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Article

On the Ternary Exponential Diophantine Equation Equating a Perfect Power and Sum of Products of Consecutive Integers

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Abstract: Consider the Diophantine equation $y^n = x + x(x+1) + \cdots + x(x+1)\cdots(x+k)$, where x, y, n, and k are integers. In 2016, a research article, entitled – 'power values of sums of products of consecutive integers', primarily proved the inequality $n = 19{,}736$ to obtain all solutions (x, y, n) of the equation for the fixed positive integers $k \leq 10$. In this paper, we improve the bound as $n \leq 10{,}000$ for the same case $k \leq 10$, and for any fixed general positive integer k, we give an upper bound depending only on k for n.

Keywords: Diophantine equation; Ternary Diophantine equation

MSC: 11D61; 11D45

1. Introduction

In 1976, Tijdeman proved that all integral solutions (x, y, n), $n > 0$ and $|y| > 1$, of the equation

$$y^n = f(x)$$

satisfy $n < c_0$, where c_0 is an effectively computable constant depending only on f if $f(x)$ is an integer polynomial with at least two distinct roots (Shorey-Tijdeman [1], Tijdeman [2], Waldschmidt [3]). In 1987, Brindza in [4] obtained the unconditional form of the result for $f(x) = f_1^{k_1}(x) + f_2^{k_2}(x) + \cdots + f_s^{k_s}(x)$, where $f_1, f_2, \ldots, f_s$ are integer polynomials and $k_1, k_2, \ldots, k_s$ are positive integers such that $\min\{k_i : 1 \leq i \leq s\} > s(s-1)$. In 2016, Hajdu, Laishram, and Tengely in [5] proved the above result for $f(x) = x + x(x+1) + \cdots + x(x+1)\cdots(x+k)$. In 2018, Subburam [6] assured that, for each positive, real $\epsilon < 1$, there exists an effectively computable constant $c(\epsilon)$ such that

$$\max\{x, y, n\} \leq c(\epsilon)(\log \max\{a, b, c\})^{2+\epsilon},$$

where (x, y, n) is a positive integral solution of the ternary exponential Diophantine equation

$$a^n = b^x + c^y$$

and a, b, and c are fixed positive integers with $\gcd(a, b, c) = 1$. In 2019, Subburam [7] provided the unconditional form of the first result for $f(x) = (x + a_1)^{r_1} + (x + a_2)^{r_2} + \cdots + (x + a_m)^{r_m}$, where $m \geq 2$; $a_1, a_2, \ldots, a_m$; $r = r_1, r_2, \ldots, r_m$ are integers such that $r_1 \geq r_2 \geq \cdots \geq r_m > 0$; $\gcd(\eta, \mathrm{cont}(f(x))) = 1$; $\eta^{1/r}$ is not an integer > 1; $r_2 < r_1 - 1$ when $r_2 < r_1$; $\eta = |\{r_i : r_1 = r_i\}|$; and $\mathrm{cont}(f(x))$ is the content of $f(x)$. For further results related to this paper, see Bazsó [8]; Bazsó, Berczes, Hajdu, and Luca [9]; and Tengely and Ulas [10].

In this paper, we consider the Diophantine equation

$$y^n = x + x(x+1) + \cdots + x(x+1)\cdots(x+k) =: f_k(x) \tag{1}$$

in integral variables x, y, and n, with $n > 0$, where k is a fixed positive integer. In Theorem 2.1 of [5], Hajdu, Laishram, and Tengely proved that there exists an effectively computable constant $c(k)$ depending only on k such that (x, y, n) satisfy

$$n \leq c(k)$$

if $y \neq 0, -1$. For the case $1 \leq k \leq 10$, they explicitly calculated $c(k)$ as

$$n \leq 19{,}736.$$

Here, we prove the following theorem. For any positive integers s, p_1, p_2, $\ldots$, p_m, we denote

$$\lambda_s(p_1, \ldots, p_m) = \sum_{\substack{i_1, \ldots, i_s \\ 1 \leq i_1 < \cdots < i_s \leq m}} p_{i_1} p_{i_2} \cdots p_{i_s}$$

and $\lambda_0(p_1, \ldots, p_m) = 1$. This elementary symmetric polynomial and its upper bound have been studied in Subburam [11].

