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Comprehensive Interval-Induced Weights Allocation with Bipolar Preference in Multi-Criteria Evaluation

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Abstract: Preferences-involved evaluation and decision making are the main research subjects in Yager's decision theory. When the involved bipolar preferences are concerned with interval information, some induced weights allocation and aggregation methods should be reanalyzed and redesigned. This work considers the multi-criteria evaluation situation in which originally only the interval-valued absolute importance of each criterion is available. Firstly, based on interval-valued importance, upper bounds, lower bounds, and the mean points of each, we used the basic unit monotonic function-based bipolar preference weights allocation method four times to generate weight vectors. A comprehensive weighting mechanism is proposed after considering the normalization of the given absolute importance information. The bipolar optimism–pessimism preference-based weights allocation will also be applied according to the magnitudes of entries of any given interval input vector. A similar comprehensive weighting mechanism is still performed. With the obtained weight vector for criteria, we adopt the weighted ordered weighted averaging allocation on a convex poset to organically consider both two types of interval-inducing information and propose a further comprehensive weights allocation mechanism. The detailed comprehensive evaluation procedures with a numerical example for education are presented to show that the proposed models are feasible and useful in interval, multi-criteria, and bipolar preferences-involved decisional environments.

Keywords: aggregation operator; bipolar preference; multi-criteria evaluation; ordered weighted averaging operator; weights allocation

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Citation: Jin, X.; Yager, R.R.; Mesiar, R.; Borkotokey, S.; Jin, L. Comprehensive Interval-Induced Weights Allocation with Bipolar Preference in Multi-Criteria Evaluation. *Mathematics* **2021**, *9*, 2002. https://doi.org/10.3390/math9162002

Academic Editor: Mariano Luque

Received: 11 August 2021
Accepted: 18 August 2021
Published: 21 August 2021

1. Introduction

Information fusion theories and techniques are important in numerous comprehensive evaluation problems [1–24] in which multiple criteria or data sources should be considered rather than a single one. Aggregation operators [7] are powerful and strict information fusion tools that have been systematically studied and developed during the past decades [2,6,9,12,15]. In computational intelligence and intelligent decision making, suitably chosen or designed aggregation operators are more efficient and flexible to address different types of comprehensive evaluations.

As a typical multi-criteria evaluation problem, when several different criteria are involved, some normalized weight vectors can be assigned to those criteria, presenting the relative importance between different criteria. We say relative importance here rather than absolute importance because those importances together should add up to a unit. This type of weight vector is usually determined before input vectors can be obtained. There are some classical methods to determine this type of weight vector, including the subjective weighting method or Analytic Hierarchy Process (AHP) method [17].

After the input vector (whose entries correspond to multiple criteria) has been obtained, according to the magnitudes of the entries of the vector, some bipolar preferences (such as the optimism–pessimism preference) can be embodied by another type of weighting method. Yager introduced the ordered weighted averaging (OWA) operator [20], which is a type of averaging aggregation operator and can well reflect the bipolar optimism–pessimism preference of decision-makers. In decision analysis, when decision-makers are faced with several individual evaluation values, the decision-makers with the strongest optimism level will prefer the maximum of them, those with the strongest pessimism will prefer the minimum of them, and those with the Laplace decision preference will take the mean of them as the merging result they believe in. OWA operators are determined by some associated normalized weight vectors which can be derived by some preferences-based weighting methods [8–10,14,19]. Therefore, as a prefect generalization, OWA operators connect the two extreme preferences and serve as a preference continuum from the strongest optimism via the Laplace preference to the strongest pessimism.

An important extension of OWA operator and related weights allocation is the induced ordered weighted averaging (IOWA) operator [22,23], in which the weighting process can be based on the magnitudes of entries in some inducing vector rather than in the input vector.

The inputs for information fusion and aggregation have become ever-increasingly more complex and with more uncertainties with diverse types [3,11,18,25]. For example, instead of the mere real values, more inputs faced in modern decision making and evaluation problems are fuzzy information granules [25] and interval values. Hence, this work will mainly concern the situation in which the relevant information, including inputs information and inducing information, are both interval-valued.

With some given inducing of the vector which is real-valued, we may use the IOWA weighting method to derive some suitable weight vectors for a collection of criteria (or a group of experts, etc.). However, this method can only embody a single type of weighting style. In practice, it is much better to comprehensively consider more possible weighting styles. To make the weighting process more comprehensive, this work will provide some different weighting styles based on given inducing vectors and then use ways to put them all together, obtaining a resulting weight vector as a comprehensive and more desired one.

