

Article

Applications of Non-Standard Analysis in Topoi to Mathematical Neurosciences and Artificial Intelligence: Infons, Energons, Receptons (I)

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Abstract: The purpose of this paper is to promote new methods in mathematical modeling inspired by neuroscience—that is consciousness and subconsciousness—with an eye toward artificial intelligence as parts of the global brain. As a mathematical model, we propose topoi and their non-standard enlargements as models, due to the fact that their logic corresponds well to human thinking. For this reason, we built non-standard analysis in a special class of topoi; before now, this existed only in the topos of sets (A. Robinson). Then, we arrive at the pseudo-particles from the title and to a new axiomatics denoted by Intuitionistic Internal Set Theory (IIST); a class of models for it is provided, namely, non-standard enlargements of the previous topoi. We also consider the genetic–epigenetic interplay with a mathematical introduction consisting of a study of the Yang–Baxter equations with new mathematical results.

Keywords: non-standard analysis; topos theory; artificial intelligence; Yang–Baxter equations; brain studies; intuitionistic logic

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1. Introduction

This paper contains both known and new results, and it addresses an auditorium of great diversity: neuroscientists, computer scientists, physicians, physicists, biologists, mathematicians, etc. We attempted to write a paper that could be read by different kinds of specialists. The authors and their collaborators belong to various scientific areas: neurosciences, mathematics, medicine, and economics; and are professors, researchers, and students.

The authors hope they have produced a self-contained and readable paper for most categories of scientists who might be interested in the topics. At the same time, the preprint [1] represents an expanded version of this article, from the mathematical point of view, collecting most of what the authors considered as being the necessary mathematical background.

Therefore, in this paper, some already known notions and results are briefly re-explained, but, at the same time, the paper contains many new original contributions. The main original contribution is the idea of considering the non-standard extension of the intuitionistic logic (in topoi) as modeling human thinking, consciousness, and considering *together* both natural and artificial intelligence, as parts of the *global brain*, as explained in Sections 2–4. This paper also contributes to the development of the non-standard analysis in set-type exponential topos, in the aria of pure mathematics, see Theorem 1.

Information and energy represent the two facets of an elementary particle in string theory (the string vibrates = energy in a specific way (amplitude, frequency, spin, etc.) = information, in order to give one or another one of the elementary particles). They do not reduce to our brain or to our body; they travel freely realizing connections between the interior (brain, sensory organs, and DNA = genetics) and the exterior (epigenetics) of the human body, generating *the mind*, mutually related to *the brain* (see [2,3]). These subjects will be considered in more detail in Sections 2–4.

In Sections 5 and 6, the basic notions of the present modeling approach (based on topos theory and its non-standard extension), are recalled with original examples in Sections 5.2.2–5.2.4. The interpretation of theories (from mathematical logic and model theory) and of the corresponding categories (models of the theories) as infons, energons, and receptons also represents an original approach. For details, see Sections 5 and 7. Finally, the use of the theory of knots and braid groups—and, as a consequence, of the Yang–Baxter equations—in neurosciences might also be a novel idea.

In Section 8, we recall the Yang–Baxter equations and produce logical solutions in topoi, finding connections between topoi theory (and, thus, intuitionistic logic) and the Yang–Baxter various equations, preparing a bridge toward the second part of this work. However, this paper uses certain notions that are not explicitly introduced here, such as: (quantum) neural networks, (quantum) Turing machines, and further developments of the non-standard analysis in more general topoi. All these things will be considered in a forthcoming paper.

For topos theory, the book [4] is a good reference, and [5] is a good reference for model theory, while, for non-standard analysis in the topos SET, the book [6] is an effective reference. The topoi model of intuitionistic logic (multi-valued) has been used in quantum physics (see [7] and its references; see, also, [8]) while non-standard analysis in SET (introduced by A. Robinson [9]) has been applied in mathematical economics (see [10] and its references). The logic of non-standard extensions in topoi is proposed as a model of human thinking (based on infons) and was inspired by the paper [11]. These theories represent recent and difficult results in abstract mathematics. As references, we recommend [2,3,12–14]; other references will appear later.

All the subjects of this paper are considered in order to justify (from the point of view of neurosciences, cybernetics, and physics) the infons and receptons, with energons between them (an energon being something similar to the corpus callosum from the human brain), to guess how these pseudo-particles should look like (Sections 2–4), and to mathematically build them, having the recognized pattern (Sections 5–7). The present theory is intended to be a kind of *Emergent Theory of Everything*.

This is a Theory of Everything because all that we will ever know passes through our (global) brain; therefore, our consciousness and subconsciousness are emergent (that is, in continuous evolution), because the structures considered here must be particularized progressively with concrete infons and Receptons from various sciences. These must be the next steps after completing the constructions in the forthcoming Part II of this work (it appears that, as S. Hawking remarked, a concrete Theory of Everything in physics—or other sciences—might be forbidden by the Gödel Incompleteness Theorem, see [15]).

Note 1.1. The topoi will model the consciousness and, together with their non-standard enlargement, the subconsciousness also. Although it is not so obvious why, the same method of modeling will be considered for all other areas of knowledge; this is because the perceived reality comes to us only through consciousness (and the subconsciousness) of the (global) brain; therefore, it must be compatibility.

Note 1.2. Let us explain further the meaning of the *Emergent Theory of Everything*, from two perspectives:

(a) Physics. There are certain theories in physics modeling the *emergent quantum mechanics* (EmQM), such as the de Broglie–Bhm theory (see [16]) and the Kleen Irwin’ theory (shortly described in Section 3.3). Although our approach is different from the previous two examples, the main idea is the same, namely to make the brains (and global

brain) parts of the same system with elementary particles (photons, electrons, etc.) to make the consciousness (and subconsciousness) and human thinking become part of physics.

In quantum mechanics (a mainly mathematical theory after the axiomatization given by J. von Neumann in [17]) the collapse of the wave function may take place at any moment between the interference with the measuring device and the conscious perception of a conscious observer. E. Wigner proposed in [18] that the consciousness itself generates the collapse of the wave function (von Neumann–Wigner interpretation). On the other hand, there are nowadays precise measurements proving that the brain takes any decision shortly *before* we become aware of it (if we ever become; for example we never become aware of the subliminal messages, they are received by subconsciousness only).

It seems in these conditions that, in fact, the collapse of the wave function takes place when subconsciously observed, that is the subconsciousness plays a very important role, producing the collapse. Thus, in this paper, one consider not only the (intuitionistic) consciousness (modeled in topoi), but the subconsciousness also, as a conservative extension (modeled by the non-standard enlargement of topoi). In order to understand physics, it is further necessary to consider concrete examples of topoi, for various physical problems (this being the subject of further work), this general theory being applied in precise topics, in an emergent way. Examples of such topoi can be found in [7], for example. Similarly for neuroscience and other possible arias.

(b) Other sciences. The atomistic concepts of infons, energons, receptons, introduced in this paper aims to unify a broad aria of empirical sciences also, as argued in the first Sections (biology, neurosciences, cybernetics, maybe others as economics, sociology, etc.). For this one must, once again, to find suitable topoi as models of the corresponding theories, in an emergent and unifying manner. For neurosciences, such examples can be found in [19,20]. See, also, Section 9 for further comments.

Note 1.3. This paper does not claim in any way that we will ever be able to prove “the everything”. This should be a continuous (emergent) task, for several categories of scientists. In fact, this paper only presents a first part of the approach (a second part will need the construction of non-standard analysis in more general topoi, of the exponential type, that is of the form $\mathbf{SET}^C$, with C a general category, not only a finite one, see Definition 21 and Theorem 1). One argue that the **IIST** axiomatics for an extended theory of sets is a *new* good approach (in fact, **IIST** is an axiomatics for enlargements of some topoi, including the topos **SET**).

The introduction of this axiomatics represents a contribution to the *Foundations of mathematics* also; a class of models for **IIST** is provided in Theorem 1, starting from **ZFC** axiomatics (so **IIST** is non-contradictory if and only if **ZFC** is so), and it is a *conservative* extension of **ZFC**, see Remark 13, 2; for several truth values, *conservative* means that the truth values are kept by Transfer, (see Step 1 of the proof of Theorem 1). We start our study with some incursions through neurosciences, physics, cybernetics in order to justify the interest. Then, we pass to concrete mathematical constructions. One consider several arias of science which push us toward the same conclusions. In fact, one possible *definition* of mathematics is used here, namely: start with examples from several arias, find common features, and then forget the words used in these examples, keep only the relations between them. If they are similar, built an (symbolic) abstract theory, which is simultaneously a model for apparently completely different arias. One can find isomorphic structures, allowing us to make a transfer of information between these apparently very different arias (there are examples of such transfer between economics and thermodynamics [21], Bayesian games, and quantum physics [22], etc.).

Certainly, the simplest example of simultaneous modeling with mathematics is something like $2 + 3 = 5$, operation following from **ZF** axiomatics, where these 2, 3, 5 might be stars (astronomy), trees (forestry), people (sociology), etc. The previous examples are at a much higher level, but in the same philosophy. This **IIST** approach generates similar unifications through abstract mathematics, seen as taking place inside the global brain (including natural and artificial intelligence, see Section 4).

Note 1.4. This paper is a review/article one, containing the following topics (the order from the paper does not necessarily coincides with the next one):

- (1) Presentation of some known results from neurosciences, physics, and cybernetics arranged and interpreted in order to prove the importance of the axiomatics from the next (2) (Sections 2–4);
- (2) Presentation of the axiomatics **IIST** (Intuitionistic Internal Set Theory) (Section 7.1);
- (3) Theoretical introduction of non-standard analysis in topoi (until now this was known only for the topos **SET**), at the level of general definition (having higher order language)) (Section 6.2);
- (4) Topoi of **SET**-type and the construction of a model class for **IIST** on them, based on non-standard analysis in **SET** (**IIST** is then as consistent as **ZF(C)** is) (Section 5.2 and Theorem 1);
- (5) Definitions of the notions of infon, energon, recepton, which would use for a simultaneous approach of the natural and artificial neural (quantum) networks, both part of the global brain (Section 7); this approach is based also on a novel study of the global brain (Section 4); and this last-one is based on some novel points of view concerning the human brain, mind, genetics, and epigenetics (Section 2).
- (6) Yang–Baxter equations in topoi (Section 8);
- (7) Recapitulation of some technical notions of mathematics (non-standard analysis, topos theory, and its associated intuitionistic logic), because our target audience is made up of specialists other than only those working in mathematical logic and category theory (neuroscientists, computer scientists, etc.) (Sections 5.1–5.3 and 6.1, with some original examples of geometric topoi in Sections 5.2.3 and 5.2.4).

The topics from (7) are review topics, the topics from (1) and (6) are partly original, partly review, while the topics from (2), (3), (4), (5), are original.

2. Information and Reception: Brain and Mind, Genetics and Epigenetics

The next three Sections try to put in evidence, using known and, by many experts recognized results from biology and physiology, physics, and cybernetics which relates between them the information and its reception with energy. In this section we deal with biology and physiology (see also [2,3,23]).

2.1. Information and Entropy

First of all, information is related to organization. Some systems or objects are more complex than others because we perceive them as containing more information. For example, it is obvious that an epithelial tissue contains more information than a cell. Epithelial cells exhibit distinct polarity, having an apical, a lateral and a basal domain. This organization involves more information, as the molecular mechanism responsible for this arrangement is required to create a functional barrier between adjacent cells.

Following Tom Stonier’s idea from [23], information is related to entropy, because the latter is a measurement of organization. They are exponentially inverse proportional: as a consequence, if—inside a complex system—the entropy decreases, meaning that matter is more organized, the amount of incorporated information increases. It is really important to see how this new concept—the incorporated information—influences our lives and more specifically the role it plays in the structure of matter. How do atoms know how to bind together and form a more complex structure? How can a simple stem cell differentiate in so many other types of cells?