Theorem 1. *Let k be any positive integer and*

$$\mathfrak{b} = 4\left|\sum_{i=0}^{k-1} (-1)^i A_{k-i-1} 2^i\right|,$$

where $A_0 = 1$, $A_1 = 1 + \alpha_1$, $A_{k-1} = 1 + \alpha_{k-2}$, and $A_j = \alpha_{j-1} + \alpha_j$ for $j = 2, 3, \ldots, k-2$ and where

$$\alpha_m = 1 + \sum_{i=0}^{m-1} \lambda_{i+1}(3, \ldots, k+i-m+1)$$

for $m = 1, 2, \ldots, k-2$. Then, all integral solutions (x, y, n), with $y \neq 0, -1, x \neq 1, n \geq 1$, of (1) satisfy

$$n \leq c_2 \log \mathfrak{b},$$

where c_2 can be bounded using the linear form of the logarithmic method in Laurent, Mignotte, and Nesterenko [12], and an immediate estimation is

$$c_2 = \begin{cases} 21{,}468 & \text{if } 21 > \log n \\ 26{,}561(\log\log \mathfrak{b})^2 & \text{if } 21 \leq \log n. \end{cases}$$

If

$$\mathfrak{b} \leq 4 \times 9 \times 11 \times 467 \times 2{,}018{,}957,$$

then all integral solutions (x, y, n), *with* $y \neq 0, -1, x \neq 1, n \geq 1$, *of* (1) *satisfy*

$$n \leq \begin{cases} \max\{1000, 824.338 \log \mathfrak{b} + 0.258\} & \text{if } \mathfrak{b} \leq 100 \\ \max\{2000, 769.218 \log \mathfrak{b} + 0.258\} & \text{if } 100 < \mathfrak{b} \leq 10{,}000 \\ \max\{10{,}000, 740.683 \log \mathfrak{b} + 0.234\} & \text{if } \mathfrak{b} > 10{,}000. \end{cases}$$

The result of Hajdu, Laishram, and Tengely in [5] is much stronger than the following corollary. They explicitly obtained all solutions for the values $k \leq 10$ using the MAGMA computer program along with two well-known methods (See Subburam [6], Srikanth and Subburam [13], and Subburam and Togbe [14]), after proving that $n \leq 19{,}736$ for $1 \leq k \leq 10$. Here, we have

Corollary 1. *If* $1 \leq k \leq 10$, *then* $n \leq 10{,}000$.

Hajdu, Laishram, and Tengely studied each of the cases "(n, k) where $n = 2$ and k is odd with $1 \leq k \leq 10$" in the proof of Theorem 2.2 of [5]. Here, we prove the following theorem for any odd k. This can be written as a suitable computer program by considering each step of the following theorem as a sub-program that can be separately and directly run.

Theorem 2. *Let k be odd. Then, we have the following:*

(i) *There uniquely exist rational polynomials* $B(x)$ *and* $C(x)$ *with* $\deg(C(x)) \leq \frac{k-1}{2}$ *such that*

$$f_k(x) = B^2(x) + C(x).$$

(ii) *Let l be the least positive integer such that* $lB(x)$ *and* $l^2C(x)$ *have integer coefficients for any nonnegative integer i and* $\delta \in \{1, -1\}$

$$P_{i,\delta}(x) = \delta(lB(x) + \delta i)^2 - \delta(lB(x))^2 - \delta l^2 C(x),$$

r is any positive integer,

$$H_1 = \{\alpha \in \mathbb{Z} : P_{i,\delta}(\alpha) = 0, \delta \in \{1, -1\}, i = 0, 1, 2, \ldots, r-1\},$$

