number of ideals from $\mathbb{J}$, which divide the ideal I. Using the above notations, we have:

$$\tau_{ext}(I) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdot \ldots \cdot (\alpha_l + 1).$$

Quickly, we obtain that:

Proposition 6.

$$\tau_{ext}(I_1 \cdot I_2) = \tau_{ext}(I_1) \cdot \tau_{ext}(I_2),$$

for any I_1 *and* I_2 *which are nonzero ideals from* $\mathbb{J}$, *such that* $I_1 + I_2 = \mathcal{O}_K$.

In this article, we obtain certain inequalities involving the functions τ, τ_{ext}, φ, φ_{ext}.

2. Results

Popovici (in [19]) obtained the following inequality:

$$\varphi^2(a \cdot b) \leq \varphi\left(a^2\right) \cdot \varphi\left(b^2\right) \ (\forall)\ a, b \in \mathbb{N}^*,$$

where φ is Euler's function. In [20] (Proposition 3.4), Minculete and Savin proved a similar inequality, for extended Euler's function:

Proposition 7. *Let n be a positive integer, $n \geq 2$, and let K be an algebraic number field of degree $[K:\mathbb{Q}] = n$. Then:*

$$\varphi_{ext}^2(I \cdot J) \leq \varphi_{ext}\left(I^2\right) \cdot \varphi_{ext}\left(J^2\right),\ (\forall)\ ideals\ I\ and\ J\ of\ \mathcal{O}_K.$$

We ask ourselves if the functions τ and τ_{ext} satisfy a similar inequality. We obtain that these functions satisfy the opposite inequality.

Proposition 8. *Let K be an algebraic number field. Then:*

$$\tau_{ext}^2(I \cdot J) \geq \tau_{ext}\left(I^2\right) \cdot \tau_{ext}\left(J^2\right),\ (\forall)\ ideals\ I\ and\ J\ of\ \mathcal{O}_K.$$

Proof. Let I and J be two nonzero ideals in the ring $\mathcal{O}_K$. Applying the fundamental theorem of Dedekind rings, $(\exists!) l, r \in \mathbb{N}^*$, the different prime ideals $P_1, P_2, \ldots, P_l, P_1'', P_2'', \ldots, P_r''$ of the ring $\mathcal{O}_K$ and $\alpha_1, \alpha_2, \ldots \alpha_l, \gamma_1, \gamma_2, \ldots, \gamma_r \in \mathbb{N}^*$ such that $I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \ldots \cdot P_l^{\alpha_l} \cdot (P_1'')^{\gamma_1} \cdot \ldots \cdot (P_r'')^{\gamma_r}$ and $(\exists!) m \in \mathbb{N}^*$, the different prime ideals $P_1', P_2', \ldots, P_m'$ of the ring $\mathcal{O}_K$ and $\beta_1, \beta_2, \ldots, \beta_m, \gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_{2r} \in \mathbb{N}^*$ such that $J = (P_1')^{\beta_1} \cdot \ldots \cdot (P_m')^{\beta_m} \cdot (P_1'')^{\gamma_{r+1}} \cdot \ldots \cdot (P_r'')^{\gamma_{2r}}$. It results that

$$\tau_{ext}^2(I \cdot J) = (\alpha_1 + 1)^2 \cdot (\alpha_2 + 1)^2 \cdot \ldots \cdot (\alpha_l + 1)^2 \cdot$$

$$\cdot (\beta_1 + 1)^2 \cdot (\beta_2 + 1)^2 \cdot \ldots \cdot (\beta_m + 1)^2 \cdot (\gamma_1 + \gamma_{r+1} + 1)^2 \cdot (\gamma_2 + \gamma_{r+2} + 1)^2 \cdot \ldots \cdot (\gamma_r + \gamma_{2r} + 1)^2$$

and

$$\tau_{ext}\left(I^2\right) \cdot \tau_{ext}\left(J^2\right) = (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) \cdot \ldots \cdot (2\alpha_l + 1) \cdot (2\beta_1 + 1) \cdot (2\beta_2 + 1) \cdot \ldots \cdot (2\beta_m + 1) \cdot$$

$$\cdot (2\gamma_1 + 1) \cdot (2\gamma_2 + 1) \cdot \ldots \cdot (2\gamma_r + 1) \cdot (2\gamma_{r+1} + 1) \cdot (2\gamma_{r+2} + 1) \cdot \ldots \cdot (2\gamma_{2r} + 1).$$