Information dictates all those processes. Everything contains infons (see Section 7, for their mathematical definition, one of the main purposes of this paper). Let us take, as an example, a sound wave. We can tell for sure that it contains some kind of information, that travels through air until it reaches the human ear. Here, it is transformed into a nervous impulse and travels to the brain, where it will be perceived. A sound wave is a mechanical wave that causes local regions of compression and rarefaction through the

medium, deviating the local pressure from the equilibrium pressure. Therefore, sound, which is made of information, changes the structure of matter.

In neuroscience, it is of extreme interest to see how information affects the brain. We, as humans, are born with unconnected neurons. In the first two years, we form synapses while experiencing the world around us. As we grow up, we lose almost 50% of the already formed synapses, but we also form others while interacting with the environment. How does information influence this process? The more we interact with something, the more powerful the synapse becomes. It is like a road through a forest.

At first, there's nothing there, but as more and more people go along it, the grass disappears and a path is formed. The same thing happens in the brain. Information extends one's life. During the first part of our lives, the human organism functions in an anti-entropic way. As it becomes more organized, the entropy decreases, instead of increasing. However, after this period, the entropy of the human organism continues to increase until it reaches a state of equilibrium with the environment, when the organism dies, because no more *energy exchange* can be done.

By increasing the amount of information one takes in, the entropy of the organism decreases, meaning that it prolongs the process of ageing. Thus, learning new things delays the decay of the brain primarily, but of other organs as a consequence as well. For a much more physico-mathematical approach of such things, see Sections 3 and 7 of this paper.

2.2. Information and Reception, Brain and Mind

For further details related to this paragraph, we refer to the books [2,3]. The *brain* is the main organ where the *mind* is born, but the brain is far from being the only entity involved in the growth of the human mind. The mind is a by-product of the neural networks from the brain (*synapses*), in continuous transformations (some synapses disappear while new ones are born), representing the so-called *plasticity* of the brain. The mind is a by-product of other kind of communications between neurons (and the glial environment), namely the *quantum* networks connected by the so-called microtubules.

Certainly, these interactions are between the brain, through the sensory organs, moderated by the information from the DNA molecule (genetics), and the environment (epigenetics). The human being has much information in its DNA, but only some of it will be used in a life-long period. Simplifying, there is information that tells us to start trembling when it is cold (or to sweat when it is hot).

However, if we spend a life-time period at the equator we will never experience the trembling (respectively, if we spend a life-time period at the North pole we will never experience sweating). Thus, the information from the around 2 m long DNA molecule is read by the environment epigenetics, the system genetics/epigenetics becoming a kind of (quantum) Turing machine, generating, in particular, the evolution of the mind, which, through plasticity, generates the evolution of the brain, and vice-versa.

We—humans—think using both consciousness (awareness) and subconsciousness. There are precise measurements proving that the brain takes a decision shortly before we became aware of it, so through subconsciousness (or, as another example, the subliminal messages). The logic of consciousness is *intuitionistic* (we have, mainly, feelings about things, such as about the truth of conjectures in mathematics, as an example), and the reunion between consciousness and subconsciousness is a *conservative* extension of consciousness (that is, the subconsciousness does not alter our awareness).

Thus, the logic of human thinking is very similar to the *logic of infons* from Section 7, having, as models, non-standard objects from non-standard enlargements of topoi—see Section 7 again (based on Sections 5 and 6 also). From a mathematical point of view, it is necessary to understand how the (*quantum*) *neural networks* from the brain (*synapses*) are influenced by the *genetic–epigenetic (quantum) Turing machine*.

In this paper, one consider that the quantum physics approach is better to be based on truth values than on probabilities, the truth values approach being used in topoi (see [8]). Thus, the reunion between consciousness and subconsciousness is modeled here by the

non-standard logic in enlarged topoi (using the **IIST** axiomatics from Section 7, with the theories as infons, and their categories of models as receptons—the details will be given in the forthcoming Section 7).

A theorem of mathematics (for example) is based on the work of many others; therefore, it is a by-product of the global brain. The global brain includes all the human brains and technology, used both for finding information from the outside of the direct human perception, and for connecting the brains (libraries, social networks, etc.)—see Section 4. In this case, the genetic–epigenetic interaction is between the global brain and the environment, generating a kind of global plasticity for the global brain, generating a global mind (for some details, see Section 4). See, also, the previous Section 2.1.

2.3. Information, Energy, and Reception

This paragraph could be introduced here, but it can also belong to the next Section 3, and we preferred to put it there (it is mainly related to physics, while this Section is mainly related to biology/physiology). Here, we only say that there are opinions in quantum physics saying that the collapse of a probability wave takes place when we become aware of it (as explained before, through subconsciousness, so the consciousness and subconsciousness should be considered together). In this paper, the waves from classical quantum physics become truth values waves, that is, infons, while the receptons represent the collapse of such kind of truth waves.

3. Information and Reception: String Theory and Emergent Quantum Mechanics

In this Section, one continue to put in evidence the relation information–energy–reception, using known and, by many experts recognized results from celestial mechanics/physics (see [12,14,16,24]).

3.1. Max Tegmark's Multiverses

Certainly, everything happens in our real world, our real universe, or our real multiverse, or etc. This means that the human brain, human being, global brain, etc., represent substructures of the highest structure which must exist. There are discussions on what this highest structure is. We recall here the theory of multiverses belonging to Max Tegmark (see [14]). Namely, he classifies the notions of multiverse in 4 categories. We do not insist too much, because there are scientists which consider it as being speculative, but the interested people might look in the previous reference.

However the quantum computers seems to work in a rank 3 multiverse (quantum), at the macroscopic level such an object being only one, but at nano-, pico-level is a whole network in superposition. First of all, what is an (not the) universe. In these theories an universe is, by definition, the *visible* universe for some observer (everything here is based on electromagnetic waves, but the things would change if these waves were replaced by the recently found gravitational waves; to detect them one needs some technology, artificial sensors, not the natural ones only; all these belong to global brain, see Section 4). Two observers might see the same universe, or not. Moreover, a visible universe is increasing (with the speed of light) and might unify with another similarly increasing one. Such a family of universes (which might become only one in some future) defines a rank 1 multiverse. On the other hand, the space can, in theory, dilate faster than the speed of light in some arias (this is not forbidden by the general relativity of Einstein), and the observers from different sides will see for ever different universes. M. Tegmark gets in such a way, a family of multiverses of rank one, that is a multiverse of rank 2 (a comment: in physics, a theory is recognized as true, when it was mathematically—or empirically—predicted AND was experimentally confirmed; three Nobel Prize awards confirm this philosophy: the theoretical existence of the Higgs boson, gravitational waves, and black holes were mathematically predicted—by Higgs and others, Einstein, and Penrose, respectively—with decades before being found, but to be found in reality was crucial; indeed, in organic chemistry, for instance, one can imagine a lot of molecules respecting all the known rules,

but which do not exist; some of them can be produced in laboratories, but this is another aspect; even the periodic table of Mendeleev was artificially completed with elements produced in laboratories; on the other hand we assumed the Tegmark's axiom for the rank 4 universe, saying that all the non-contradictory theory must have a physical model, see below; however, it is not obligatory that such a model exists in our universe; this is the end of the comment).

Further, there are discussions concerning the interpretations of quantum physics.

One interpretation is based on the Copenhagen program, based on the collapse of the waves of probabilities when measured (observed), and another one belongs to Hugh Everett III [25] saying that the probability waves never collapse, but what happens is in fact a *bifurcation* (*multifurcation*) of the (rank 2) multiverse depending on all the possible situations (so, Schrodinger's cat is alive in an universe and dead in another one, at the same time). This is the so-called many-worlds interpretation, giving the Tegmark's rank 3 multiverse. Finally, a rank 4 multiverse is represented by pure mathematics, with the axiom saying that any non-contradictory mathematical theory should have real model(s) in the rank 4 multiverse (see also Definitions 6 and 30). This attempts to explain why our universe allows life to exist (all the universal constants should be very, very carefully chosen: the number of Avogadro, the speed of light, the Planck distance or Planck time, etc.). We will restrict ourselves to a rank 3 multiverse, as being the quantum family of our observable universe. Although the quantum logic of John von Neumann is considered as being the quantum logic in the universe (in the sense of the previous definition), the quantum logic in topoi is considered as being the quantum logic in the quantum multiverse (rank 3).

This is the logic that we will use as it is the most appropriate to the human thinking, which is a product of the brain, a substructure of the uni (multi) verse (considered one of the most complex such substructures, along with black holes and dark matter). However, the best theories from physics, namely the Einstein's theory of (general) relativity (modeling the universe at the macro-level), and the quantum physics (modeling the smallest substructures on the uni (multi) verse, at the nano-, pico-, etc. level) are in contradiction in some situations (the quantum entanglement is one of them).

At the same time, the two theories should coexist in black holes, for instance. Thus, a new unifying theory is required, that is a theory unifying the standard model (which unifies the electromagnetic, strong, and electroweak forces), a quantum model, with the gravitation (modeled by the general relativity). There are currently several approaches in this direction: string theory, emergent quantum mechanics, loop gravity, etc. We will restrict ourselves to some comments relative to the first two of them.

3.2. *Infons and Energons in String Theory*

In *string theory* (see [12]) one consider that any elementary particle is a tiny vibrating chord—string—in a 10-dimensional space, representing the direct product between the Einsteinian 4-dimensional space-time universe (Einstein–Minkowski real 4-fold) and a complex (projective algebraic) Calabi–Yau 3-fold (so a 6-dimensional real manifold). The Calabi–Yau 3-fold is considered because it has very many symmetries, giving the so-called supersymmetric string theory, modeling not only bosons (as the initial string theory did)—that is particles with integer spin, but the fermions as well—particles with half integer but not integer spin.

In such a situation, the number of elementary particles (quarks, photons, electrons, etc.) doubled, and one still look for the new elementary particles at CERN, in Geneva. The 6 additional real dimensions of the universe (multiverse) are supposed to be very compacted, and give only very limited possibilities to the string to vibrate, imposing conditions, as wave length, amplitude, etc. Thus, a string needs *energy*, in order to vibrate, and *information*, in order to vibrate in a specific way to give one or another of the elementary particles (or antiparticles).

In [23], the author considers the existence of some hypothetical elementary particles related to information, calling them *infons*. It is for the first time when the notion appeared

in the literature. However, the present paper is based on the fact that an infon represents only one facet of an elementary particle, the other being represented by the *energons*, so an elementary particle is like a coin, with two facets: the infon, and the energon. The theory of infons has been already discussed a little in Section 2.

After [23], another theory of infons (propositional calculus) has been built in [11]. The authors came to the conclusion that this theory is a *conservative* extension of the *intuitionistic* logic. However, one can show that the axiomatics of non-standard analysis in an environment using intuitionistic (many-valued) logic represents another similar notion of *infon* which will be used in this paper. Thus, an *infon* is given here by a non-standard enlargement of a topos of the form $\mathbf{SET}^{cop}$ (see Section 5).

In string theory, an infon is a non-standard enlargement of a topos of the form $\mathbf{SET}^{cop}$ (see Section 5), hence the theory of infons is the theory associated a category of such non-standard enlargements (so a category of the form $\mathbf{CAT}$, see Remark 3 from Section 5). to while in emergent quantum mechanics an infon remains the theory itself, meaning and the enlargement of the topos of the previous type will be called *recepton*. In string theory the notion of *energon* is also necessary, but for now, we move on:

3.3. Infons and Receptons in Emergent Quantum Mechanics

In *emergent quantum mechanics* [16] we consider a different interpretation of the previous notion of infon. Namely, an infon remains the theory itself, meaning a triplet $\mathcal{I} = (\mathcal{L}, \mathcal{A}, \mathcal{D})$, where $\mathcal{L}$ is the usual language of the intuitionistic non-standard context (see Sections 6 and 7.1), $\mathcal{A}$ is a set of propositions written in the previous language considered as being true (axioms), and $\mathcal{D}$ is a set of deduction rules (as modus ponens is), having as models non-standard enlargements of categories (topoi) of exponential type, as in the the previous subsection (i.e., $\mathbf{SET}^{cop}$)—see Sections 5–7. These categories of models are the *receptons* (different such theories represent different infons, while different such categories represent different receptons). Moreover, the potential energy is information, so is represented by infons (the information allowing us to know that something can move from one place to another, producing actual energy—as an example, while the actual energy (kinetic, caloric, electromagnetic, etc.) is represented by receptons. The matter is considered a condensed form of (actual) energy (under the action of the strong nuclear force) related to the (actual) energy by the well-known formula $E = mc^2$. Both the nuclear fission, which takes place under the action of the electroweak force—from the atomic power plants, and the nuclear fusion, which takes place under the combined action of the strong and electroweak forces—from the Sun, generates loss of mass, producing actual energy from condensed energy, these operations taking place at the level of receptons; the information that this is possible—but not yet effective is potential energy, so infon. Therefore, an energon is either an infon (when virtual, potential), or an recepton (when we deal with actual energy).