and

$$H_2 = \{\alpha \in \mathbb{R} : P_{r,1}(\alpha) = 0 \text{ or } P_{r,-1}(\alpha) = 0\},$$

where $\mathbb{R}$ *and* $\mathbb{Z}$ *are the sets of all real numbers and integers, respectively. If* H_1 *and* H_2 *are empty, then* (1) *has no integral solution* $(x, y, 2)$. *Otherwise, all integral solutions* $(x, y, 2)$ *of* (1) *satisfy* $x \in H_1$ *or*

$$\min H_2 \leq x \leq \max H_2.$$

2. Proofs

Lemma 1. *Let* $k \geq 3$. *Then, all integral solutions* (x, y, n), $n > 0$ *and* $y \neq 0$, *of* (1) *satisfy the equation*

$$a_2 b_1 y_2^n - b_2 a_1 y_1^n = 2 b_1 a_1,$$

where a_1, a_2, b_1, *and* b_2 *are positive integers such that*

$$a_1 a_2 b_1 b_2 \mid 4 \sum_{i=0}^{k-1} (-1)^i A_{k-i-1} 2^i,$$

A_i *is the coefficient of* x^{k-i-1} *in the polynomial* $f_k(x)/x(x+2)$,

$$x = \left(\frac{b_2}{b_1}\right) y_1^n \text{ , and } x + 2 = \left(\frac{a_2}{a_1}\right) y_2^n$$

for some nonzero integers y_1 and y_2.

Proof. Let $k \geq 3$. Let (x, y, n), with $n > 0$ and $y \neq 0$, be any integral solution of the Diophantine equation

$$y^n = x + x(x+1) + \cdots + x(x+1)\cdots(x+k).$$

This can be written as

$$y^n = x(x+2)g_k(x)$$

for some integer polynomial $g_k(x)$, which is not divided by x and $x+2$, since $k \geq 3$. Let d and q be positive integers such that

$$\gcd(x, (x+2)g_k(x)) = d \text{ and } \gcd((x+2), xg_k(x)) = q.$$

Let d_1, d_2, q_1, and q_2 be positive integers such that $d_1 d_2 = d$ $\gcd(d_1, d_2) = 1$, $\gcd(d_2^2, (x/d)) = \gcd(d_1^2, ((x+2)g_k(x)/d)) = 1$, $q_1 q_2 = q$, and $\gcd(q_1, q_2) = 1 = \gcd(q_2^2, ((x+2)/q)) = \gcd(q_1^2, (xg_k(x)/q)) = 1$. Then,

$$\left(\frac{d_1^2}{d}\right)x = y_1^n \text{ and } \left(\frac{q_1^2}{q}\right)(x+2) = y_2^n$$

for some nonzero integers y_1 and y_2, since $y \neq 0$ and $n \geq 1$. From this, we have

$$qd_1^2 y_2^n - dq_1^2 y_1^n = 2q_1^2 d_1^2 \qquad \text{and so} \qquad q_2 d_1 y_2^n - d_2 q_1 y_1^n = 2q_1 d_1.$$

Let

$$g_k(x) = f_k(x)/(x(x+2)) = x^{k-1} + A_1 x^{k-2} + \cdots + A_{k-1}$$

and

$$g(x) = x^2 + 2x.$$

Then, for each integer l with $0 \leq l \leq k-1$,

$$h_l(x) = \left(\sum_{i=0}^{l} (-1)^i A_{l-i} 2^i\right) x^{k-l-1} + A_{l+1} x^{k-l-2} + \cdots + A_{k-1}.$$

In particular,

$$h_{k-1}(x) = \sum_{i=0}^{k-1} (-1)^i A_{k-i-1} 2^i.$$

This implies that

$$\gcd(g(x), g_k(x)) \mid \sum_{i=0}^{k-1} (-1)^i A_{k-i-1} 2^i,$$

where A_i is the coefficient of x^{k-i-1} in the polynomial $g_k(x)$.