It immediately follows that

$$(\alpha_i + 1)^2 \geq 2\alpha_i + 1,\ (\forall)\ i = \overline{1, l},$$

$$(\beta_i + 1)^2 \geq 2\beta_i + 1,\ (\forall)\ i = \overline{1, m}$$

and

$$(\gamma_i + \gamma_{r+i} + 1)^2 \geq (2\gamma_i + 1) \cdot (2\gamma_{r+i} + 1) \Leftrightarrow (\gamma_i - \gamma_{r+i})^2 \geq 0\ (\forall)\ i = \overline{1, r}.$$

Thus, we obtain that

$$\tau_{ext}^2(I \cdot J) \geq \tau_{ext}\left(I^2\right) \cdot \tau_{ext}\left(J^2\right),\ (\forall)\ ideals\ I\ and\ J\ of\ \mathcal{O}_K.$$

□

Sivaramakrishnan (in [21]) obtained the following inequality involving Euler's function and the function τ:

Proposition 9. *For any positive integer n, the following inequality is true*

$$\varphi(n) \cdot \tau(n) \geq n.$$

Now, we generalize Proposition 9, for an extended Euler's function and the function τ_{ext}.

Proposition 10. *Let K be an algebraic number field and let $\mathcal{O}_K$ be the ring integers of the field K. Then, the following inequality is true:*

$$\varphi_{ext}(I) \cdot \tau_{ext}(I) \geq N(I),\ (\forall)\ a\ nonzero\ ideal\ I\ of\ \mathcal{O}_K.$$

Proof. Let I be a nonzero ideal of the ring $\mathcal{O}_K$. According to the fundamental theorem of Dedekind rings, $(\exists!) l \in \mathbb{N}^*$, the different ideals $P_1, P_2, \ldots, P_l \in \mathrm{Spec}(\mathcal{O}_K)$ and $\alpha_1, \alpha_2, \ldots \alpha_l \in \mathbb{N}^*$ such that $I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \ldots \cdot P_l^{\alpha_l}$. Using the properties of the functions φ_{ext}, N and τ_{ext} which we specified in the introduction and preliminaries section, we have:

$$\varphi_{ext}(I) \cdot \tau_{ext}(I) = \prod_{i=1}^{l} \varphi_{ext}\left(P_i^{\alpha_i}\right) \cdot \tau_{ext}\left(P_i^{\alpha_i}\right) =$$

$$\prod_{i=1}^{l} (N(P_i))^{\alpha_i} \cdot \left(1 - \frac{1}{N(P_i)}\right) \cdot (\alpha_i + 1).$$

It results that

$$\varphi_{ext}(I) \cdot \tau_{ext}(I) = N(I) \cdot \prod_{i=1}^{l} \frac{N(P_i) - 1}{N(P_i)} \cdot (\alpha_i + 1). \tag{1}$$

It is easy to see that

$$\frac{N(P_i) - 1}{N(P_i)} \cdot (\alpha_i + 1) \geq \frac{N(P_i) - 1}{N(P_i)} \cdot 2 \geq 1,\ (\forall)\ \alpha_i \in \mathbb{N}^*,\ (\forall)\ P_i \in Spec(\mathcal{O}_K). \tag{2}$$

From (1) and (2), it results that

$$\varphi_{ext}(I) \cdot \tau_{ext}(I) \geq N(I),\ (\forall)\ a\ nonzero\ ideal\ I\ of\ \mathcal{O}_K.$$

□

We are giving another result involving Euler's function and the function τ.

Proposition 11. *For any positive integer n, the following inequality*

$$\frac{3\sqrt{15}}{2} \cdot \varphi(n) \geq \tau(n) \cdot \sqrt{n} \tag{3}$$

holds. The equality is obtained only for $n = 60$.