Moreover, a stronger argument connecting the (potential) energy with information is represented by the *Landauer Principle*, quantifying the energy necessary to erase a bit of information (for details see [24]). It means that both energy and information are quantised, and they are somehow inverse proportional (not exponentially, as in the case of entropy, see Section 2, but in the same direction, the relation being given by a precise formula, namely $E_{(min)} = k_B T \ln(2)$, where k_B is the Boltzmann constant, T is the temperature of the environment and $\ln$ is the natural logarithm.

Thus, in emergent quantum mechanics everything primordially exists as waves of information = theories in a conservative extension of the intuitionistic logic, all of them being *infons*. Their associated categories (of models) represent, then, the other facets of the coin, called *receptons* so, in this setting the two facets are *the infon*, and *the recepton*, the category of models of the theory which gives us the chosen infon (which is a non-standard enlargement of a topos of a special type, see Section 7).

We, the human beings, are, at the level of our brains, or to the level of the global brain (which includes both all the networks of the human brains, and the technological

discoveries, artificial intelligence included, but also other kind of sensory-type machines) like the radio-sets travelling in a landscape with plenty of electromagnetic waves, and discovering them progressively, function of the increasing quality of the receptor, improved due to the technologies. Thus, we become progressively aware of the surrounding informational waves (theories) improving progressively our consciousness (both at the individual and global level).

At the same time, the subconsciousness works, impacting at every step on our lives, on our level of knowledge. Thus, the *receptons* become the *elementary pseudoparticles of the union between consciousness and subconsciousness*, the two facets of the coin being the theory and the associated category (which must be a non-standard topos). In this version of an unification theory, the uni (quantum-multi) verse is emerging, being stepwise discovered by brain and global brain through the measurement (collapse) of the intuitionistic informational waves.

In fact, emergent quantum mechanics is essentially based on *information* and *consciousness* (which becomes part of physics), but there are other important facts around, namely the *non-determinism* (with the meaning that not only the past influences us, but also the future), the *discrete space*, *Gosset Polytope* (see Section 5 also, for a topos representing it), *E8-crystal*, and *golden ratio* $= \Phi = \lim_{n \rightarrow \infty} (F_{n+1}/F_n) = (1 + \sqrt{5})/2$, where $(F_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence.

The 4D space-time is "pixelated" by the projection on $\mathbf{R}^4$ of the Gosset polytope from $\mathbf{R}^8$ with an angle which produce on $\mathbf{R}^4$ two similar 4-dimensional polytopes, such that the quotient of their 4-volumes is equal to Φ . We recall that an *energon* is either a theory about a potential work (potential energy), so an infon, or a real work (actual energy), which is a *repton*.

4. Cybernetical Principles of the Global Brain

In this Section, we consider a notion of the global brain, using a cybernetical approach, relating again information–reception–energy. The approach is both original and based on ideas coming from [26–29].

4.1. What Is the Global Brain?

The global brain is a bio-technological system composed by physical receptons organized to generate (electromagnetic, gravitational, etc.)—fields of energons, fields which interact to the higher levels of infons and create consciousness. The purpose for humans to create a global consciousness can be achieved by a smart global architecture of a physical human-machine system to implement cybernetic principles of information. As Mihai Drăgănescu proposed in 1990 there is a frontier between physical level of matter and its informational dimension called InfoMatter [29]. At this intersection we identify cybernetic laws which appear in all information systems detected by human sensors (natural and artificial).

4.2. Cybernetic Principles of Information in the Global Brain

1. *Autonomy of information*: information pre-exists at the highest level of consciousness and operates individually (finite) and globally (infinite);
2. *Universal connectivity of information* manifests on the matrix of all data connections between and inside physical and non-physical phenomena on all dimensions;
3. *Free will versus information*: human being relates to information in terms of lists of possible actions and non-actions, a basis for the free will;
4. *Authorization access to information*: at physical level quantum fields interact to create support for information, the access to information on macro level of human beings consists in permissions for entities as humans, robots or other automatic to interchange data according to the free will principle;
5. *Exception handling for information ambiguities*: natural phenomena of information error consists of contradictions inside information with associated correction choices (multiple choices).

Eric Kandel [30], a recipient of the 2000 Nobel Prize in physiology or medicine for his research on the memory storage in neurons, proved that information is linked inside brain to physical level.

4.3. Global Brain Functions Regarding Information

Global brain functions regarding information are derived from data cybernetic principles [31] with the difference that information is a metadata relevant in terms of utility for a recipient (human or robotic).

1. *Information acquisition*: metadata are identified from perception of existing data inside and outside entities (humans/robots, at the reception level);
2. *Information processing*: metadata are created through algorithmic processing of data perceived through sensors (at the infon–energon level);
3. *Information Storage*: metadata are physically stored in human brain and IT recipients (natural/artificial neurons and machine storage recipients) (infor level);
4. *Information sending and receiving*: metadata interchange between sender and receiver in request-answer process (reception level).

4.4. Limits and Out Bounds of the Global Brain

We start with the Figure 1 (it is related to Figure 7 from [32]):

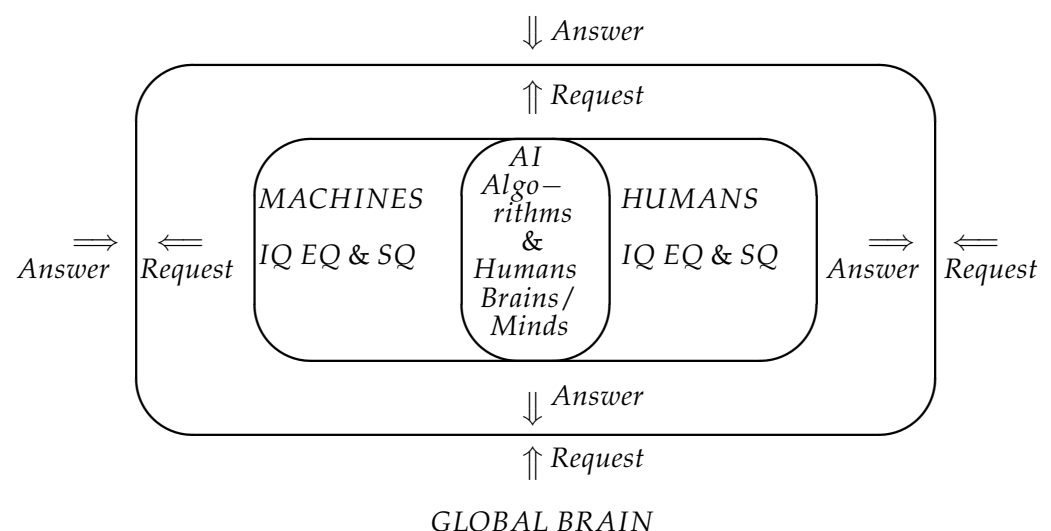


Figure 1. Machines (AI algorithms) vs. Humans: a global brain architecture.

The architecture of global brain generated by machines (artificial intelligence algorithms)—human beings interactions are inspired by Norbert Wiener’s cybernetic communication principles [28], and have a possible graphical representation as above (M-Hemisphere, H-Hemisphere, corpus AIH, analogous to the left hemisphere, right hemisphere and corpus callosum of the human brain). Based on Stefan Odobleja’s vision [26] on analogy and models of mental psychology, a global brain should attain at least the psychological functions of the human brain.

Therefore: request-answer is a cybernetic model to create bio-technological feedback between global brain entities at the knowledge frontier. In the figure above, the membrane of the global brain is, in fact, an invisible knowledge frontier toward global consciousness, continuously in activity inside the system, and as in Figure 1. Many scientists claim for unified theory in science for the explanations of natural phenomena, one of them is Nassim Haramein [33] with unified field theory (see also Section 3), which connects the standard model and gravitation in a single unifying framework (see also [26,34,35]).

From this viewpoint, the global brain is the instrument to apply and study multiple theories from multiple disciplines: psychology, physiology, informatics, cybernetics,

physics, mathematics, and others in one brain framework. Nikola Tesla stated also that we live in the age of electricity [36], where electricity is the meeting point for chemists, physicists, and physiologists, as the electromagnetic energy can be transmitted wireless and its power conducts mankind to progress.

Electricity is a basis for any global system and the global brain can be viewed as an application of Tesla's scientific work. Therefore, Tesla's vision of communication services in a global system have become possible to implement almost a hundred years later due to Internet and machines [37]. Therefore, the previous Figure 1 is a vision of a complex adaptive system composed by artificial machines able to manifest rational, emotional, and social intelligence (levels of IQ, EQ, SQ) and humans with, also given by nature, same characteristics, the communication being done at the level of artificial intelligence based on cybernetic algorithms (certainly, the relatively recent discoveries in the area of gravitational waves can relate the things with the gravitation/graviton also).

The final scope of global brain is to be a knowledge instrument used by humans to develop more self consciousness in their natural habitat. To preserve the habitat of life with other forms of biological and non-biological entities, humans should design a smart and self adaptive system with ethics and human psychological components. Mihai Golu [38,39] studied the field of neuropsychology and its cybernetic dimension and his work is a startpoint to define global brain to be able to manifest consciousness at a higher level of performance in comparison to nowadays human society manifestations on all aspects of development.

Regarding the mathematical instrument to design and operate inside global brain, the non-standard analysis in topoi is a research startpoint achieved in the following Sections of this work (for previous works see [19,20]). In this setting, we propose a starting definition for *infons*, *energons*, and *receptons*, as working tools used to understand the things better and better.

The things will be partially formalized in a mathematical language, in accordance with the previous characteristics sketched for them, coming from biology and physiology, physics, and cybernetics. The formalism aims to produce a new mathematical approach for the problems of neuroscience, physics, and artificial intelligence. A list with problems from the of neuroscience is contained in the book [40]. These can be also considered a list of problems from physics in emergent quantum mechanics.

5. Theories versus Categories; Topoi and Mathematical Logic

The results from this Section are mostly known, so it is essentially a review section, representing useful preliminaries for the original mathematical part of this paper consisting in the construction of the non-standard extensions in topoi of the SET-type, from the next Section 6, and for the original definitions from Section 7. Here, we present, in Section 5.1, the interplay between theories and categories in the classical context of the bivalent logic and Zermelo–Fraenkel (Choice) axiomatics, as precursors of the mathematical definitions of the infons and receptons from Section 7.

Later, in Sections 5.2 and 5.3 we will pass to a special kind of categories, named topoi, with their intuitionistic logic detailed in Section 5.3. For more details concerning the topics from this Section, we refer to [4,5,13,41,42]. See also the extended version [1], which have the same structure as this work, but the mathematical notions are presented in much more detail.

5.1. Theories versus Categories in $\mathbf{ZF}(\mathbf{C})$ Axiomatics as Precursors of the Mathematical Infons and Mathematical Receptons

Definition 1. A class is an arbitrary collection of objects. A set is a class which belongs to another class (one avoid in such a way the well-known Russell's paradox, 5.2 from [1]).