If x is odd, then $d \mid x$, $d \mid g_k(x)$, $q \mid (x+2)$, $q \mid g_k(x)$ and so $dq \mid \gcd(g(x), g_k(x))$. Suppose that x is even. Then,

$$\frac{dq}{4} \mid \frac{x(x+2)}{4} \text{ and } \frac{dq}{4} \mid g_k(x).$$

Hence, we have

$$dq \mid 4\gcd(g(x), g_k(x)) \text{ and so } dq \mid 4\sum_{i=0}^{k-1} (-1)^i A_{k-i-1} 2^i.$$

This proves the lemma. □

Lemma 2 (Hajdu, Laishram, and Tengely [5]). *Let a, b, and c be positive integers with $a < b \leq 4 \times 2{,}018{,}957 \times 99 \times 467$ and $c \leq 2ab$. Then, the Diophantine equation*

$$au^n - bv^n = \pm c,$$

in integral variables $u > v > 1$, implies

$$n \leq \begin{cases} \max\{1000, 824.338 \log b + 0.258\} & \text{if } b \leq 100 \\ \max\{2000, 769.218 \log b + 0.258\} & \text{if } 100 < b \leq 10{,}000 \\ \max\{10{,}000, 740.683 \log b + 0.234\} & \text{if } b > 10{,}000 \end{cases}$$

Lemma 3 (Szalay [15]). *Suppose that $p \geq 2$ and $r \geq 1$ are integers and that*

$$F(x) = x^{rp} + a_{rp-1}x^{rp-1} + \cdots + a_0$$

is a polynomial with integer coefficients. Then, rational polynomials

$$B(x) = x^r + b_{r-1}x^{r-1} + \cdots + b_0$$

and $C(x)$ with $\deg(C(x)) \leq rp - r - 1$ uniquely exist for which

$$F(x) = B^p(x) + C(x).$$

Lemma 4 (Srikanth and Subburam [13]). *Let p be a prime number, $B(x)$ and $C(x)$ be nonzero rational polynomials with $\deg(C(x)) < (p-1)\deg(B(x))$, l be a positive integer such that $lB(x)$ and $l^pC(x)$ have integer coefficients for any nonnegative integer i and $\delta \in \{1, -1\}$:*

$$P_{i,\delta}(x) = \delta(lB(x) + \delta i)^p - \delta(lB(x))^p - \delta l^p C(x),$$

r be any positive integer,

$$H_1 = \{\alpha \in \mathbb{Z} : P_{i,\delta}(\alpha) = 0, \delta \in \{1, -1\}, i = 0, 1, 2, \ldots, r-1\},$$

and

$$H_2 = \{\alpha \in \mathbb{R} : P_{r,1}(\alpha) = 0 \text{ or } P_{r,-1}(\alpha) = 0\}.$$

If H_1 and H_2 are empty, then the Diophantine equation

$$y^p = B(x)^p + C(x)$$

has no integral solution (x, y). Otherwise, all integral solutions (x, y) of the equation satisfy $x \in H_1$ or

$$\min H_2 \leq x \leq \max H_2.$$

In some other new way as per Note 2, using Laurent's result leads to a better result. For our present purpose, the following lemma is enough.

Lemma 5 (Laurent, Mignotte, and Nesterenko [12]). *Let l, m, α_1, α_2, β_1, and β_2 be positive integers such that $l\log(\alpha_1/\alpha_2) - m\log(\beta_1/\beta_2) \neq 0$. Let*

$$\Gamma = \left| \left(\frac{\alpha_1}{\alpha_2}\right)^l \left(\frac{\beta_1}{\beta_2}\right)^m - 1 \right|.$$

Then, we have

$$|\Gamma| > 0.5 \exp\left\{-24.34 \log\alpha \log\beta (\max\{\gamma + 0.14, 21\})^2\right\},$$

where $\alpha = \max\{3, \alpha_1, \alpha_2\}$, $\beta = \max\{3, \beta_1, \beta_2\}$ *and* $\gamma = \log\left(\frac{l}{\log\beta} + \frac{m}{\log\alpha}\right)$.