With an input real vector and an obtained weight vector, in multi-criteria evaluation, the two basic deciding factors have already been satisfied to perform the weighted arithmetic mean over the input vector using the weight vector. However, in practice, there may be more complex decisional situations. On the one hand, as there are multiple criteria and multiple inputs from different sources, it provides space to take bipolar preference into account. Therefore, decision-makers' optimism–pessimism preferences (as some embodiments of their working experience, etc.) often should be embodied in the weighted arithmetic mean. One effective method to model such preferences is to use weighted ordered weighted averaging (WOWA) operators [19]. Nevertheless, when the input vector is interval-valued, the WOWA cannot work in general. This is mainly because WOWA is based on a linear order of inputs and it is possible that a collection of interval numbers cannot form a linear order but rather a partial order. On the other hand, the bipolar optimism–pessimism preference should also be embodied in the interval itself; that is, with more optimism, a higher value in that interval is more preferred and vice versa. To solve these problems, this work will apply some newly proposed techniques and propose some integrated preference-involved models to appropriately embody all such concerns.

The remainder of this work is organized as follows. Section 2 reviews, rephrases, or redefines some bipolar preferences-involved aggregations and related weight allocations. In Section 3, we present the analysis for bipolar preferences-involved weighting and the comprehensive evaluation. Section 4 provides the detailed comprehensive weighting and evaluation model with application in evaluation for university teachers. Section 5 concludes and comments on this work.

2. Bipolar Preferences-Involved Aggregations and Related Weight Allocations

Without the loss of generality, the real input a concerned in this work is within the unit interval $a \in [0,1]$. A real input vector (of dimension n) is denoted by $\mathbf{a} = (a_i)_{i=1}^n \in [0,1]^n$. The interval inputs considered in this work are closed intervals (also called interval numbers) within unit interval [0, 1], i.e., $[a,b] \subseteq [0,1]$. When no confusion arises, sometimes $[a,a]$ is identified with real number $a \in [0,1]$, which reserves all related interval operations. The set of all such interval numbers is denoted by $\mathcal{I}$. A vector of interval numbers is denoted by $[\mathbf{a},\mathbf{b}] = ([a_i,b_i])_{i=1}^n \in \mathcal{I}^n$, where $\mathbf{a},\mathbf{b} \in [0,1]^n$ are real input vectors. For any two interval numbers, namely $[a_i,b_i],[a_j,b_j] \in \mathcal{I}$, we adopt the well-known interval order $\leq_{Int}$ such that $[a_i,b_i] \leq_{Int} [a_j,b_j]$ occurs only if $a_i \leq a_j$ and $b_i \leq b_j$; we write $[a_i,b_i] <_{Int} [a_j,b_j]$ if $[a_i,b_i] \leq_{Int} [a_j,b_j]$ and $[a_i,b_i] \neq [a_j,b_j]$.

The weighted arithmetic mean is one of the most representative aggregation operators, which is widely applied in multi-criteria evaluations. For any weight vector $\mathbf{w} = (w_i)_{i=1}^n \in [0,1]^n$ ($\sum_{i=1}^n w_i = 1$), recall that a real-valued weighted arithmetic mean (with $\mathbf{w}$) $WA_{\mathbf{w}} : [0,1]^n \to [0,1]$ is defined by

$$WA_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^n w_i a_i. \tag{1}$$

An ordered weighted averaging (OWA) operator [20] $OWA_{\mathbf{w}} : [0,1]^n \to [0,1]$ is defined by

$$OWA_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^n w_i a_{\sigma(i)} \tag{2}$$

where $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is any suitable permutation such that $x_{\sigma(i)} \geq x_{\sigma(j)}$ whenever $i < j$. Note that both of the above operators can be equipped with a same weight vector, but in the OWA operator, the input vector $\mathbf{a} = (a_i)_{i=1}^n$ is in a reordered form of $\mathbf{a}_\sigma = (a_{\sigma(i)})_{i=1}^n$.

Yager proposed an important generalization of the OWA operator called the induced ordered weighted averaging (IOWA) operator [22,23]. In IOWA aggregation, a new vector $\mathbf{d} = (d_i)_{i=1}^n$ (called the inducing vector) is attached to the input vector $\mathbf{a} = (a_i)_{i=1}^n$. Then, with a different permutation, which is in direct relation to $\mathbf{d}$, the IOWA operator with weight vector $\mathbf{w}$ $IOWA_{\mathbf{w}} : [0,1]^n \to [0,1]$ is defined by

$$IOWA_{\mathbf{w};\mathbf{d}}(\mathbf{a}) = \sum_{i=1}^n w_i a_{\sigma(i)} \tag{3}$$

where $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is any suitable permutation such that $d_{\sigma(i)} \geq d_{\sigma(j)}$ whenever $i < j$.

Considering the involved weight vector $\mathbf{w}$ will mainly decide the represented bipolar preference, Yager defined the orness [20] of any weight vector $\mathbf{w}$ by $orness(\mathbf{w}) \triangleq \sum_{i=1}^{n} \frac{n-i}{n-1} w_i$ and dually the andness of it is defined by $andness(\mathbf{w}) \triangleq 1 - orness(\mathbf{w})$. Cognitively, a weight vector with larger orness will be considered to embody larger optimism in an OWA-based evaluation (or larger preference extent in an IOWA-based evaluation). For some new development and strict analysis in relation to orness/andness, we recommend the Reference [16].