Definition 2. A language $\mathcal{L}$ is the family of all formulae, obtained inductively using conjunctions, disjunctions, negations, implications, and quantifications (existential, universal) of the atomic

formulae (represented by equality and membership of variables, sets, t -uples, $\tau(\sigma)$ with τ and σ from before) (see also 5.3 from [1]).

If $\mathcal{N}$ is a family of classes stable to " $\in$ " (that is $\forall y(y \in \mathcal{N} \wedge x \in y \rightarrow x \in \mathcal{N})$), then we denote by $\mathcal{L}_{\mathcal{N}}$ the restriction of $\mathcal{L}$ to $\mathcal{N}$ (this means that we replace in the classes with those classes from $\mathcal{N}$ only).

Definition 3. A sentence (or proposition) is a formula having all the variables bounded (quantified), so there is no free variable. A sentence is true or false (bivalent logic, see Definition 7).

Definition 4. A set of axioms is a set $\mathcal{A}$ of sentences, supposed true.

Definition 5. The deduction rules are the rules of inference from a set $\mathcal{D}$.

Example 1. 1. Modus ponens (MP): $\frac{p, p \rightarrow q}{q}$;

2. Particularization (P): $\frac{\forall x p(x)}{p(y)}$;

3. Generalization (G): $\frac{p(y)}{\forall x p(x)}$, etc.

Definition 6. A theory is a triplet $\mathcal{T} = (\mathcal{L}, \mathcal{A}, \mathcal{D})$, with $\mathcal{L}, \mathcal{A}, \mathcal{D}$ as in Definitions 2, 4, 5, respectively; in the classical context $\mathcal{A}$ contains the axioms of classical logic (bivalent) **CL** (see Definition 7) and the well-known Zermelo–Fraenkel (Choice) **ZF(C)** axiomatics (see Remark 1). A model of a theory is a concrete mathematical object on which all the sentences from set $\mathcal{A}$ of axioms of a given theory are true. We assume here, by definition, that, in our context, a theory has at least one model (this represents a compatibility with Tegmark's rank 4 multiverse, see Section 3), which is not automatic for the theories of order at least 2 (see Definition 11). However, for the theories of order 1 (i.e., having a first-order language) it is true that such a theory is non-contradictory if it has a model.

Definition 7. A classical logic (**CL**) is a theory having the following axioms:

Then, -1 : $A \rightarrow (B \rightarrow A)$.

Then, -2 : $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$.

And -1 : $A \wedge B \rightarrow A$.

And -2 : $A \wedge B \rightarrow B$.

And -3 : $A \rightarrow (B \rightarrow (A \wedge B))$.

Or -1 : $A \rightarrow A \vee B$.

Or -2 : $B \rightarrow A \vee B$.

Or -3 : $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.

Not -1 : $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.

Not -2 : $A \rightarrow (\neg A \rightarrow B)$ (or $\perp \rightarrow B$, $\perp$ false).

Not -3 : $A \vee \neg A$.

Remark 1. We recall from the list of **ZF** the following axioms only (see 5.10–5.14 from [1]):

Extensionality (**E**): $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

Regularity (**R**): $\forall x [\exists a (a \in x) \rightarrow \exists y (y \in x \wedge \neg (\exists z (z \in y \wedge z \in x)))]$

Specification (**S**): $\forall z \forall \omega_1 \forall \omega_2 \dots \forall \omega_n \exists y \forall x [x \in y \leftrightarrow ((x \in z) \wedge \varphi)]$, where φ is a formula in $\mathcal{L}_{\mathcal{M}}$, with the free variables among $z, \omega_1, \dots, \omega_n$ and having y as a bounded variable. In particular, there are sets of the form $\{x | \varphi(x)\}$; note that this axiom allow us to consider such kind of sets only if they are subsets of a bigger set, avoiding in such a way the Russell's paradox. Moreover, this axiom allow us to built the empty set, specifically

Definition 8. $\emptyset = \{x \in y | x \in x \wedge \neg(x \in x)\}$ (the empty set). Moreover, the empty set is unique (does not depend on y) by the Axiom (**E**).

Infinity (I) (Peano): $\exists X[\emptyset \in X \wedge \forall y (y \in X \rightarrow S(y) \in X)]$, where $S(\omega) := \omega \cup \{\omega\}$ (the successor of ω).

Definition 9. $\mathbb{N} :=$ the smallest X as before (=the set of natural numbers).

Definition 10. Let T be a set (possibly $T = \emptyset$). Then, $\mathcal{U} = \mathcal{U}(T) := \bigcup_{n=0}^{\infty} \mathcal{U}_n(T)$, where $\mathcal{U}_0(T) = T$ and $\mathcal{U}_{n+1}(T) = \mathcal{U}_n(T) \cup \mathcal{P}(\mathcal{U}_n(T))$, $\forall n \geq 0$. This is the Universe generated by T (one can consider a bigger universe, where the union is taken over all the ordinals, by transfinite recurrence—the von Neumann Universe—but we do not need it now). It is constructed from the set T , using **ZF** axiomatics. We remark here that, even when $T = \emptyset$, we have $0 = \emptyset \in \mathcal{U}_0$, and, if $\omega \in \mathcal{U}_n$, then $S(\omega) = \omega \cup \{\omega\} \in \mathcal{U}_{n+1}$, so $\mathbb{N} \subset \mathcal{U}(\emptyset)$. Then, take $T = \mathbb{N}$. We find, using the standard constructions of the number fields, that all of them ($\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) are contained in $\mathcal{U}(\mathbb{N})$, by the definition of $\mathcal{U}(\mathbb{N})$ and the **ZF** axioms. Continuing, it is easy to see that, in fact, all the necessary mathematical structures can be progressively built starting with the empty set $\emptyset$ (whose existence is postulated by **ZF**), using the **ZF(C)** axioms, in higher and higher universes as before.

Definition 11. A language $\mathcal{L}_{\mathcal{U}}$, associated with the previous Universe $\mathcal{U}$ can be defined as in Definition 2. If the quantifiers applies to $\mathcal{U}_0(T) = T$ only, we call the resulting language, denoted by $\mathcal{L}_T$, a first order language, if the quantifiers applies to $\mathcal{U}_1(T)$ only, we call the resulting language a second order language, denoted by $\mathcal{L}_{\mathcal{U}_1(T)}$, etc., if the quantifiers applies to $\mathcal{U}_{n-1}(T)$ only, we call the resulting language a n -th order language, etc., in this terminology, $\mathcal{L}_{\mathcal{U}}$ being called a higher order language.

Example 2 (Examples of theories and models). $\mathcal{T} = (\mathcal{L}, \mathcal{A}, \mathcal{D})$.

1. The theory of sets: (Set) = $(\mathcal{L}, \mathcal{CL} + \text{ZF}, \text{MP})$;
2. The theory of groups: $\mathcal{G} = (\mathcal{L}, \mathcal{A} = \text{ZFC} + \mathcal{CL} + 1) - 4, \text{MP})$, where:
 - (1) $*$: $G \times G \rightarrow G$, function;
 - (2) $\forall x \forall y \forall z [(x * y) * z = x * (y * z)]$;
 - (3) $\exists e \forall x (x * e = e * x = x)$;
1. $(\mathbb{Z}, +)$ is a model for the theory of groups; ($\mathbb{Z}$ = the set of integers);
2. $(\mathbb{Z}, +, \cdot)$ is a model for the theory of rings; and so long.

Definition 12. The family of all the models (or of a set of models) of a theory, together with the “dictionaries” between them (functions, or other objects, compatible with the structure of the theory) gives a category $\mathcal{C}_{\mathcal{T}}$ associated to the theory $\mathcal{T}$.

Example 3. (1). **Gr** (category of groups): contains all (or a set of) groups, denoted by $\text{Ob}(\mathbf{Gr})$, and the morphisms between them, that is $\text{Hom}_{\mathbf{Gr}}(G, H) = \{f : G \rightarrow H \mid f(x *_G y) = f(x) *_H f(y), \forall x, y \in G\}$.

(2). **Ring** (category of rings): $\text{Ob}(\mathbf{Ring})$ = a class (or set) of rings, $\text{Hom}_{\mathbf{Ring}}(R, S) = \{f : R \rightarrow S \mid \forall x \forall y [f(x + y) = f(x) + f(y)] \wedge [f(x \cdot y) = f(x) \cdot f(y)], f(1_R) = 1_S\}$. Similarly for fields (the category **Field**), vector spaces (the category **Vect**), ordered sets (the category **POSET**), and so long.

Definition 13 (Theory of categories). A category $\mathcal{C}$ is a 4-uple $(\text{Ob}(\mathcal{C}), (\text{Hom}_{\mathcal{C}}(A, B))_{A, B \in \text{Ob}(\mathcal{C})}, \circ, (\mathbf{1}_X)_{X \in \text{Ob}(\mathcal{C})})$ satisfying the axioms:

- (i) $\circ : \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$
 $(f, g) \mapsto g \circ f$, such that $(h \circ g) \circ f = h \circ (g \circ f) \forall f, g, h$ for which all the compositions are defined;
- (ii) $\forall X \forall Y, X, Y \in \text{Ob}(\mathcal{C}), \forall f \in \text{Hom}_{\mathcal{C}}(X, Y)$, the following diagrams are commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathbf{1}_X \downarrow & & \downarrow \mathbf{1}_Y \quad (\text{i.e. } \mathbf{1}_Y \circ f = f \circ \mathbf{1}_X) \\ X & \xrightarrow{f} & Y \end{array}$$

Remark 2. Any category (the models of a theory) is a model of the theory of categories, as are, for instance, the last two examples from Example 2.

Remark 3. The category associated to the theory of categories is a category of categories, named **CAT**, that is:

$Ob(\mathbf{CAT}) = \text{categories}$

$Hom_{\mathbf{CAT}}(\mathcal{C}_1, \mathcal{C}_2) = \text{the functors (see Definition 14) from } \mathcal{C}_1 \text{ to } \mathcal{C}_2, \text{ with "}\circ\text{" and } \mathbf{1}_{\mathcal{C}} \text{ defined canonically.}$

Definition 14. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. A functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is acting on objects: $F(A) \in Ob(\mathcal{C}_2)$, $\forall A \in Ob(\mathcal{C}_1)$ and on morphisms: $F(f) \in Hom_{\mathcal{C}_2}(F(X), F(Y))$, $\forall f \in Hom_{\mathcal{C}_1}(X, Y)$, being compatible with compositions and identities, that is:

- (i) $F(g \circ f) = F(g) \circ F(f)$, $\forall f, g$ such that the composition is defined; and
- (ii) $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$, $\forall X \in Ob(\mathcal{C}_1)$.

Remark 4. The previous definition introduce both the notion of morphism in **CAT**, and the theory of functors also. Thus, one expects to obtain the category of functors also. Indeed,

Definition 15. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. We define the category $\mathcal{C}_2^{\mathcal{C}_1}$ as follows:

- $Ob(\mathcal{C}_2^{\mathcal{C}_1}) = \{F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 | F \text{ functor}\}$
- $Hom_{\mathcal{C}_2^{\mathcal{C}_1}}(F_1, F_2) = \{h : F_1 \rightarrow F_2 | h \text{ is a natural transformation}\}$ (see Definition 16).

The composition and the identities are defined canonically.

Here we need the following:

Definition 16. If $F_1, F_2 \in Ob(\mathcal{C}_2^{\mathcal{C}_1})$, a natural transformation $h : F_1 \rightarrow F_2$ is a collection $h = (h_X)_{X \in Ob(\mathcal{C}_1)}$ of morphisms $h_X : F_1(X) \rightarrow F_2(X)$, $h_X \in Hom_{\mathcal{C}_2}(F_1(X), F_2(X))$, $\forall X \in$

$Ob(\mathcal{C}_1)$, such that $\forall f \in Hom_{\mathcal{C}_1}(X, Y)$ the diagrams

$$\begin{array}{ccc} F_1(X) & \xrightarrow{h_X} & F_2(X) \\ F_1(f) \downarrow & & \downarrow F_2(f) \\ F_1(Y) & \xrightarrow{h_Y} & F_2(Y) \end{array}$$

commutes (i.e., $h_Y \circ F_1(f) = F_2(f) \circ h_X$).