Proof of Theorem 1. Assume that $k \geq 3$. Then, by Lemma 1, all integral solutions (x, y, n), $y \neq 0, -1$ and $n \geq 1$, of (1) satisfy the equation

$$ay_2^n - by_1^n = c, \tag{2}$$

where y_1 and y_2 are nonzero integers, a and b are positive integers such that $c \leq 2ab$,

$$ab \mid 4\sum_{i=0}^{k-1}(-1)^i A_{k-i-1}2^i,$$

and A_i is the coefficient of x^{k-i-1} in the polynomial $f_k(x)/x(x+2)$. Without loss of generality, we can take $y_1 > y_2$ to prove the result. From (2), we write

$$\left|1 - \left(\frac{a}{b}\right)\left(\frac{y_2}{y_1}\right)^n\right| = \frac{c}{by_1^n}.$$

Next, take $\alpha_1 = a$, $\alpha_2 = b$, $\beta_1 = y_2$, $\beta_2 = y_1$, $l = 1$, and $m = n$ in Lemma 5. Then, by the lemma, we obtain

$$\frac{c}{by_1^n} \geq \exp\{-24.3414(\log\max\{3, a, b\})(\log\max\{3, y_1\})\max\{21, (\log n)\}^2.$$

From this, we obtain the required bound. Next, assume that $1 \leq k \leq 2$. Then, we can write Equation (1) as

$$y_1^n = c_1 x$$

and

$$y_2^2 = c_2(x+2)^i,$$

where

$$c_1, c_2 \in \{1/4, 1/2, 1, 2, 4\}$$

and $i \in \{1, 2\}$. In the same way, we can obtain the required bound. To find the exact values of A_0, $A_1, \ldots, A_{k-1}$, equate the coefficients of the polynomials

$$g_k(x) = 1 + (x+1)(1 + (x+3) + \cdots + (x+3)(x+4)\cdots(x+k)).$$

and

$$g_k(x) = x^{k-1} + A_1 x^{k-2} + \cdots + A_{k-1}.$$

Then, we obtain $A_0 = 1$, $A_1 = 1 + \alpha_1$, $A_{k-1} = 1 + \alpha_{k-2}$, and $A_j = \alpha_{j-1} + \alpha_j$ for $j = 2, 3, \ldots, k-2$ and

$$\alpha_m = 1 + \sum_{i=0}^{m-1} \lambda_{i+1}(3, \ldots, k+i-m+1)$$

for $m = 1, 2, \ldots, k-2$. □

Next, we consider the case that

$$\mathfrak{b} \leq 4 \times 9 \times 11 \times 467 \times 2{,}018{,}957.$$

If $y_1 = 1$, $y_2 = 1$, or $y_1 = y_2$, then we have

$$x = \frac{d_2}{d_1} = 1, x = \frac{q_2}{q_1} - 2 = -1, x = \frac{2q_1 d_2}{d_1 q_2 - q_1 d_2},$$

where d_1, d_2, q_1 and q_2 are positive integers such that $d_1 d_2 q_1 q_2 = ab$. These three equations give the required upper bound. Hence, Lemma 2 completes the theorem.

Proof of Corollary 1. Take $k = 10$ in Theorem 1. Then, $A_0 = 1$, $A_1 = 54$, $A_2 = 1258$, $A_3 = 16{,}541$, $A_4 = 134{,}716$, $A_5 = 700{,}776$, $A_6 = 2{,}309{,}303$, $A_7 = 4{,}589{,}458$, $A_8 = 4{,}880{,}507$, $A_9 = 2{,}018{,}957$, and $b/4 = 46{,}233$ and so