A BUM function $Q : [0,1] \to [0,1]$ is monotonic and non-decreasing with $Q(0) = 0$ and $Q(1) = 1$. Yager [21] proposed a convenient method and effective mechanism to derive weight vector $\mathbf{w} = (w_i)_{i=1}^n$ from a given BUM function, namely $Q : [0,1] \to [0,1]$, such that:

$$w_i = Q(i/n) - Q((i-1)/n). \tag{4}$$

In general, a larger BUM will usually generate a weight vector with larger orness and vice versa. For example, $Q(y) = 1 - (1-y)^2$ represents an optimism preference while $Q(y) = y^2$ indicates a pessimism preference. The orness of any BUM Q is defined by $orness(Q) \triangleq \int_0^1 Q(t)dt$, while the andness of it is defined dually by $andness(Q) \triangleq 1 - orness(Q)$ [13]. BUM function-based weights allocation has some important extensions, one of which is the three-set method [9] used to perform OWA aggregation on poset values.

In interval decision making and evaluation environments, the corresponding weighted arithmetic mean can be defined with slight modification. For any weight vector $\mathbf{w}$, interval-valued weighted arithmetic mean (with $\mathbf{w}$) (IvWA) $IvWA_{\mathbf{w}} : \mathcal{I}^n \to \mathcal{I}$ is defined by

$$IvWA_{\mathbf{w}}([\mathbf{a},\mathbf{b}]) = \sum_{i=1}^{n} w_i[a_i, b_i] = \sum_{i=1}^{n} [w_i a_i, w_i b_i]. \tag{5}$$

However, OWA and IOWA operators are both based on some linearly ordered set being inducing information and thus some new methods or formulations should be introduced to provide the interval-valued OWA and IOWA. Here, we rephrase those definitions using the three-set formulation [9].

The interval-induced ordered weight averaging (IvIOWA) operator $IvIOWA_{\mathbf{w};\mathbf{d}} : \mathcal{I}^n \to \mathcal{I}$ with the inducing interval vector $[\mathbf{u},\mathbf{v}] = ([u_i, v_i])_{i=1}^{n}$ and the BUM function $Q : [0,1] \to [0,1]$ is defined by the interval-valued weighted arithmetic mean (with $\mathbf{w}$) $IvWA_{\mathbf{w}} : \mathcal{I}^n \to \mathcal{I}$,

$$IvIOWA_{Q;[\mathbf{u},\mathbf{v}]}([\mathbf{a},\mathbf{b}]) = \sum_{i=1}^{n} w_i[a_i, b_i] = \sum_{i=1}^{n} [w_i a_i, w_i b_i] \tag{6}$$

in which $\mathbf{w}$ is defined in the following steps:

Step 1: for each $[u_i, v_i]$, define three disjoint subsets of $\{1, \ldots, n\}$: $A_i, B_i, E_i \subseteq \{1, \ldots, n\}$ such that

$A_i = \{j \in \{1, \ldots, n\} : [u_i, v_i] <_{Int} [u_j, v_j]\}$,
$B_i = \{j \in \{1, \ldots, n\} : [u_j, v_j] <_{Int} [u_i, v_i]\}$, and
$E_i = \{1, \ldots, n\} \backslash (A \cup B)$.

Step 2: form the intermediate vector $\mathbf{v} = (v_i)_{i=1}^{n} \in [0,1]^n$ (which is not necessarily normalized) such that

$$v_i = \frac{Q(1 - \frac{|B_i|}{n}) - Q(\frac{|A_i|}{n})}{|E_i|} \tag{7}$$

where $|S|$ denotes the cardinality of any finite set S.

Step 3: it can be shown that $\mathbf{v} \neq 0 = (0, \ldots, 0)$ [9] and then after normalizing $\mathbf{v}$, we obtain the normalized weight vector $\mathbf{w} = (w_i)_{i=1}^{n}$ by

$$w_i = \frac{v_i}{\sum_{k=1}^{n} v_k}. \tag{8}$$

When the inducing interval vector $[\mathbf{u},\mathbf{v}] = [\mathbf{a},\mathbf{b}]$, the IOWA defined in Equation (6) is called the interval-ordered weight averaging (IvOWA) operator $IvOWA_{\mathbf{w}} : \mathcal{I}^n \to \mathcal{I}$ with the BUM function $Q : [0,1] \to [0,1]$.

3. Some Analysis for Bipolar Preferences-Involved Weighting and Comprehensive Evaluation

This section firstly analyzes the comprehensive weighting method for criteria with a bipolar preference of inducing information that is an absolute importance vector for criteria. Then, with the obtained weight vector for criteria, we will adjust it via the bipolar preference of inducing information, which is the input vector.