Proof. For $n = 1$, we have $\frac{3\sqrt{15}}{2}\varphi(1) = \frac{3\sqrt{15}}{2} > 1 = \tau(1)\sqrt{1}$. We take $n \geq 2$. By mathematical induction, we proved the inequality

$$\sqrt{p^d}\left(1 - \frac{1}{p}\right) \geq d + 1,$$

for every $d \geq 1$, where $p \geq 7$ is a prime number. This inequality is in fact the following:

$$\varphi(p^d) \geq \tau(p^d)\sqrt{p^d}. \tag{4}$$

We consider the decomposition in prime factors of n given by $n = 2^a 3^b 5^c \prod_{i=1}^{s} p_i^{a_i}$, $p_i \neq 2, 3, 5$. We know that if the functions φ and τ are multiplicative arithmetic functions, then the inequality of the statement becomes

$$\frac{3\sqrt{15}}{2}\varphi(2^a)\varphi(3^b)\varphi(5^c)\prod_{i=1}^{s}\varphi\left(p_i^{a_i}\right) \geq \tau(2^a)\sqrt{2^a}\tau(3^b)\sqrt{3^b}\tau(5^c)\sqrt{5^c}\prod_{i=1}^{s}\tau(p_i^{a_i})\sqrt{p_i^{a_i}}$$

$$= (a+1)(b+1)(c+1)\sqrt{2^a3^b5^c}\prod_{i=1}^{s}(a_i+1)\sqrt{p_i^{a_i}}.$$

It is easy to see, by mathematical induction, that for every $a, b, c \geq 1$, we have the following inequalities:

$$2^a \geq \frac{4}{9}(a+1)^2,\ 3^b \geq \frac{3}{4}(b+1)^2,\ 5^c \geq \frac{5}{4}(c+1)^2,$$

which are equivalent to

$$3\sqrt{2^a}\frac{1}{2} \geq a+1,\ \sqrt{3}\sqrt{3^b}\frac{2}{3} \geq b+1,\ \frac{\sqrt{5}}{2}\sqrt{5^c}\frac{4}{5} \geq c+1,$$

which means that

$$3\varphi(2^a) \geq \tau(2^a)\sqrt{2^a},\ \sqrt{3}\varphi(3^b) \geq \tau(3^b)\sqrt{3^b},\ \frac{\sqrt{5}}{2}\varphi(5^c) \geq \tau(5^c)\sqrt{5^c}.$$

Using the above inequalities and (4), we deduce the inequality of the statement. In the case when $c = 0$, we have $n = 2^a3^b\prod_{i=1}^{s}p_i^{a_i}$, so the inequality of the statement becomes

$$\frac{3\sqrt{15}}{2}\varphi(2^a)\varphi(3^b)\prod_{i=1}^{s}\varphi\left(p_i^{a_i}\right) = \frac{\sqrt{5}}{2}3\varphi(2^a)\sqrt{3}\varphi(3^b)\prod_{i=1}^{s}\varphi\left(p_i^{a_i}\right) \geq \frac{\sqrt{5}}{2}\tau(n)\sqrt{n} > \tau(n)\sqrt{n}.$$

Analogously, the cases are treated when at least one of the numbers a, b, c is equal to 0. Therefore, the inequality of the statement is true.

Now, we prove that the equality in (3) is obtained only for $n = 60$. For this, we study the equality

$$\frac{3\sqrt{15}}{2}\cdot\varphi(n) = \tau(n)\cdot\sqrt{n}. \tag{5}$$

If $n \not\equiv 0 \pmod{15}$, then $n = 15k + r$, where $k \in \mathbb{N}$, $r \in \{1, \ldots, 14\}$, which means that

$$\sqrt{\frac{15}{15k+r}} = \frac{2\tau(15k+r)}{3\varphi(15k+r)} \in \mathbb{Q},$$