Remark 5. We insisted until now to obtain categories from theories, but the converse also works, through axiomatization. Thus, the theories and the categories are the two faces of the same thing, but they represent different notions.

Definition 17. Let $\mathcal{C}$ be a category. The dual category of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$ is the category having:

- $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$;
- $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$, $\forall X, Y \in Ob(\mathcal{C})$ (reversing the arrows).

Definition 18. Let us assume further that the reader is familiar with parts of the theory of categories. If not, we produced the manuscript [1] (Sections 5 and 6) exactly for those readers (from neuroscience, or some physicists, economists) which feel better to have all the necessary things collected in only one place. A much more detailed treatment is presented in [4]. In particular, we assume as known the notions of initial object (usually denoted by $\mathbf{0}$), final object (usually denoted by $\mathbf{1}$), monomorphism, epimorphism, isomorphism, subobject of an object, direct product, direct sum, equalizer, co-equalizer, inductive limit, projective limit, pullbacks, pushouts, exponentiation, (usually denoted by $\mathbf{X}^{\mathbf{Y}}$, with $\mathbf{X}, \mathbf{Y}$ objects), subobjects classifier (usually denoted by Ω). The notions product and sum, initial and final, epi and mono, co, etc., and without co, pull and push, inductive, and projective, etc., are dual notions. An example is given by the next

Definition 19. A functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2^{op}$ is called a cofunctor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$. It satisfies $F(g \circ f) = F(f) \circ F(g)$, $\forall f, g$ (instead of $F(g \circ f) = F(g) \circ F(f)$, $\forall f, g$ as a functor does). This is the first example of a pair of dual notions (functor vs. cofunctor).

5.2. Topos Theory

Now let us pass to the main notions used in this paper:

Definition 20. An elementary topos is a category $\mathcal{E}$ such that:

- (T1) $\mathcal{E}$ has finite inductive limits;
- (T2) $\mathcal{E}$ has finite projective limits;
- (T3) $\mathcal{E}$ has exponentiation;
- (T4) $\mathcal{E}$ has subobjects classifier (see [1,4]).

Remark 6. It can be shown that the conditions required in the definition of a topos, implies the existence in such categories of the objects having the propriety to be any of the objects from Definition 18 (Therefore, there exists initial and final objects, (finite) direct products and sums, equalizer, coequalizer, pullbacks, pushouts, power object of any two objects, etc.).

5.2.1. Examples of Topoi

- (1). The category **SET** of sets;
- (2). The category of n -uples of sets with the morphisms defined on components. This can be written as $\mathbf{SET}^{\mathbf{n}}$, where $\mathbf{n}$ is the discrete category having $Ob(\mathbf{n}) = \{1, 2, \dots, n\}$, $1, 2, \dots \in \mathbb{N}$ as objects and only the identities as morphisms (this category is of type from Definition 15);
- (3). $\mathbf{SET}^{C^{op}}$ is also a topos. The previous example is a particular case, when $C = \mathbf{n}$.

Definition 21. A topos of the form (3) will be called exponential topos. If the topos from (3) is obtained for a category C having a finite number of objects and morphisms, we will call it a **SET-type** topos.

Remark 7. In Section 6, we will construct non-standard extensions for **SET-type** topoi, as finite-generalization of the non-standard analysis in the topos **SET**, introduced in [9], see also [1,6]. This will be an original mathematical contribution. The general case of the exponential topos will be considered in Part II of this work.

5.2.2. Oriented Graphs as Topoi of SET-Type

An oriented graph $\vec{\mathcal{G}}$ consists in a finite number of points $A = \{A_1, A_2, \dots, A_m\}$ (the word point, is only formally used, the elements of A are not necessarily geometric entities), some (or all) of them being joined with some others (or all) of them by oriented segments (arrows); we note that the elements of A can be categories, and the arrows can be functors (for example).

Let us suppose here, additionally, that for each i, j , $i, j \in \{1, \dots, n\}$ there is at most one arrow, either from A_i to A_j or from A_j to A_i , that the path of two consecutive arrows can be replaced with an arrow, and implicitly assume the existence of an unique arrow from A_i to itself, for any i as before, namely the identity arrow (which has no orientation). This graph can be organized as a category of type $\mathbf{n}$ with some additional morphisms: add to $\mathbf{n}$ a morphism from i to j , $i, j \in \mathbf{n}$ if there is an arrow from A_j to A_i in $\vec{\mathcal{G}}$. Let us denote this new category by $\mathbf{n}_{\vec{\mathcal{G}}}$. Then, the **SET-type** topos $\mathbf{SET}^{\mathbf{n}_{\vec{\mathcal{G}}}^{op}}$ can replace the given graph. In fact one can work with a category of n -uples of sets, $(X_1, \dots, X_n)$ requesting additionally that $X_i \subset X_j$, whenever there is an arrow from A_i to A_j in $\vec{\mathcal{G}}$. This approach can be used to study various kind of graphs, in particular *graph neural (quantum) networks* of various kinds (optical, convolutional, etc.) with the quantum aspects modeled in topoi.

Such a development is related to *deep learning*, *artificial intelligence*, *quantum computers* (starting with the *quantum Turing machine*). Moreover, mathematical game theory is also related to graph theory, and some economical applications can be obtained, for instance related to the new area of *neuroeconomics*. There are also connections between graphs and

the evolutionary games (with applications in the study of climate change, evolutionary biology, etc.).

5.2.3. Geometric Topoi

One can use the previous topos-interpretation of the oriented graphs to build geometric topoi. Let us consider the following oriented tetrahedron, as in Figure 2:

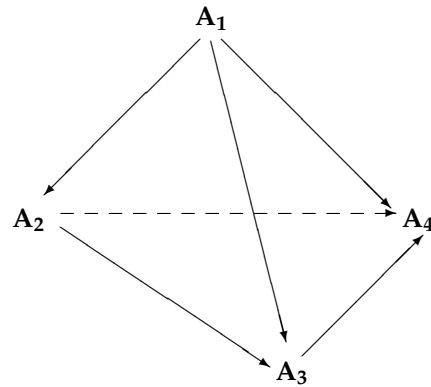


Figure 2. Tetrahedron.

Let us denote by $\vec{\mathcal{T}}$ the oriented graph given by the previous tetrahedron, and by $\mathbf{n}_{\vec{\mathcal{T}}}$ the discrete category $\mathbf{n}$ at which we added the additional morphisms $2 \rightarrow 1$, $3 \rightarrow 1$, $4 \rightarrow 1$, $3 \rightarrow 2$, $4 \rightarrow 2$, $4 \rightarrow 3$. Then, the previous tetrahedron can be recognized in the topos $\mathbf{SET}^{\mathbf{n}_{\vec{\mathcal{T}}}^{op}}$ or, alternatively, in the topos whose objects are the quadruplets $\{(A_1, A_2, A_3, A_4) | A_1, A_2, A_3, A_4 \text{ sets, with } A_1 \subset A_2, A_1 \subset A_3, A_1 \subset A_4, A_2 \subset A_3, A_2 \subset A_4, A_3 \subset A_4\}$ and having as morphisms quadruplets of functions compatible with restrictions. In such a way, one can represent in topoi any polygon, polyhedron, and polytope from $\mathbf{R}^n$, $n \geq 2$.

5.2.4. Gosset Polytope

An important polytope in emergent quantum mechanics (where the consciousness become part of physics) is the Gosset polytope (see the final part of Section 4). It is a polytope $P \subset \mathbf{R}^8$ having 240 vertices, with coordinates $(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0)$, with any combination of signs and the 2s on any two positions (112 vertices) and $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$, by taking an even number of minus signs (128 more vertices).

The vertices of the analogue of the Gosset polytope in $\mathbf{R}^3$ has the coordinates $(\pm 2, \pm 2, 0)$, $(\pm 2, 0, \pm 2)$, $(0, \pm 2, \pm 2)$ (12 vertices), and $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, -1)$ so, this polyhedron has 16 vertices. They form 3 squares of side 4, with the centers in the center of the axes of coordinates and sitting in the planes xOy , yOz , zOx , respectively (O is the center of the axes), and having the edges parallel with the axes of coordinates, and a regular tetrahedron.

The 12 vertices of the three squares are the middle of the edges of a cube having edges of length 4, the center in O, and the faces parallel with the planes of coordinates. Thus, this polyhedron is determined by a cube and a (regular) tetrahedron. It can be represented as a topos using the method from Section 5.2.3. However, even the Gosset polytope from $\mathbf{R}^8$ can be represented in such a way once we give orientations to its edges. Moreover, the group of symmetries of the Gosset polytope is a group which can be found in the structure of the (-2) curves from a degenerate del Pezzo surface of degree 1. We will deal with this subject in a forthcoming paper, based on [43,44].

5.3. Intuitionistic Logic in Topoi

It is well known the connection between **CL** (Definition 7) and the Boolean algebras (respectively lattices), $\mathcal{A} = (A, +, \cdot, -, 0, 1)$ (respectively $= (A, \vee, \wedge, ', 0, 1)$), with the operations defined canonically: sup, inf, etc.), see [13] ([1] might be also useful). We will not insist here.

Definition 22. The axioms of intuitionistic logic (**IL**) are (notations from Definition 7):

Then—1, Then—2;
And—1, And—2, And—3;
Or—1, Or—2, Or—3;
Not 2.

Definition 23. A poset $(H, \leq)$ is called a Heyting lattice if it is a bounded lattice (so, for any $a, b \in H$ there exists $a \wedge b = \inf\{a, b\}$, $a \vee b = \sup\{a, b\}$, there exists $0, 1 \in H$, such that $0 \leq x \leq 1$ for any $x \in H$) satisfying the following conditions (a)–(e), when replacing $\vee$ and $\wedge$ by $+$ and $\cdot$ ($0, 1$ remain the same), and having an operation $\rightarrow$ satisfying:

$$\forall a, b, c, (a \wedge b \leq c \Leftrightarrow a \leq (b \rightarrow c)) \quad (1).$$

Here: (a) $\forall x \forall y (x + y = y + x)$; $\forall x \forall y : (x \cdot y = y \cdot x)$;

(b) $\forall x \forall y \forall z [(x + y) + z = x + (y + z)]$; $\forall x \forall y \forall z [(x \cdot y) \cdot z = x \cdot (y \cdot z)]$;

(c) $\forall x \forall y \forall z [x \cdot (y + z) = x \cdot y + x \cdot z]$; $\forall x \forall y \forall z [x + (y \cdot z) = (x + y) \cdot (x + z)]$;

(d) $\forall n (x + 0 = 0 + x = x)$; $\forall n (x \cdot 0 = 0 \cdot x = 0)$;

(e) $\forall n (x + 1 = 1 + x = 1)$; $\forall n (x \cdot 1 = 1 \cdot x = x)$;

Thus, a Heyting lattice is a 6-uple $(H, \vee, \wedge, \rightarrow, 0, 1)$ as before. A Heyting algebra is a 6-uple $(H, +, \cdot, \rightarrow, 0, 1)$ obtained by replacing $\vee$ and $\wedge$ with $+$ and $\cdot$. Given a Heyting algebra we obtain from it a poset which is a Heyting lattice, putting $a \leq b \Leftrightarrow a \rightarrow b = 1$.