3.1. Comprehensive Weights Determination for Criteria with Relative Importance Information

In a multi-criteria evaluation, given a collection of n criteria $\{C_i\}_{i=1}^{n}$ without further order information for criteria, we cannot use the bipolar preference weighting method. When each criterion C_i is given an absolute importance degree of $[c_i, d_i] \in \mathcal{I}$ which is the interval number and determined independently from the importance degrees of other criteria, we can use the three-set method in Equations (7) and (8) to determine the weight vector $\mathbf{s}^{<I>} = (s_i^{<I>})_{i=1}^{n}$ for the collection of criteria.

Alternatively, we may also allocate weights to the criteria according to the real-valued inducing information that is derived from $[\mathbf{c},\mathbf{d}] = ([c_i, d_i])_{i=1}^n$. As three representative real-value vectors serving as real-inducing information are derived from $([c_i, d_i])_{i=1}^n$, we firstly consider $\mathbf{c} = (c_i)_{i=1}^n$ and $\mathbf{d} = (d_i)_{i=1}^n$ obtained from the upper and lower bounds of $([c_i, d_i])_{i=1}^n$. We also consider $\mathbf{z} = (z_i)_{i=1}^n = (\frac{c_i+d_i}{2})_{i=1}^n$, obtained from the mean of $([c_i, d_i])_{i=1}^n$. Note those weight vectors derived from inducing vectors $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{z}$ by $\mathbf{s}^{<c>} = (s_i^{<c>})_{i=1}^n$, $\mathbf{s}^{<d>} = (s_i^{<d>})_{i=1}^n$, and $\mathbf{s}^{<z>} = (s_i^{<z>})_{i=1}^n$, respectively.

As another judging method, we may consider directly normalizing $\mathbf{z}$ to obtain the weight vector $\mathbf{s}^* = (s_i^*)$ for criteria. When $\mathbf{z} = 0$ (i.e., $[\mathbf{c},\mathbf{d}] = [0, 0]$), we may directly set the corresponding weight vectors obtained from them using the Laplace principle with $\mathbf{s}^* = (s_i^*) = (1/n, \ldots, 1/n)$.

Considering that both real-valued inducing information and interval-valued inducing information can reasonably reflect some absolute importance extents of each criterion, as mentioned above, we can comprehensively consider the weighting results obtained from them by assigning some weights to these inducing sources. For example, by taking a weight vector $(0.3, 0.1, 0.1, 0.1, 0.4)$, we may have a comprehensive weighting result for criteria $\mathbf{s} = 0.3\mathbf{s}^{<I>} + 0.1\mathbf{s}^{<c>} + 0.1\mathbf{s}^{<d>} + 0.1\mathbf{s}^{<z>} + 0.4\mathbf{s}^*$; note that in practice, decision-makers can adopt any weight vectors according to their preferences or by some voting results from a collection of decision-makers in group decision making.

Note that as it is reasonable that the criterion with a larger absolute importance should obtain a larger weight, the BUM function Q_1, used as preference indicator, should have $Q_1(y) \geq y$. For example, we may take the concave BUM function $Q_1(y) = 1 - (1-y)^2$, indicating a moderate preference.

3.2. Comprehensive Weights Determination for Criteria with Optimism–Pessimism Preference

When the input vector for aggregation is $[\mathbf{a},\mathbf{b}] = ([a_i, b_i])_{i=1}^n$ and the original weight vector for criteria is not known, it is ideal to derive a weight vector from the IvIOWA operator with the three-set method (that is, using Equations (7) and (8) with inducing interval vector $[\mathbf{u},\mathbf{v}] = [\mathbf{a},\mathbf{b}]$). We denote the weight vector obtained in such way by $\mathbf{r}^{<I>} = (r_i^{<I>})_{i=1}^n$. Similar to what has been discussed previously, we may consider the real vectors as inducing information, namely $\mathbf{a} = (a_i)_{i=1}^n$, $\mathbf{b} = (a_i)_{i=1}^n$, and $\mathbf{q} = (q_i)_{i=1}^n = (\frac{a_i+b_i}{2})_{i=1}^n$, and obtain the corresponding three weight vectors $\mathbf{r}^{<a>} = (r_i^{<a>})_{i=1}^n$, $\mathbf{r}^{<b>} = (r_i^{<b>})_{i=1}^n$, and $\mathbf{r}^{<q>} = (r_i^{<q>})_{i=1}^n$, respectively. Analogously, we may consider directly normalizing $\mathbf{q}$ to obtain the weight vector $\mathbf{r}^* = (r_i^*)$ from inputs. When $\mathbf{q} = 0$ (i.e., $[\mathbf{a},\mathbf{b}] = [0, 0]$), we may directly set the corresponding weight vectors obtained from them using the Laplace principle with $\mathbf{r}^* = (r_i^*) = (1/n, \ldots, 1/n)$. Similarly, we may also take a combinational form to obtain a comprehensive weighting result embodying the optimism–pessimism preference, i.e., $\mathbf{r} = 0.3\mathbf{r}^{<I>} + 0.1\mathbf{r}^{<a>} + 0.1\mathbf{r}^{<b>} + 0.1\mathbf{r}^{<q>} + 0.4\mathbf{r}^*$.