which is false, because $\sqrt{\frac{15}{15k+r}} \notin \mathbb{Q}$. To prove this, we assume by absurdity that $\sqrt{\frac{15}{15k+r}} \in \mathbb{Q}$, so there are $a, b \in \mathbb{N}^*$, $(a, b) = 1$ such that $\sqrt{\frac{15}{15k+r}} = \frac{b}{a}$. This implies the equality

$$b^2\cdot(15k+r) = 15a^2.$$

Since $(15k + r, 15) \in \{1, 3, 5\}$, we obtain that $b \equiv 0 \pmod{15}$ when $(15k + r, 15) = 1$, $b \equiv 0 \pmod 5$ when $(15k + r, 15) = 3$, respectively, $b \equiv 0 \pmod 3$ when $(15k + r, 15) = 5$. In the first case, when $(15k + r, 15) = 1$, we find $b = 15b'$, where $b' \in \mathbb{N}$. Therefore, the above equality becomes

$$15\left(b'\right)^2\cdot(15k+r) = a^2.$$

It results that $a \equiv 0 \pmod{15}$, which is false because $(a, b) = 1$.

In the second case, when $(15k + r, 15) = 3$, we find $r = 3r'$ and $b = 5b'$, where $r \in \{3, 6, 9, 12\}$, $b' \in \mathbb{N}$. We obtain that

$$5\left(b'\right)^2\cdot\left(5k+r'\right) = a^2.$$

It results that $a \equiv 0 \pmod 5$, which is false because $(a, b) = 1$.

Analogously, we obtain a contradiction in the third case, when $(15k + r, 15) = 5$.

If $n \equiv 0 \pmod{15}$, then we have $n = 15k$, with $k \in \mathbb{N}^*$. Replacing it in equality (5), we obtain

$$2\sqrt{k} \cdot \tau(15k) = 3\varphi(15k),$$

which can be written as

$$\sqrt{k} = \frac{3\varphi(15k)}{2\tau(15k)} \in \mathbb{Q},$$

thus, there exists $q \in \mathbb{N}^*$ such that $k = q^2$. Replacing it in the above equality, we deduce the following relation

$$3\varphi\left(15q^2\right) = 2q\tau\left(15q^2\right). \tag{6}$$

If $(15, q) = 1$, then relation (6) becomes

$$3\varphi\left(q^2\right) = q\tau\left(q^2\right). \tag{7}$$

We study equality (7) in two cases:

Case I: when q is a prime number, we obtain

$$3q(q-1) = 3q,$$

it follows that $q = 2$, so $k = 4$. Therefore, we have $n = 60$.

Case II: when q is a compose number,

$\varphi(q^2)$ is an even number and $\tau(q^2)$ is an odd number. It follows from relation (7) that q is an even number, so $q = 2^s \cdot v$, where $s, v \in \mathbb{N}^*$, v is an odd number. Relation (7) becomes

$$3\varphi\left(2^{2s} \cdot v^2\right) = 2^s \cdot v \cdot \tau\left(2^{2s} \cdot v^2\right),$$

which implies, taking into account that $(2, v) = 1$, the following inequality holds:

$$3 \cdot 2^{s-1}\varphi\left(v^2\right) = v(2s+1) \cdot \tau\left(v^2\right). \tag{8}$$

For $s \geq 2$, the term from the left part of the equality (8) is an even number and the term $(2s+1) \cdot \tau(v^2)$ is an odd number, so v is an even number, which is false, because $(2, v) = 1$. The case $q = 2v$ then remains, where $v \geq 3$ is an odd number. Relation (7) becomes

$$\varphi\left(v^2\right) = v\tau\left(v^2\right),$$

but $\varphi(v^2)$ is an even number and $\tau(v^2)$ is an odd number; thus, we deduce that the number v is an even number, which is false.