$$740.683 \log \mathfrak{b} \leq 8982.9.$$

In a similar way, for the case $k < 10$, we have

$$\max\{10{,}000, 740.683 \log \mathfrak{b} + 0.23\} \leq 10{,}000.$$

Hence, Lemma 2 confirms the result. □

Proof of Theorem 2. Take $F(x) = x + x(x+1) + \cdots + x(x+1)\cdots(x+k)$ in Lemma 3. Since k is odd, so $2 \mid \deg(F(x))$, $p = 2$, and $r = \frac{k+1}{2}$. Then, by Lemma 3, there uniquely exist rational polynomials $B(x)$ and $C(x)$ with $\deg(C(x)) \leq \frac{k-1}{2}$ such that

$$F(x) = B^2(x) + C(x).$$

Now, by Lemma 4, we have the theorem. □

Note 1. *First, find the values of the elementary symmetric forms $\lambda_{i+1}(3, \ldots, k+i-m+1)$ for $i = 0, \ldots, m-1$ and $m = 1, 2, \ldots, k-2$. Next, obtain $\alpha_1, \alpha_2, \ldots, \alpha_{k-2}$ and so $A_0, A_1, \ldots, A_{k-1}$. Using this, calculate $|A_{k-i-1} - 2A_{k-i-2}|$ and so*

$$2^i |A_{k-i-1} - 2A_{k-i-2}| = |A_{k-i-1} 2^i - A_{k-i-2} 2^{i+1}|$$

for $i = 0, 2, 4, \ldots$. In this way, for any positive integer k, we can find the exact value of $\mathfrak{b}$ in Theorem 1. Therefore, it is not so hard to decide for which k is

$$\mathfrak{b} \leq 4 \times 9 \times 11 \times 467 \times 2{,}018{,}957$$

as in Theorem 1. For this work, we can use a suitable computer program.

Note 2. *The result of Laurent [16] is an improvement on the result of Laurent, Mignotte, and Nesterenko [12]. From the proof, using the result of Laurent [16] and Proposition 4.1 in Hajdu, Laishram, and Tengely [5], we write the following:*

Let A, B, and C be positive integers with $C \leq 2AB$, $B > A$ and $B \leq 4 \times 9 \times 11 \times 467 \times 2{,}018{,}957$. Then, the equation

$$Au^n - Bv^n = \pm C$$

in integer variables $u > v > 1, n > 3$ implies

$$n \leq C_m (\max\{m, h_n\})^2 (\log B) \left(2 + \frac{(\tau - 1) q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{\log u_0},$$

where

$$h_n = \log\left(\frac{n}{(\tau+1)\log B} + \frac{1}{2\log u + (\tau - 1) q_0}\right) + \epsilon_m,$$

in which q_0, u_0, C_m, m, τ, and ϵ_m are positive real numbers such that $u \geq u_0$, $\log(u/v) \leq q_0$, $C_m > 1, \epsilon_m > 1$, and $\tau > 1$.

If we use the above observation in Lemma 1 of this paper, then we obtain the bound

$$n \leq c_2' (\log n - \log\log \mathfrak{b})^2 \log \mathfrak{b}$$

and so an immediate estimation is

$$n \leq c_2 \log \mathfrak{b},$$

where c_2 is as in Theorem 1 and c_2' is a positive real number depending on u_0, q_0, C_m, m, τ, and ϵ_m. Though there are better bounds in the literature than what the linear form of the logarithmic method in Laurent, Mignotte, and Nesterenko [12] gives, it is sufficient to obtain an explicit bound only in terms of k using our method, which simplifies the arguments in Section 5 of [5] as well.

3. Conclusions

This article implied a method to obtain an upper bound for all n where (x, y, n) is an integral solution of (1) and to improve the method and algorithm of [4]. The same method can be applied to study the general Diophantine equation (see [8–10]),

$$y^n = a_0 x + a_1 x(x+1) + \cdots + a_k x(x+1)\cdots(x+k),$$

where $k, a_0, a_1, \cdots, a_k$ are fixed integers and x, y, n are integral variables in obtaining a better upper bound (depending only on $k, a_0, a_1, \cdots, a_k$) for all $max\{x, y, n\}$, where (x, y, n) is an integral solution of the general equation.

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