Note that with this type of optimism–pessimism-inducing formation, the BUM function Q_2, adopted as a preference indicator, can be any function without restriction to $Q_2(y) \geq y$. That is, the criterion corresponding to a larger input will be generally assigned a larger weight and vice versa. For example, we may take the convex BUM function $Q_2(y) = y^2$ whose orness is $orness(Q_2) = 1/3$, indicating a moderately pessimistic preference.

3.3. Adjusted Weights with the Optimism–Pessimism Preference under Known Weights

In spite of the fact that the previously obtained weight vectors $\mathbf{s}$ and $\mathbf{r}$ are reasonable from different types of inducing information, it is necessary to consider the weights allocation with the optimism–pessimism preference under the situation in which an original weight vector for criteria has already been known and will matter.

An ideal method is to use weighted OWA allocation on a convex poset [9]. With the background of the interval input vector, suppose the original known weight vector for criteria is the previously obtained vector $\mathbf{s} = (s_i)_{i=1}^n$; we rephrase this method in the following steps.

Step 1: for each $[a_i, b_i]$, define three disjoint subsets of $\{1, \ldots, n\}$: $A_i, B_i, E_i \subseteq \{1, \ldots, n\}$ such that

$A_i = \{j \in \{1, \ldots, n\} : [a_i, b_i] <_{Int} [a_j, b_j]\}$,

$B_i = \{j \in \{1, \ldots, n\} : [a_j, b_j] <_{Int} [a_i, b_i]\}$, and

$E_i = \{1, \ldots, n\} \backslash (A \cup B)$.

Step 2: form the intermediate vector $\mathbf{v} = (v_i)_{i=1}^n \in [0,1]^n$ (which is not necessarily normalized) such that

$$v_i = s_i \cdot \frac{Q_2(1 - \sum_{k \in B_i} s_k) - Q_2(\sum_{k \in A_i} s_k)}{1 - \sum_{k \in B_i} s_k - \sum_{k \in A_i} s_k} \text{ (with the convention } \sum_{k \in \varnothing} s_k = 0). \tag{9}$$

Step 3: it can be shown that $\mathbf{v} \neq 0 = (0, \ldots, 0)$ [9] and then after normalizing $\mathbf{v}$, we obtain the normalized weight vector $\mathbf{w} = (w_i)_{i=1}^n$ by

$$w_i = \frac{v_i}{\sum_{k=1}^n v_k}. \tag{10}$$

With the obtained weight vectors $\mathbf{s}$, $\mathbf{r}$, and $\mathbf{w}$ from different perspectives, we may take a weighted average of them to yield a final resulting weight vector for criteria $\mathbf{u} = 0.3\mathbf{s} + 0.3\mathbf{r} + 0.4\mathbf{w}$ (note that the involved weight vector $(0.3, 0.3, 0.4)$ can be changed by any other weight vector according to different situations in practice) and finally perform the interval-valued weighted arithmetic mean (with $\mathbf{u}$) (IvWA) $IvWA_{\mathbf{u}} : \mathcal{I}^n \to \mathcal{I}$ with

$$IvWA_{\mathbf{u}}([\mathbf{a}, \mathbf{b}]) = \sum_{i=1}^n u_i [a_i, b_i] = \sum_{i=1}^n [u_i a_i, u_i b_i].$$

Such evaluation results comprehensively and organically reflects two types of bipolar preferences with known absolute importance for criteria.

4. Detailed Comprehensive Weighting and Evaluation Model with Bipolar Preferences

This section will provide the detailed evaluation procedures for what has been discussed in the previous section with a numerical evaluation case of university teachers.

Comprehensive evaluation for university teachers is important because in general, there is a wider diversity in university teachers than in middle school teachers; for example, some university teachers and educators mainly focus on teaching and education, while some other scholars mostly conduct research or academic work. We cannot view scholars as better and more important than educators and vice versa. For the illustrative purpose, we consider three important roles of a university teacher: teaching, conducting research, and providing social service. The detailed evaluation procedures are as follows.

Stage 1. Evaluation background determination and evaluation information collection

Step 1: list $n = 4$ criteria for evaluating the performance of a university teacher as follows:

C_1: teaching attitude and time;

C_2: teaching effect;

C_3: social service effect; and

C_4: academic performance.