If $(15, q) \neq 1$, then $q = 3^a 5^b$ or $q = 3^a 5^b \prod_{p \text{ prime } p \geq 7} p^c$, where $a, b \in \mathbb{N}$, with $a + b \geq 1$ and $c \in \mathbb{N}^*$. We note $P = \prod_{p \text{ prime, } p \geq 7} p^c$. For $q = 3^a 5^b$, relation (6) becomes

$$3\varphi\left(3^{2a+1}5^{2b+1}\right) = 2 \cdot 3^a 5^b \tau\left(3^{2a+1}5^{2b+1}\right),$$

which is equivalent to $3^{a+1}5^b = (a+1)(b+1)$, which is false, because $3^{a+1}5^b > (a+1)(b+1)$, and it is easy to see by mathematical induction for $a, b \in \mathbb{N}$, with $a + b \geq 1$. For $q = 3^a 5^b P$, relation (6) becomes

$$3\varphi\left(3^{2a+1}5^{2b+1}P^2\right) = 2 \cdot 3^a 5^b P\tau\left(3^{2a+1}5^{2b+1}P^2\right),$$

which is equivalent to $3^{a+1}5^b\varphi(P^2) = (a+1)(b+1)P\tau(P^2)$, so we obtain

$$3^{a+1}5^b \prod_{p \text{ prime, } p \geq 7} (p^c - 1) = (a+1)(b+1)\prod(2c+1).$$

However, by mathematical induction, we have $p^c - 1 > 2c + 1$, where $p \geq 7$ and $c \geq 1$. Combining the above inequalities, we prove that $3^{a+1} 5^b \prod_{p\ prime, p \geq 7} (p^c - 1) > (a+1)(b+1) \prod (2c+1)$. Consequently, the statement is true. □

Now, we generalize Proposition 11, for extended Euler's function and the function τ_{ext}.

Proposition 12. *Let K be an algebraic number field of degree* $[K : \mathbb{Q}] = n$*, where n is a positive integer,* $n \geq 2$*. Then:*

$$3^{r_1} \cdot \sqrt{3}^{r_2} \cdot \left(\frac{4}{3}\right)^{r_3} \cdot \left(\frac{\sqrt{5}}{2}\right)^{r_4} \cdot \varphi_{ext}(I) \geq \tau_{ext}(I) \cdot \sqrt{N(I)},\ (\forall)\ a\ nonzero\ ideal\ I\ of\ \mathcal{O}_K,$$

where r_1 *is the number of prime ideals of norm* 2*, which divides I;* r_2 *is the number of prime ideals of norm* 3*, which divides I;* r_3 *is the number of prime ideals of norm* 4*, which divides I; and* r_4 *is the number of prime ideals of norm* 5*, which divides I.*

Proof. Let I be a nonzero ideal of the ring $\mathcal{O}_K$. Applying the fundamental theorem of Dedekind rings, Propositions 3 and 4, it results that $(\exists!) r_1, r_2, r_3, r_4, l \in \mathbb{N}$, $l \geq 5$, the different prime ideals $P_{11},...,P_{1r_1}$, $P_{21},...,P_{2r_2}$, $P_{31},...,P_{3r_3}$, $P_{41},...,P_{4r_4}$, P_5, P_6,..., P_l of the ring $\mathcal{O}_K$ and $\alpha_{11}, ..., \alpha_{1r_1}, \alpha_{21}, ..., \alpha_{2r_2}, \alpha_{31}, ..., \alpha_{3r_3}, \alpha_{41}, ..., \alpha_{4r_4} \in \mathbb{N}$, $\alpha_5, \ldots \alpha_l \in \mathbb{N}^*$ such that

$$I = \prod_{i=1}^{r_1} P_{1i}^{\alpha_{1i}} \cdot \prod_{i=1}^{r_2} P_{2i}^{\alpha_{2i}} \cdot \prod_{i=1}^{r_3} P_{3i}^{\alpha_{3i}} \cdot \prod_{i=1}^{r_4} P_{4i}^{\alpha_{4i}} \cdot P_5^{\alpha_5} \cdot \ldots \cdot P_l^{\alpha_l},$$

with $N(P_{1i}) = 2$, $i = \overline{1, r_1}$, $N(P_{2i}) = 3$, $i = \overline{1, r_2}$, $N(P_{3i}) = 4$, $i = \overline{1, r_3}$, $N(P_{4i}) = 5$, $i = \overline{1, r_4}$ and $N(P_i) \geq 7$, $(\forall)\ i = \overline{5, l}$.