A monadic Heyting algebra (abbreviated MHA) is an algebraic structure $(H, +, \cdot, \rightarrow, 0, 1, \exists, \forall)$ where $(H, +, \cdot, \rightarrow, 0, 1)$ is a Heyting algebra, $\exists$ is a closure operator on H (this means, it satisfies the following (i)–(iv)), $\forall$ is an interior operator on H (this means, it satisfies the following (i)'–(iv)'), and

(v) $\forall \exists a = \exists a$, for any $a \in H$;

(vi) $\exists \forall a = \forall a$, for any $a \in H$;

(vii) $\exists (\exists a \wedge b) = \exists a \wedge \exists b$, for any $a, b \in H$, where

(i) $\exists 0 = 0$;

(ii) $a \leq \exists a$, for any $a \in B$ (recall that $a \leq b$ iff $a \cdot b = a$, iff $a + b = b$, iff $a \rightarrow b = 1$; we recall that $a \rightarrow b = \bar{a} + b$);

(iii) $\exists a = \exists \exists a$, for any $a \in B$;

(iv) $\exists (a \vee b) = (\exists a) \vee (\exists b)$, for any $a, b \in B$, and

(i)' $\forall 1 = 1$;

(ii)' $\forall a \leq a$, for any $a \in B$;

(iii)' $\forall a = \forall \forall a$, for any $a \in B$;

(iv)' $\forall (a \wedge b) = (\forall a) \wedge (\forall b)$, for any $a, b \in B$.

Note that, here $a \wedge b = a \cdot b$, $a \vee b = a + b$ (we preferred to use the logic related symbols).

Remark 8. The condition (1) from Definition 23 is satisfied in the Boolean algebra case for $b \rightarrow c = b' \vee c$. If, in the definition of MHA we have $H = B =$ Boolean algebra, then we use the notation monadic Boolean algebra (MBA). In this case, the existence of the closure operator is equivalent to the existence of the interior operator, and the conditions (v)–(vii) from before are automatically satisfied.

Remark 9. Any Heyting algebra (lattice) satisfies the axioms of **IL** from Definition 22.

Example 4. (1). All the Boolean algebras (lattices) are Heyting algebras (lattices). Any MBA is MHA;

(2). An example of a Heyting algebra which is not Boole can be found in [1], at 6.5. (2);

(3). Another example of Heyting non-Boole algebra is: If (X, τ) is a topological space. If $A \subset X$, we define the interior of A as being the set $\{a \in A \mid \exists D_a \in \tau \text{ s.t. } a \in D_a \subset A\} := \overset{\circ}{A}$. Then, $(X, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra which is not Boolean in general, where: for $A, B \in \tau$, $A \vee B := A \cup B \in \tau$, $A \wedge B := A \cap B \in \tau$, $A \rightarrow B := \overset{\circ}{B \setminus A} \in \tau$, $0 = \emptyset \in \tau$, $1 = X \in \tau$.

Definition 24. We can define a negation in Heyting algebra, putting $\neg a := a \rightarrow 0$ (this is compatible with the Boolean case); however, it does not satisfy the principle of double negation and other principles (tertium non datur, reductio ad absurdum, De Morgan's laws).

Remark 10. Given a topos $\mathcal{E}$, it is possible to define, for any $X \in \text{Ob}(\mathcal{E})$, operations on its subobjects, which are similar, in some sense, with the union, intersection, complementary, empty set, total set from the case **SET** (and are exactly these operations for **SET**). In a general topos, for any $a, b \in \text{Sub}_{\mathcal{E}}(X)$ (= the family of subobjects of X) one defines: union $a \cup b$ (using the existence of the equalizers and direct sums), intersection $a \cap b$ (using the existence of pullbacks), implication $a \rightarrow b$ (using the existence of the exponentiations), minimal object $= 0 \cap X$, and maximal object $= 1 \cap X$. In the case of **SET**, the usual operations between sets gives a structure of Boolean algebra. In a general topos, the previous operations gives a structure of a Heyting algebra. We obtain an analogue of the power set of X . We suppose all these things as known. For details, see [4,42] (the constructions are included in 6.37–6.42 from [1] also).

Remark 11. For the topos **SET**, the subobjects classifier is $\Omega = \{0, 1\}$ (via the usual characteristic function), and, in **CL** and **ZF(C)** axiomatics, the number of truth values is the cardinal of $\text{Hom}_{\text{SET}}(\mathbf{1}, \Omega)$ (here $\mathbf{1}$ =any singleton), that is 2. For a general topos, on $\text{Hom}_{\text{SET}}(\mathbf{1}, \Omega)$ one can define logic operators $\vee, \wedge, \rightarrow, \perp, \top$ using the so called Ω -axiom appearing in the definition of the subobjects classifier. Then, $(\Omega, \vee, \wedge, \rightarrow, \perp, \top)$ is a Heyting algebra, in general. When it is a Boole algebra, one says that the topos $\mathcal{E}$ is a Boolean topos. We again assume these things as known. Details, again in [4,42] (the constructions are also included in 6.43, 6.44 from [1]). Moreover, one can give to Ω a structure of MHA also. See [45,46] (referred in [4], p. 457). See also 6.51–6.53 from [1].

Remark 12. Fix a language $\mathcal{L}$, and, for a given topos $\mathcal{E}$ having Ω as subobjects classifier, we assign to any atomic sentence a from $\mathcal{L}$ (see Definitions 2 and 3) an element $\mathbf{V}(a)$ from $\text{Hom}_{\mathcal{E}}(\mathbf{1}, \Omega)$. Then, extend this association $\mathbf{V}$ to all the sentences from $\mathcal{L}$, using the logical connectors for $\mathcal{L}$, and, correspondingly, the operators from Remark 11 in $\text{Hom}_{\mathcal{E}}(\mathbf{1}, \Omega)$ ($\mathbf{V}(p \vee q) = \mathbf{V}(p) \vee \mathbf{V}(q)$, with the $\vee$ from the left being the logical connector, and $\vee$ from the right being the connector from Remark 11, similarly for the other operations $\wedge$, etc.). This extends the known Boolean valuations. We remark that $\text{card}(\text{Hom}_{\mathcal{E}}(\mathbf{1}, \Omega))$ represents the number of truth values in $\mathcal{E}$ (possible infinite). They are called the global sections of Ω .

Example 5. Consider the **SET**-type topos $\mathcal{E}$, whose objects are $\{(A_1, A_2, A_3, A_4) \mid A_1 \subset A_2, A_3 \subset A_4, A_1 \subset A_3, A_2 \subset A_4\}$, with morphisms quadruplets of functions, compatible with the existing inclusions. In this cases $\text{card}(\text{Hom}_{\text{SET}_2}(\mathbf{1}, \Omega)) = 6$, the sections being $\{(0, 0, 0, 0) < (0, 0, 0, 1) < (0, 0, 1, 1) < (0, 1, 0, 1) < (0, 1, 1, 1) < (1, 1, 1, 1)\}$ (lexicographical order); the characteristic function of the quadruplet of sets is the corresponding quadruplet of usual characteristic functions (for sets).

Thus, the intuitionistic logic in this topos has six truth values. In such a way, we can build as many examples as we want. In particular we can prove that, for any natural number n , there is a topos whose intuitionistic logic has exactly n truth values. One example of such a topos is the topos having as objects the n -uples of sets $\{(A_1, \dots, A_n) \mid A_1 \subset A_2 \subset \dots \subset A_n\}$ and as morphisms n -uples of functions, compatible with the inclusions. The truth values are $\{(0, \dots, 0) < (0, \dots, 0, 1) < (0, \dots, 0, 1, 1) < \dots < (1, \dots, 1)\}$, so $(n + 1)$ truth values. For more details and examples, see [4,42] (see also 5.46–5.52 and 6.45–6.53 from [1]).

6. Non-Standard Analysis in SET-Type Topoi

Let us begin this Section by shortly recalling (in Section 6.1) the notion of *non-standard enlargement* of a standard universe, denoted by $\mathcal{U} \subset {}^*\mathcal{U}$, where $\mathcal{U} = \mathcal{U}(T)$ is as in Definition 10 (with its mathematics), as introduced by Abraham Robinson in [9]. This works in the category (topos) **SET**. Modern details can be found in [6]. See also [47,48]. Most of the results from [6] are also presented in 7.20–7.45 from Section 7 of [1]. Then, in Section 6.2, we extend the things to **SET**-type topoi, introducing for the first time non-standard extensions for such kind of topoi. These are original results from pure mathematical point of view.

6.1. Non-Standard Enlargement in SET

Let $\mathcal{U} = \mathcal{U}(T) \xrightarrow{*} \mathcal{U}' = \mathcal{U}(T')$ ($*(T) \subset T'$) a monomorphism (injective map) of universes. Let us denote by ${}^*\mathcal{U} := {}^*(\mathcal{U}) \subseteq \mathcal{U}'$ and by *a the image of a , ${}^*a := {}^*(a)$, $\forall a \in \mathcal{U}$. For a sentence φ in $\mathcal{L}_{\mathcal{U}}$, let us denote by ${}^*\varphi$ the sentence in $\mathcal{L}_{\mathcal{U}'}$ obtained by keeping unchanged the primary relations $\in$ and $=$, as well as the logical quantifiers $\forall, \wedge, \neg, \rightarrow, \exists, \forall, \exists$, and replacing all the constant symbols $a \in \mathcal{U}$ by ${}^*a \in \mathcal{U}'$. One requests that $*$ satisfies the axioms:

- ${}^*a = a, \forall a \in T$ (if $\mathcal{U} = \mathcal{U}(T)$, for $T = \emptyset$ this condition does not exist);
- ${}^*\emptyset = \emptyset$;
- S, T : An $\mathcal{L}_{\mathcal{U}}$ -sentence φ is true $\Leftrightarrow {}^*\varphi$ is true as $\mathcal{L}_{\mathcal{U}'}$ -sentence; the $\Rightarrow$ is called **transfer (T)**, and $\Leftarrow$ is called **standardization (S)**;
- **I (idealization)**: If $\mathcal{R} \in \mathcal{U}$ is any *finitely satisfiable* binary relation (i.e., $\forall n \forall x_1, \dots, x_n \exists y (yRx_1 \wedge \dots \wedge yRx_n)$), then ${}^*\mathcal{R}$ is *globally satisfiable* (i.e., $\exists y' \forall x (y' \mathcal{R} x)$). We should remark that the x s and y run inside $\mathcal{U}$, while y' runs through $\mathcal{U}'$ ($*x = x, \forall x \in \text{dom } \mathcal{R}, y \in \mathcal{U}, y' \in \mathcal{U}'$). We note that the order of the variables is not important, because, if $\mathcal{R} = \mathcal{R}(x, y)$ is a binary relation, so is $\mathcal{R}(y, x) = \mathcal{R}^{-1}$.

In the previous situation, one call the extension $\mathcal{U} \xrightarrow{*} \mathcal{U}'$, a non-standard extension of $\mathcal{U}$. We note that the inclusion $\mathcal{U}(T) \subset \mathcal{U}(*T)$, ($*T \in \mathcal{U}'$) is a non-standard extension ([6], Enlargement Theorem, p. 185).

Remark 13. 1. If $\mathcal{U} = \mathcal{U}(T)$ and $\mathcal{U}' = \mathcal{U}(T')$ are universes, with $T \subset T'$, then take T'' such that $T \subset T'', T' \subset T''$ and let $\mathcal{U}'' = \mathcal{U}(T'')$. Then $\mathcal{L} := \mathcal{L}_{\mathcal{U}} = \mathcal{L}_{\mathcal{U}''}|_{\mathcal{U}}$ and $\mathcal{L}' := \mathcal{L}_{\mathcal{U}'} = \mathcal{L}_{\mathcal{U}''}|_{\mathcal{U}'}$ (notations from Definition 11). We call **ZFC + CL + I + S + T := IST** (internal set theory), with **CL** from Definition 7, **ZFC** from Remark 1, and **I, S, T** from before. The triplet $(\mathcal{L} = (\mathcal{L}_{\mathcal{U}}, \mathcal{L}_{\mathcal{U}'}), \text{IST}, \mathcal{D} = \{\text{MP}, G, P\})$ (see Example 1) represents the Theory of non-standard extensions. We point out that this theory has a higher order logic, that is, it is possible to quantify over any entity.

2. If $\mathcal{U}$ has **ZFC** then ${}^*\mathcal{U}$ has **IST** as axiomatics. Moreover, a sentence φ in $\mathcal{L}_{\mathcal{U}}$ conserves its truth value (0 or 1) when passing to ${}^*\varphi$ in ${}^*\mathcal{U}$. We say that the extension $\mathcal{U} \xrightarrow{*} {}^*\mathcal{U}$ is conservative. Moreover, $\mathcal{U}$ is not contradictory if ${}^*\mathcal{U}$ is (because we can construct models for ${}^*\mathcal{U}$ starting from $\mathcal{U}$ with **ZFC**, see [6], or look at [1] 7.38–7.40).