Step 2: Decision-maker invites some experts to access individual performance of each criterion of that teacher with the interval vector $[\mathbf{a}, \mathbf{b}] = ([a_i, b_i])_{i=1}^4 = ([0.3, 0.7], [0.5, 0.8], [0.1, 0.9], [0.4, 0.8])$. That is, the individual performance of C_1 is $[a_1, b_1] = [0.3, 0.7]$ and so forth.

Step 3: Decision-maker invites some experts to judge the absolute importance for each criterion, which can be represented by the interval vector $[\mathbf{c}, \mathbf{d}] = ([c_i, d_i])_{i=1}^4 =$

$([0.2, 0.4], [0.2, 0.6], [0.1, 0.5], [0.5, 0.9])$. That is, the absolute importance of C_1 is $[c_1, d_1] = [0.2, 0.4]$ and so forth.

Step 4: Decision-maker determines two types of bipolar preferences expressed by two BUM functions. BUM function Q_1 is chosen to be concave with $Q_1(y) = 1 - (1-y)^2$, indicting the decision-maker prefers (also assigns more weight to) the criterion with higher absolute importance. BUM function Q_2 is chosen to reflect a moderate pessimistic preference with $Q_2(y) = y^2$.

Stage 2. Comprehensive weights determination for criteria with relative importance information

Step 1: Determine the weight vector $\mathbf{s}^{<I>} = (s_i^{<I>})_{i=1}^4$ for criteria with the BUM function Q_1 and inducing information $[\mathbf{c}, \mathbf{d}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$v_1 = \frac{Q_1(1-\frac{0}{4}) - Q_1(\frac{2}{4})}{2} = \frac{1-(1-(1-\frac{1}{2})^2)}{2} = \frac{1}{8}$,

$v_2 = \frac{Q_1(1-\frac{2}{4}) - Q_1(\frac{1}{4})}{1} = \frac{\left[1-(1-\frac{1}{2})^2\right] - \left[1-(1-\frac{1}{4})^2\right]}{1} = \frac{5}{16}$,

$v_3 = \frac{1}{8}, v_4 = \frac{15}{16}$.

Hence, after normalizing $\mathbf{v}$, we obtain the weight vector $\mathbf{s}^{<I>} = \frac{1}{24}(2, 5, 2, 15) \doteq (0.0833, 0.2084, 0.0833, 0.625)$.

Step 2: Determine the weight vector $\mathbf{s}^{<c>} = (s_i^{<c>})_{i=1}^4$ for criteria with the BUM function Q_1 and inducing information $[\mathbf{c}, \mathbf{c}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{1}{4},\ v_2 = \frac{1}{4},\ v_3 = \frac{1}{16},\ v_4 = \frac{7}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{s}^{<c>} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{16}, \frac{7}{16}) = (0.25, 0.25, 0.0625, 0.4375)$.

Step 3: Determine the weight vector $\mathbf{s}^{<d>} = (s_i^{<d>})_{i=1}^4$ for criteria with the BUM function Q_1 and inducing information $[\mathbf{d}, \mathbf{d}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{1}{16},\ v_2 = \frac{5}{16},\ v_3 = \frac{3}{16},\ v_2 = \frac{7}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{s}^{<d>} = (\frac{1}{16}, \frac{5}{16}, \frac{3}{16}, \frac{7}{16}) = (0.0625, 0.3125, 0.1875, 0.4375)$.

Step 4: Determine the weight vector $\mathbf{s}^{<z>} = (s_i^{<z>})_{i=1}^4$ for criteria with the BUM function Q_1 and inducing information $[\mathbf{z}, \mathbf{z}] = ([0.3.0.3], [0.4, 0.4], [0.3, 0.3], [0.7, 0.7])$ (with $\mathbf{z} = (z_i)_{i=1}^4 = (\frac{c_i + d_i}{2})_{i=1}^4$) using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{2}{16},\ v_2 = \frac{5}{16},\ v_3 = \frac{2}{16},\ v_2 = \frac{7}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{s}^{<z>} = (\frac{2}{16}, \frac{5}{16}, \frac{2}{16}, \frac{7}{16}) = (0.125, 0.3125, 0.125, 0.4375)$.

Step 5: directly normalize $\mathbf{z}$ into the weight vector $\mathbf{s}^* = (s_i^*)_{i=1}^4 = \frac{1}{17}(3, 4, 3, 7) \doteq (0.1765, 0.2352, 0.1765, 0.4118)$.