Applying the inequality $\sqrt{a^d} \cdot \left(1 - \frac{1}{a}\right) \geq d + 1$, $(\forall)\ d, a \in \mathbb{N}^*$, $a \geq 7$ for $a = N(P_i)$ we obtain:

$$\sqrt{(N(P_i))^d} \cdot \left(1 - \frac{1}{N(P_i)}\right) \geq d + 1,\ (\forall)\ d \in \mathbb{N}^*,\ (\forall)\ P_i \in \mathbb{J},\ N(P_i) \geq 7.$$

The last inequality is equivalent with

$$\sqrt{(N(P_i))^{\alpha_i}} \cdot \left(1 - \frac{1}{N(P_i)}\right) \geq \alpha_i + 1,\ (\forall)\ P_i | I,\ (\forall)\ i = \overline{5, l}.$$

It results that

$$\prod_{i=5}^{l} \sqrt{(N(P_i))^{\alpha_i}} \cdot \left(1 - \frac{1}{N(P_i)}\right) \geq \prod_{i=5}^{l} \tau_{ext}\left(P_i^{\alpha_i}\right). \tag{9}$$

Applying the inequality $3\sqrt{2^d} \cdot \left(1 - \frac{1}{2}\right) \geq d + 1$, $(\forall)\ d \in \mathbb{N}^*$, for $N(P_{1i}) = 2$ and for $d = \alpha_{1i}$, $(\forall)\ i = \overline{1, r_1}$, we obtain:

$$3\sqrt{(N(P_{1i}))^{\alpha_{1i}}} \cdot \left(1 - \frac{1}{N(P_{1i})}\right) \geq \alpha_{1i} + 1,\ (\forall)\ i = \overline{1, r_1}.$$

From this last inequality, it results that

$$3^{r_1} \cdot \prod_{i=1}^{r_1} \sqrt{(N(P_{1i}))^{\alpha_{1i}}} \cdot \left(1 - \frac{1}{N(P_{1i})}\right) \geq \prod_{i=1}^{r_1} \tau_{ext}\left(P_{1i}^{\alpha_{1i}}\right). \tag{10}$$

Applying the inequality $\sqrt{3} \cdot \sqrt{3^d} \cdot \left(1 - \frac{1}{3}\right) \geq d + 1$, $(\forall)\ d \in \mathbb{N}^*$, for $N(P_{2i}) = 3$ and for $d = \alpha_{2i}$, $(\forall)\ i = \overline{1, r_2}$, we obtain:

$$\sqrt{3}\sqrt{(N(P_{2i}))^{\alpha_{2i}}} \cdot \left(1 - \frac{1}{N(P_{2i})}\right) \geq \alpha_{2i} + 1,\ (\forall)\ i = \overline{1, r_2}.$$

From this last inequality, it results that

$$\sqrt{3}^{r_2} \cdot \prod_{i=1}^{r_2} \sqrt{(N(P_{2i}))^{\alpha_{2i}}} \cdot \left(1 - \frac{1}{N(P_{2i})}\right) \geq \prod_{i=1}^{r_2} \tau_{ext}\left(P_{2i}^{\alpha_{2i}}\right). \tag{11}$$

Applying the inequality $\frac{4}{3} \cdot \sqrt{4^d} \cdot \left(1 - \frac{1}{4}\right) \geq d + 1$, $(\forall)\ d \in \mathbb{N}^*$, for $N(P_{3i}) = 4$ and for $d = \alpha_{3i}$, $(\forall)\ i = \overline{1, r_3}$, we obtain:

$$\frac{4}{3}\sqrt{(N(P_{3i}))^{\alpha_{3i}}} \cdot \left(1 - \frac{1}{N(P_{3i})}\right) \geq \alpha_{3i} + 1,\ (\forall)\ i = \overline{1, r_3}.$$