Remark 14. One can define the sets ${}^*\mathbb{N}, {}^*\mathbb{Z}, {}^*\mathbb{Q}, {}^*\mathbb{R}, {}^*\mathbb{C}$ of the non-standard number sets. In particular, in ${}^*\mathbb{R}$ one obtains the notions of infinitesimal, unlimited, infinite, etc., numbers. We also can obtain further properties of non-standard numbers. See [6], or look at 7.28 from [1].

6.1.1. Some Properties of $*$

(For more of them, see [6], or look at 7.29 from [1]):

- $a = b$ if ${}^*a = {}^*b$ so $a \mapsto {}^*a$ is injective;
- $a \in B$ if ${}^*a \in {}^*B$;
- $A \subseteq B$ if ${}^*A \subseteq {}^*B$;
- ${}^*(a_1, \dots, a_m) = ({}^*a_1, \dots, {}^*a_m), \forall m$;
- ${}^*\{a_1, \dots, a_m\} = \{{}^*a_1, \dots, {}^*a_m\}, \forall m$.

Definition 25. A set $B \in \mathcal{U}'$ is called standard if $\exists A \in \mathcal{U}$ with $B = {}^*A$, otherwise it is called non-standard. A non-standard set $B \in {}^*\mathcal{U}$ is called internal if $\exists A \in \mathcal{U}$ with $B \in {}^*A$ and external otherwise.

6.1.2. Some Properties of Internal sets

(For more proprieties, examples and further developments, see [6], or look at 7.31–7.45 from [1]):

- A, B internal $\Rightarrow A \cap B, A \cup B, A \setminus B, A \times B$ internal;
- The union and intersection of any internal collection of sets are internal: if $\{A_i | i \in I\} \in {}^*\mathcal{U}$ and A_i internal for any i , then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are internal;
- If $a_1, \dots, a_m$ are internal, then both $\{a_1, \dots, a_m\}$ and $(a_1, \dots, a_m)$ are internal;
- If $f : A \rightarrow B$ is internal (that is, the graph of f is internal, as a set) $a \in \text{dom } f = A$ internal, then $f(a)$ is internal; $D \subset B$ internal, then $f^{-1}(D)$ is internal.

Example 6. 1. $\mathbb{N}$ is an external set in $\mathcal{U}'$.

2. ${}^*(A) := \{^*a | a \in A\}$, A infinite is external (note that ${}^*(A) \neq {}^*A$).

6.2. Non-Standard Analysis in SET-Type Topoi

We start now to achieve our main objective from the mathematical point of view, namely to build a non-standard analysis (NSA) in topoi, extending the previous theory from the classical setting (in the topos **SET**) to more general topoi. We will be only interested in building NSA in the exponential topoi of the form **SET**^{cop} (such kind of topoi were also used in quantum physics). However, for the moment we will give the general definitions, but we will exemplify the ideas on the **SET**-type topoi (see Definition 21).

Remark 15. Let $\mathcal{E}$ be a topos. In Remark 11, we recalled the existence of all the logical operations in the space of sections $\text{Hom}_{\mathcal{E}}(1, \Omega)$, such that this space become a MHA (see Definition 23). Hence, we can interpret the notion of theory (in the sense of the Definition 6) in any topos.

Definition 26. Let $\mathcal{E}$ be a topos, and $X \in \text{Ob}(\mathcal{E})$. Then, we can associate to any two morphisms $\alpha \in \text{Hom}_{\mathcal{E}}(1, X)$ and $\beta \in \text{Hom}_{\mathcal{E}}(1, X)$ their direct product $(\alpha, \beta) \in \text{Hom}_{\mathcal{E}}(1, X \times X)$ (1 is (the) final object, so $1 \times 1 = 1$). Then, a binary relation $\mathcal{R}$ on X is a set of pairs (α, β) as before. As per usual, we write $\alpha \mathcal{R} \beta$ instead of $(\alpha, \beta) \in \mathcal{R}$. We say that a binary relation on X is finitely satisfiable if for any positive integer n and $\alpha_1, \dots, \alpha_n \in \text{Hom}_{\mathcal{E}}(1, X)$ there is $\beta \in \text{Hom}_{\mathcal{E}}(1, X)$, such that $\alpha_1 \mathcal{R} \beta, \dots, \alpha_n \mathcal{R} \beta$.

Definition 27. Let $\mathcal{E}$ be a topos. An enlargement of $\mathcal{E}$ is a monomorphism ${}^* : \mathcal{E} \hookrightarrow {}^*\mathcal{E}$ in a category **CAT** containing both categories $\mathcal{E}$ and ${}^*\mathcal{E}$, satisfying:

- 1. ${}^*\mathcal{E}$ is a topos;
- 2. If $\mathbf{V} : \mathcal{P} \rightarrow \text{Hom}_{\mathcal{E}}(1, \Omega)$ is any valuation (see Remark 12) (here $\mathcal{P}$ is the set of sentences in the language $\mathcal{L}_{\mathcal{E}}$, 1 is the final object in $\mathcal{E}$, and Ω is the subobjects classifier from $\mathcal{E}$), then ${}^*\mathbf{V}({}^*p) = \mathbf{V}(p)$ for any $p \in \mathcal{P}$ (here ${}^*\mathbf{V} : {}^*\mathcal{P} \rightarrow {}^*\text{Hom}_{\mathcal{E}}(1, {}^*\Omega)$ is the non-standard image of the original map $\mathbf{V}$, $1 = {}^*1$ is the final object, and ${}^*\Omega$ is the subobjects classifier from ${}^*\mathcal{E}$);
- 3. For any $X \in \text{Ob}(\mathcal{E})$ and any finitely satisfiable relation $\mathcal{R}$ on X there is $g \in {}^*\text{Hom}_{\mathcal{E}}(1, {}^*X)$, such that $\alpha {}^*\mathcal{R} g$ for any $\alpha \in \text{Hom}_{\mathcal{E}}(1, X)$ (we say that any finitely satisfiable relation is globally satisfiable) (if $\mathcal{R} \subseteq \text{Hom}_{\mathcal{E}}(1, X \times X)$ then ${}^*\mathcal{R} \subseteq {}^*\text{Hom}_{\mathcal{E}}(1, {}^*X \times {}^*X)$).

Definition 28. The inequality ${}^*\mathbf{V}({}^*p) \leq \mathbf{V}(p)$ from 2. is called standardization (**S**), the converse inequality is called transfer (**T**), and the propriety 3. is called idealization (**I**). They generalize the similar proprieties from **CL** appearing in Section 6.1, and can be formalized in the appropriate language.

Theorem 1. For any SET - type topos $\mathcal{E}$ (see Definition 21) there is a non-standard extension (in the sense of Definition 27) $*$: $\mathcal{E} \hookrightarrow * \mathcal{E}$.

Proof. Step 1. Let us consider the topos $\mathbf{SET}^n$ having as objects n-uples of sets, and as morphisms n-uples of functions, with the canonical compositions and identities, defined component-wise. At this step, we will build a non-standard extension for this particular topos. This will be the topos $*\mathbf{SET}^n$, having as objects n-uples $(*A_1, \dots, *A_n)$ where $(A_1, \dots, A_n)$ runs through the objects of $\mathbf{SET}^n$, and as morphisms n-uples of internal functions $(*f_1, \dots, *f_n)$, where $(f_1, \dots, f_n)$ runs through the morphisms of $\mathbf{SET}^n$ (with $*$ being the usual non-standard extension from Section 6.1). Due to the property of transfer (T) of $*$ (see Section 6.1) and because of Sections 6.1.1 and 6.1.2 we obtain that all the proprieties of $\mathbf{SET}^n$ contributing to its structure of topos transfers to $*\mathbf{SET}^n$. In particular, $\Omega_{*\mathbf{SET}^n} = *\Omega_{\mathbf{SET}^n} = (\{0, 1\}, \dots, \{0, 1\})$ (so both topoi have 2^n truth values).

Then, $\mathbf{SET}^n \xrightarrow{*} *\mathbf{SET}^n$ and this is a conservative extension of topoi.

More precisely, one can build a NSA on $\mathbf{SET}^n$ by imitating the case SET. Namely, in $\mathbf{SET}^n$ we define a union, intersection, and power set on components, namely:

$$(A_1, \dots, A_n) \cup (B_1, \dots, B_n) := (A_1 \cup B_1, \dots, A_n \cup B_n)$$

$$(A_1, \dots, A_n) \cap (B_1, \dots, B_n) := (A_1 \cap B_1, \dots, A_n \cap B_n)$$

$$\mathcal{P}(A_1, \dots, A_n) := (\mathcal{P}(A_1), \dots, \mathcal{P}(A_n)).$$

Then, we can build a universe $\mathcal{U}(X)$ using the same procedure as in Definition 10 (here $X = (X_1, \dots, X_n)$). In fact, we obtain that $\mathcal{U}(X) = (\mathcal{U}(X_1), \dots, \mathcal{U}(X_n))$, where $\mathcal{U}(X_i)$ are universes as in Definition 10. We can also build the non-standard extension $\mathcal{U}(X) \xrightarrow{*} \mathcal{U}(*X) = \mathcal{U}(*X_1, \dots, *X_n) = (\mathcal{U}(*X_1), \dots, \mathcal{U}(*X_n))$.

We extend the language to $\mathcal{L}_{\mathcal{U}(X)}$ (Definition 11) and we obtain that $\mathcal{L}_{\mathcal{U}(X)} = (\mathcal{L}_{\mathcal{U}(X_1)}, \dots, \mathcal{L}_{\mathcal{U}(X_n)})$; this can be further component-wise extended to $\mathcal{L}_{\mathcal{U}(*X)}$.

With Definition 25 in mind, one can define in $\mathbf{SET}^n$ the notions of standard and non-standard object on components (the standard objects are of the form $(*X_1, \dots, *X_n)$, with $X_1, \dots, X_n$ sets; the non-standard objects have a hierarchy of non-standardness, depending on how many components are non-standard, and which ones—one can rank them using the lexicographical order on n-uples); one can define, for the non-standard objects, the internal and the external objects, namely an object $(X_1, \dots, X_n)$ is internal if all the sets X_i are internal; and, $(X_1, \dots, X_n)$ is external if at least one of the X_i is external (as in the case of the non-standard sets, one finds several degrees of externality, lexicographically ordered).

A sentence in $\mathcal{U}(X)$ consists of an n-uple of sentences from $\mathcal{U}(X_1), \dots, \mathcal{U}(X_n)$, in this order. The degree of truth of a sentence in $\mathcal{U}(X)$ depends on the number of the true sentences from the components, and on their order, with the lexicographical order on n-uples. The degree of truth of a sentence ϕ in $\mathcal{U}(X)$ is kept in $\mathcal{U}(*X)$ and vice-versa, due to T (transfer) and S (standardization) from the NSA on SET. We got in such a way a conservative extension of the intuitionistic (2^n valued) logic of the topos $\mathbf{SET}^n$. The idealization property (I) from NSA remains unchanged.

Step 2. It is easy to see that $\mathbf{SET}^n = \mathbf{SET}^{n^{op}}$, with n from Section 5.2.3, (2) (note that n is self-dual).

Step 3. Cf. Definition 21 a category of SET-type has the form $\mathbf{SET}^{\mathcal{C}^{op}}$, with $\mathcal{C}$ finite (that is having a finite number of objects and morphisms), so, there is $n \in \mathbb{N}$, such that n is a subcategory of $\mathcal{C}^{op}$ ($n = \text{card}(\text{Ob}(\mathcal{C})) = \text{card}(\text{Ob}(\mathcal{C}^{op}))$). Thus, our SET-type category is a subcategory of $\mathbf{SET}^n = \mathbf{SET}^n$. Now, it is not hard to check that the restriction of the monomorphism $*$ from Step 1 gives a non-standard extension of our SET-type topos. $\square$

7. Infons, Energons, and Receptons: Mathematical Definitions

This is a concluding section of all the previous work. As already written in the Introduction (Section 1) the next Section 8 represents an opening toward applications of braid groups in the context, which will be one of the subjects of the Part II of this work. In Section 7.1 we define the intuitionistic context, and in Section 7.2 we define mathematically the infons, receptons, and energons, providing some examples of receptons.