Step 6: obtain a comprehensive weighting result for criteria

$$\mathbf{s} = 0.3\mathbf{s}^{<I>} + 0.1\mathbf{s}^{<c>} + 0.1\mathbf{s}^{<d>} + 0.1\mathbf{s}^{<z>} + 0.4\mathbf{s}^* = (0.13934, 0.2441, 0.13309, 0.48347).$$

Stage 3. Comprehensive weights determination for criteria with the optimism–pessimism preference

Step 1: Determine the weight vector $\mathbf{r}^{<I>} = (r_i^{<I>})_{i=1}^4$ for criteria with the BUM function Q_2 and inducing information $[\mathbf{a}, \mathbf{b}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$v_1 = \frac{Q_1(1-\frac{0}{4})-Q_1(\frac{2}{4})}{2} = \frac{1-(\frac{1}{2})^2}{2} = \frac{3}{8}$,
$v_2 = \frac{Q_1(1-\frac{2}{4})-Q_1(\frac{0}{4})}{2} = \frac{(\frac{1}{2})^2-0^2}{2} = \frac{1}{8}$,
$v_3 = \frac{Q_1(1-\frac{0}{4})-Q_1(\frac{0}{4})}{4} = \frac{1}{4}$, $v_4 = \frac{1}{4}$.

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{r}^{<I>} = (\frac{3}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}) = (0.375, 0.125, 0.25, 0.25)$.

Step 2: Determine the weight vector $\mathbf{r}^{<a>} = (r_i^{<a>})_{i=1}^4$ for criteria with the BUM function Q_2 and inducing information $[\mathbf{a}, \mathbf{a}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{5}{16},\ v_2 = \frac{1}{16},\ v_3 = \frac{7}{16},\ v_4 = \frac{3}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{r}^{<a>} = (\frac{5}{16}, \frac{1}{16}, \frac{7}{16}, \frac{3}{16}) = (0.3125, 0.0625, 0.4375, 0.1875)$.

Step 3: Determine the weight vector $\mathbf{r}^{<b>} = (r_i^{<b>})_{i=1}^4$ for criteria with the BUM function Q_2 and inducing information $[\mathbf{b}, \mathbf{b}]$ using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{7}{16},\ v_2 = \frac{4}{16},\ v_3 = \frac{1}{16},\ v_4 = \frac{4}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{r}^{<b>} = (\frac{7}{16}, \frac{4}{16}, \frac{1}{16}, \frac{4}{16}) = (0.4375, 0.25, 0.0625, 0.25)$.

Step 4: Determine the weight vector $\mathbf{r}^{<q>} = (r_i^{<q>})_{i=1}^4$ for criteria with the BUM function Q_1 and inducing information $[\mathbf{q}, \mathbf{q}] = ([0.5.0.5], [0.65, 0.65], [0.5, 0.5], [0.6, 0.6])$ (with $\mathbf{q} = (q_i)_{i=1}^4 = (\frac{a_i+b_i}{2})_{i=1}^4$) using Equations (7) and (8). We have the intermediate vector $\mathbf{v}$ with

$$v_1 = \frac{6}{16},\ v_2 = \frac{1}{16},\ v_3 = \frac{6}{16},\ v_4 = \frac{3}{16}.$$

Considering $\mathbf{v}$ is already normalized, we have $\mathbf{r}^{<q>} = (\frac{6}{16}, \frac{1}{16}, \frac{6}{16}, \frac{3}{16}) = (0.375, 0.0625, 0.375, 0.1875)$.

Step 5: directly normalize $\mathbf{q}$ into the weight vector $\mathbf{r}^* = (r_i^*)_{i=1}^4 = \frac{1}{2.25}(0.5, 0.65, 0.5, 0.6) \doteq (0.2222, 0.2889, 0.2222, 0.2667)$.

Step 6: obtain a comprehensive weighting result for criteria

$$\mathbf{r} = 0.3\mathbf{r}^{<I>} + 0.1\mathbf{r}^{<a>} + 0.1\mathbf{r}^{<b>} + 0.1\mathbf{r}^{<q>} + 0.4\mathbf{r}^* = (0.31388, 0.19056, 0.25138, 0.24418).$$

Stage 4. Determine an adjusted weight vector with the optimism–pessimism preference with the known weight vector $\mathbf{s} = (0.13934, 0.2441, 0.13309, 0.48347)$.

Step 1: For each $[a_i, b_i]$, define three disjoint subsets of $\{1, \ldots, n\}$: $A_i, B_i, E_i \subseteq \{1, \ldots, n\}$ such that

$A_i = \{j \in \{1, \ldots, n\} : [a_i, b_i] <_{Int} [a_j, b_j]\}$,
$B_i = \{j \in \{1, \ldots, n\} : [a_j, b_j] <_{Int} [a_i, b_i]\}$, and
$E_i = \{1, \ldots, n\} \backslash (A \cup B)$.

In detail, $A_1 = \{2, 4\}$, $B_1 = \varnothing$, and $E_1 = \{1, 3\}$; $A_2 = \varnothing$, $B_2 = \{1, 4\}$, and $E_2 = \{2, 3\}$; $A_3 = \varnothing$, $B_3 = \varnothing$, and $E_3 = \{1, 2, 3, 4\}$; and $A_4 = \{2\}$, $B_4 = \{1\}$, and $E_4 = \{3, 4\}$.