From this last inequality, it results that

$$\left(\frac{4}{3}\right)^{r_3} \cdot \prod_{i=1}^{r_3} \sqrt{(N(P_{3i}))^{\alpha_{3i}}} \cdot \left(1 - \frac{1}{N(P_{3i})}\right) \geq \prod_{i=1}^{r_3} \tau_{ext}\left(P_{3i}^{\alpha_{3i}}\right). \tag{12}$$

Applying the inequality $\frac{\sqrt{5}}{2} \cdot \sqrt{5^d} \cdot \left(1 - \frac{1}{5}\right) \geq d + 1$, $(\forall)\ d \in \mathbb{N}^*$, for $N(P_{4i}) = 5$ and for $d = \alpha_{4i}$, $(\forall)\ i = \overline{1, r_4}$, we obtain:

$$\frac{\sqrt{5}}{2}\sqrt{(N(P_{4i}))^{\alpha_{4i}}} \cdot \left(1 - \frac{1}{N(P_{4i})}\right) \geq \alpha_{4i} + 1,\ (\forall)\ i = \overline{1, r_4}.$$

From this last inequality, it results that

$$\left(\frac{\sqrt{5}}{2}\right)^{r_4} \cdot \prod_{i=1}^{r_4} \sqrt{(N(P_{4i}))^{\alpha_{4i}}} \cdot \left(1 - \frac{1}{N(P_{4i})}\right) \geq \prod_{i=1}^{r_4} \tau_{ext}\left(P_{4i}^{\alpha_{4i}}\right). \tag{13}$$

Multiplying member-by-member inequalities (9)–(13) and applying Propositions 5 and 6, we obtain that

$$3^{r_1} \cdot \sqrt{3}^{r_2} \cdot \left(\frac{4}{3}\right)^{r_3} \cdot \left(\frac{\sqrt{5}}{2}\right)^{r_4} \cdot \varphi_{ext}(I) \geq \tau_{ext}(I) \cdot \sqrt{N(I)},$$

$(\forall)$ a nonzero ideal I of the ring $\mathcal{O}_K$.

□

3. Conclusions

Regarding the Number Theory, many papers studied the properties of the Euler totient function and the function that characterizes the number of divisors of a natural number. In this paper, we have presented some arithmetic inequalities that can be extended to inequalities in the algebraic fields theory. If K is an algebraic number field, then we deduce:

$$\tau_{ext}^2(I \cdot J) \geq \tau_{ext}\left(I^2\right) \cdot \tau_{ext}\left(J^2\right),\ (\forall)\ \textit{ideals } I \textit{ and } J \textit{ of } \mathcal{O}_K.$$

For any positive integer n, the following inequality $\varphi(n) \cdot \tau(n) \geq n$ holds. This inequality has been extended to an algebraic number field K:

$$\varphi_{ext}(I) \cdot \tau_{ext}(I) \geq N(I),\ (\forall)\ \textit{a nonzero ideal } I \textit{ of } \mathcal{O}_K,$$

where $\mathcal{O}_K$ is the ring integers of the field K. Another interesting arithmetic inequality is proven, namely:

$$\frac{3\sqrt{15}}{2} \cdot \varphi(n) \geq \tau(n) \cdot \sqrt{n},$$

for any positive integer n. This generates the following inequality in an algebraic number field K with the degree $[K : \mathbb{Q}] = n, n \geq 2$:

$$3^{r_1} \cdot \sqrt{3}^{r_2} \cdot \left(\frac{4}{3}\right)^{r_3} \cdot \left(\frac{\sqrt{5}}{2}\right)^{r_4} \cdot \varphi_{ext}(I) \geq \tau_{ext}(I) \cdot \sqrt{N(I)},$$

for all nonzero ideal I of $\mathcal{O}_K$, where r_1 is the number of prime ideals of norm 2, which divides I, r_2 is the number of prime ideals of norm 3, which divides I, r_3 is the number of prime ideals of norm 4, which divides I, and r_4 is the number of prime ideals of norm 5, which divides I.