Step 2: form an intermediate vector $\mathbf{v} = (v_i)_{i=1}^4 \in [0, 1]^4$ by Equation (9) such that

$v_1 = s_1 \cdot \frac{Q_2(1)-Q_2(s_2+s_4)}{1-(s_2+s_4)} \doteq 0.2407$,
$v_2 = s_2 \cdot \frac{Q_2(1-s_1-s_4)-Q_2(0)}{1-s_1-s_4} \doteq 0.0921$,
$v_3 = s_3 \cdot \frac{Q_2(1-0)-Q_2(0)}{1-0} = s_3 = 0.13309$, and
$v_4 = s_4 \cdot \frac{Q_2(1-s_1)-Q_2(s_2)}{1-s_1-s_2} \doteq 0.48347 \cdot \frac{0.7407-0.0596}{0.61656} \doteq 0.5341$.

Step 3: by normalizing $\mathbf{v}$, we obtain the normalized weight vector $\mathbf{w} = (w_i)_{i=1}^4 = (0.2407, 0.0921, 0.1331, 0.5341) \doteq \mathbf{v}$.

Stage 5. Obtain the final resulting weight vector for criteria and perform the interval-valued weighted arithmetic mean (with $\mathbf{u}$)

Step 1: obtain the final resulting weight vector for criteria $\mathbf{u} = 0.3\mathbf{s} + 0.3\mathbf{r} + 0.4\mathbf{w} = (0.232246, 0.167238, 0.168581, 0.431935)$.

Step 2: perform the interval-valued weighted arithmetic mean (with $\mathbf{u}$) (IvWA) $IvWA_{\mathbf{u}} : \mathcal{I}^4 \to \mathcal{I}$ with

$$IvWA_{\mathbf{u}}([\mathbf{a}, \mathbf{b}]) = \sum_{i=1}^4 [u_i a_i, u_i b_i] = [0.3429249, 0.7936335].$$

Step 3: report the evaluation result $[0.3429249, 0.7936335]$ to help with further decision making.

5. Conclusions

Both of the multiple criteria and the intervals have uncertainties in evaluation value determinations and thus uncertainties provide space for further decision preferences, especially bipolar preferences. Therefore, the information fusion techniques and comprehensive evaluation methods that can well-embody such bipolar preferences are important in both theoretical studies and applications.

This work mainly discussed bipolar preference-involved weights allocation for involved multiple criteria from three respects. The derived weight vector $\mathbf{s}$ for criteria is comprehensively generated from five aspects with a given concave BUM function, four of which are concerned with the absolute importance being inducing information and the fifth with direct normalization of the absolute importance. In a similar way, we comprehensively generate another weight vector, namely $\mathbf{r}$, for criteria with any BUM function using interval inputs as inducing information without the intervention of $\mathbf{s}$. As the third suggested method and in a tangled way, we successfully applied the weighted OWA allocation on a convex poset in an interval environment and obtained the weight vector $\mathbf{w}$ for criteria with the intervention of $\mathbf{s}$. Finally, the resulting weight vector $\mathbf{u}$ for criteria is obtained by comprehensively considering all the three types using a weighted form.

Considering interval information is one of the most representative uncertain information types and the most commonly known uncertain data-type in real life, this work actually proposed some paradigmatic preferences-involved evaluation models that can well-handle interval information and multiple criteria. This work provided some prescriptive and suggestive preference-involved comprehensive multi-criteria evaluation models for both practitioners and theorists interested in aggregation and evaluation theory.

There are also some limitations of the proposed methods. For example, concerning situations when the involved uncertain information is not interval information but rather some other uncertain information (such as rough numbers, fuzzy numbers, or neutrosophic numbers, as in the recently proposed basic uncertain information [11,15]), at present, we have not discussed the corresponding evaluation models and frames that can handle those types of uncertain information.

Author Contributions: Conceptualization, X.J. and L.J.; methodology, X.J. and L.J.; validation, X.J., R.R.Y., R.M., S.B. and L.J.; formal analysis, X.J., R.R.Y., R.M., S.B. and L.J.; investigation, X.J., R.R.Y., R.M., S.B. and L.J.; resources, X.J., R.R.Y., R.M., S.B. and L.J.; data curation, X.J., R.R.Y., R.M., S.B. and L.J.; writing—original draft preparation, X.J. and L.J.; writing—review and editing, X.J., R.R.Y., R.M., S.B. and L.J.; visualization, X.J., R.R.Y., R.M., S.B. and L.J.; supervision, X.J., R.R.Y., R.M., S.B. and L.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work is partly supported by the grant APVV-18-0052 and VEGA 1/0006/19, and by the grant IGAPrF2021 from Palacky University Olomouc.

Conflicts of Interest: The authors declare no conflict of interest.

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