In future research, we will search for other arithmetic inequalities that can extend to an algebraic field. We can see how some calculations are transferred from the elementary number theory to algebraic fields theory. It should be mentioned that these calculations cannot always be done by analogy.

Author Contributions: Conceptualization, N.M. and D.S.; methodology, N.M. and D.S.; validation, N.M. and D.S.; formal analysis, N.M. and D.S.; investigation, N.M. and D.S.; resources, N.M. and D.S.; writing—original draft preparation, N.M. and D.S.; writing—review and editing, N.M. and D.S.; visualization, N.M. and D.S.; supervision, N.M. and D.S. All authors have read and agreed to the published version of the manuscript.

Funding: Both authors acknowledges the financial support from Transilvania University of Braşov.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors want to thank the anonymous reviewers and editor for their careful reading of the manuscript and for many valuable remarks and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Andreescu, T.; Andrica, D. *Number Theory (Structures, Examples, and Problems)*; Birkhauser: Boston, MA, USA, 2009.
2. Lemmermeyer, F. *Reciprocity Laws, from Euler to Eisenstein*; Springer: Heidelberg, Germany, 2000.
3. Nathanson, M. *Elementary Methods in Number Theory*; Springer: New York, NY, USA, 2006.
4. Sierpinski, W. *Elementary Theory of Numbers*; Panstwowe Wydawnictwo Naukowe: Warsaw, Poland, 1964.
5. Subbarao, M.V. *On Some Arithmetic Convolutions in: The Theory of Arithmetic Functions*; Lecture Notes in Mathematics; Springer: New York, NY, USA, 1972.
6. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA, 1976.
7. Iwaniek, H.; Kowalski, E. *Analytic Number Theory*; American Mathematical Society, Colloquium Publications: Providence, RI, USA, 2004
8. Postnikov, A.G. *Introduction to Analytic Number Theory*; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1988.
9. Rassias, M. From a Cotangent Sum to a Generalized Totient Function. *Appl. Anal. Discret. Math.* **2017**, *11*, 369–385. [CrossRef]
10. Albu, T.; Ion, I.D. *Chapters of the Algebraic Number Theory*; Academy Publishing House: Bucharest, Romania, 1984. (In Romanian)
11. Alexandru, V.; Goşoniu, N.M. *Elements of Number Theory*; Bucharest University Publishing House: Bucharest, Romania, 1999. (In Romanian)
12. Ireland, K.; Rosen, M. *A Classical Introduction to Modern Number Theory*; Springe: New York, NY, USA, 1992.
13. Panaitopol, L.; Gica, A. *An Introduction to Arithmetic and Number Theory*; Bucharest University Publishing House: Bucharest, Romania, 2001. (In Romanian)
14. Ribenboim, P. *My Numbers, My Friends (Popular Lectures on Number Theory)*; Springer: New York, NY, USA, 2000.
15. Ribenboim, P. *Classical Theory of Algebraic Numbers*; Springer: New York, NY, USA, 2001.

16. Sándor, J.; Crstici, B. *Handbook of Number Theory II*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2004.
17. Savin, D.; Ştefănescu, M. *Lessons of Arithmetics and Number Theory*; Matrix Rom Publishing House: Bucharest, Romania, 2008. (In Romanian)
18. Minculete, N.; Savin, D. Some Generalizations of the Functions τ and $\tau^{(e)}$ in Algebraic Number Fields. *Expo. Math.* **2020**, *39*. Available online: https://www.sciencedirect.com/science/article/abs/pii/S0723086920300347 (accessed on 20 July 2021). [CrossRef]
19. Popovici, C.P. *Number Theory*; Didactic and Pedagogical Publishing House: Bucharest, Romania, 1973. (In Romanian)
20. Minculete, N.; Savin, D. Some Properties of Extended Euler's Function and Extended Dedekind's Function. *Mathematics* **2020**, *8*, 1222. [CrossRef]
21. Sivaramakrishnan, R. *Classical Theory of Arithmetic Functions, Monographs* & *Textbooks in Pure and Applied Mathematics No. 126*; M. Dekker: New York, NY, USA, 1989.