7.1. The Intuitionistic Context

Definition 29. A set is a class which is an element of another class (see Definition 1). Let us denote by $\mathcal{M}$ the class of sets. The intuitionistic set theory (**IZF**) is the triplet $(\mathcal{L}_{\mathcal{M}}, \mathbf{IL} + \mathbf{IZF}, \mathcal{D} = \{\mathbf{MP}\})$, where $\mathbf{IL}$ are the axioms from Definition 22, $\mathbf{MP}$ is as in Example 1, and **IZF** (Intuitionistic Zermelo–Fraenkel) consists in the following axioms (recall that “ $\in$ ” and “ $=$ ” are primary relations) **IZF**: Axioms of **ZF** (see Remark 1 and 5.10–5.13 from [1]), where the axiom **Regularity (R)** is replaced by the weaker axiom (**Induction**):

$$\forall x((\forall y \in x \wedge \phi(y)) \rightarrow \phi(x)) \rightarrow \forall x \phi(x), \text{ for every formula } \phi.$$

Note that this in setting the double negation, de Morgan’s relations, and tertium non datur, do not hold.

Definition 30. An intuitionistic theory is an usual theory $\mathcal{T} = (\mathcal{L}, \mathcal{A}, \mathcal{D})$ (see Definition 6) in an intuitionistic context, so assuming that $\mathcal{A}$ contains the axioms **IL** and **IZF** from before, and that $\mathcal{D} = \{\mathbf{MP}\}$ (see Example 1). As in Definition 6, we assume, by definition, that a theory has at least one model otherwise it is not considered a theory in our sense).

Definition 31. Using the same notations as in Remark 13, but with **IZF** from Definition 29 for intuitionistic set theory, we denote by **IIST** = **IL** + **IZF** + **I** + **S** + **T** with **IL** from Definition 22 and **I**, **S**, **T** from Definition 28. As in Remark 13 we define the theory of intuitionistic non-standard extensions, replacing **IST** by **IIST** and taking only $\mathcal{D} = \{\mathbf{MP}\}$ from Example 1. The propositional calculus of this theory is a conservative extension of the intuitionistic logic (that is by **T** and **S** one conserve the truth values).

7.2. Infons, Receptons, Energons

Here are presented the essential definitions for which this work was done, namely:

Definition 32. A generalized infon is a Theory $\mathcal{I} = (\mathcal{L}, \mathcal{A}, \mathcal{D})$ in the context of **IIST**. An infon is an entity of the previous form, whose category of models form a non-standard enlargements of a topos of the form $\mathbf{SET}^{\mathcal{C}^{op}}$, where $\mathcal{C}$ is any category (see Section 5 also) and a preliminary defined infon is the infon whose associated category is a non-standard enlargement of a topos of **SET**-type (see Definition 21).

Correspondingly,

Definition 33. A generalized recepton is the category of models of a generalized infon, an recepton any non-standard enlargement of a topos of exponential type $\mathbf{SET}^{\mathcal{C}^{op}}$, where $\mathcal{C}$ is any category (see Section 5 also) and a preliminary defined recepton is any non-standard enlargement of a topos of **SET**-type (see Definition 21).

We recall in the context that the logic from topoi is intuitionistic, and its non-standard enlargement is a conservative extension (see Sections 5 and 6), so the infon and the recepton are compatible entities, representing the two facets of the same entity.

An energon is also generalized, normal and preliminary, as before. It is partially infon (when dealing with theories related to potential energy) and partially recepton (when dealing with the kinetic, caloric, electromagnetic actual energies), see also Section 3.

Example 7 (of preliminary defined receptons). (1). By definition, any non-standard enlargement of a **SET**-type topos is a recepton. Particular receptons are represented by the enlargement of the graph-type topos $\mathbf{SET}^{\mathbf{n}\vec{\mathcal{G}}^{op}}$ or by the enlargement of a geometric topos (see Sections 5.2.2 and 5.2.3).

(2). The Nicolas Bourbaki group of prominent mathematicians from France considered that the fundamental structures in mathematics are (both in the language of theories and of categories) the algebraic structures, topological structures, and order structures, all the others (like the differential ones, for example) being combinations between them.

Thus, in order to apply mathematics in various arias, one needs to consider collections of such structures, communicating between them. This means that we need to consider various theories and categories together. In the case of categories, the structures are collected in categories, and the communication between them is done with the help of the functors. This means that, in fact, one needs to work with categories of type **CAT** (see Remark 3), and, as a consequence, with appropriate infons and receptons.

This means that an important family of receptons will have the form $\mathbf{SET}^{\mathbf{CAT}^{op}}$ and similarly for the corresponding theories. For example, in multiversal quantum physics several mathematical structures appearing in the usual quantum physics were put together in such kind of topoi, such as Hilbert spaces with compact self adjoint operators, von Neumann algebras, and others (see [7]). However, when studying the human and global brain, is necessary to enlarge the things to the non-standard environment, due to the role played by the subconsciousness (see [1,19]).

8. The Yang–Baxter Equations

In forthcoming Part II of this work, we will present, in more detail, the connections between DNA, quantum computers, and the sequence knots, braid groups, and Yang–Baxter equations (YBE). Here, we begin with an introduction to the Yang–Baxter equation. After that we will present related results and applications of this equation.

Let V be a vector space over k . Let $I = I_V : V \rightarrow V$ be the identity map of the space V . We denote by $\tau : V \otimes_B V \rightarrow V \otimes_B V$ the twist map defined by $\tau(v \otimes_B w) = w \otimes_B v$.

Now, we will give the main notations for introducing the Yang–Baxter equation.

For $R : V \otimes_B V \rightarrow V \otimes_B V$ a k -linear map, let $R^{12} = R \otimes_B I$, $R^{23} = I \otimes_B R : V \otimes_B V \otimes_B V \rightarrow V \otimes_B V \otimes_B V$. In a similar manner, we denote by R^{13} a linear map acting on the first and third component of $V \otimes_B V \otimes_B V$. It turns out that $R^{13} = (I \otimes_B \tau)(R \otimes_B I)(I \otimes_B \tau)$.

Definition 34. A **Yang–Baxter operator** is a k -linear map $R : V \otimes_B V \rightarrow V \otimes_B V$, which satisfies the braid condition (the Yang–Baxter equation):

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \quad (1)$$

We also require that the map R is invertible.

Remark 16. Some examples of Yang–Baxter operators are the following: $R = \tau$ (i.e., $R(a \otimes_B b) = b \otimes_B a$), and $R = I \otimes_B I$ (i.e., $R(a \otimes_B b) = a \otimes_B b$).

Remark 17. An important observation is that if R satisfies (1) then both $R \circ \tau$ and $\tau \circ R$ satisfy the QYBE (the quantum Yang–Baxter equation):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \quad (2)$$

Remark 18. The Equations (1) and (2) are equivalent.

The Yang–Baxter equation is an important tool for the theory of quantum groups, knot theory, the theory of universal quantum gates, and the theory of quantum computers (see [49–55]).

Remark 19. There is a similar terminology for the set-theoretical Yang–Baxter equation. In this case V is replaced by a set X and the tensor product by the Cartesian product. We will explain this definition in the next examples below.

Remark 20. Let X be a set containing three (bivalent) logical sentences p, q, r (i.e., $p, q, r \in X$). We can choose X as rich as we wish for the moment. Later, we will try to find the smallest X which fits for our theory.

Let $R : X \times X \rightarrow X \times X$, be defined by $R(p, q) = (p \vee q, p \wedge q)$.

It follows that

$$(R \times I) \circ (I \times R) \circ (R \times I) = (I \times R) \circ (R \times I) \circ (I \times R). \quad (3)$$

One way to check that (3) holds is to make a table with values for p, q, r .

Remark 21. Another interesting solution to the set-theoretical Yang–Baxter equation is the following.

Let $R : X \times X \rightarrow X \times X$, be defined by $R(p, q) = (p \rightarrow q, p)$.

Again, one way to check the above statement is to make a table with values for p, q, r .

Let us denote $R(p, q) = (p_1, q_2)$. The set-theoretical Yang–Baxter equation can be expressed as $(p_{101}, q_{212}, r_{020}) = (p_{010}, q_{121}, r_{212})$.

It is now the moment to discuss about X . What can be said about X in general (what is the smallest X for which R is well-defined)? One could consider a set X containing more than three logical sentences. What is the interpretation of the set-theoretical Yang–Baxter solutions in this case?

For example, if $\mathcal{B}$ is a Boolean topos (see Remark 11) the pairs from Remark 21 gives solutions for the YBE (1) from the global sections of subobjects classifier Ω (i.e., $\text{Hom}_{\mathcal{B}}(1, \Omega)$) of $\mathcal{B}$, while the pairs from Remark 20 gives solutions from the global sections of subobjects classifier Ω (i.e., $\text{Hom}_{\mathcal{E}}(1, \Omega)$) for both equations YBE and QYBE ((1) and (2)) and for any topos $\mathcal{E}$.

We can go a step further and consider an algebra of type (2,2), $(A, *, \circ)$, and call the operations $*$ and $\circ$ YB conjugated if $R(a, b) = (a * b, a \circ b)$ satisfies the set-theoretical Yang–Baxter equation. We propose the study of algebras with two operations of type (2,2) which are YB conjugated. (Distributive lattices, groups and self-distributive structures fall into this category.)

Remark 22. We now consider other equations for a k -linear map $R : V \otimes_B V \rightarrow V \otimes_B V$:

$$R^{12} \circ R^{23} = R^{13} \circ R^{12} = R^{23} \circ R^{13} \quad (4)$$

$$R^{23} \circ R^{12} = R^{12} \circ R^{13} \quad (5)$$

$$R^{12} \circ R^{13} \circ R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{13} \circ R^{12} \circ R^{12} \circ R^{23} \quad (6)$$

$$(R^{12} \circ R^{23} \circ R^{12} - R^{23} \circ R^{12} \circ R^{23}) \circ (R^{12} \circ R^{13} \circ R^{23} - R^{23} \circ R^{13} \circ R^{12}) = 0 \quad (7)$$

Theorem 2. If a k -linear map $R : V \otimes_B V \rightarrow V \otimes_B V$ verifies (4) and (5) then R is a common solution for (1) and (2).

If a k -linear map $R : V \otimes_B V \rightarrow V \otimes_B V$ verifies (1) and (2) then R is a solution for (6).

If a k -linear map $R : V \otimes_B V \rightarrow V \otimes_B V$ verifies (1) or (2) then R is a solution for (7).

Proof. We only prove the first claim: $R^{23} \circ R^{12} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} = R^{12} \circ R^{23} \circ R^{12}$
 $R^{23} \circ R^{13} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} = R^{12} \circ R^{13} \circ R^{23}$

The other claims follow in a similar manner. $\square$

We will write the above results in the following manner:

$$(4) \wedge (5) \rightarrow (1) \wedge (2);$$

$$(1) \wedge (2) \rightarrow (6);$$

$$(1) \vee (2) \rightarrow (7).$$

As a direct application of this theorem, one can check which of the functions presented in this section are common solutions for the braid condition and the quantum Yang–Baxter equation.

We are now preparing the framework for the next theorem.

For an equation p (for example, $R^{12} \circ R^{23} \text{etc.} = 0$), we can associate $\neg p$ (for example, $R^{12} \circ R^{23} \text{etc.} \neq 0$).

Now we can consider the equation $\top$ the equation always true ($p \vee \neg p$) and by $\perp = \neg(\top)$ the equation with no solutions.

We now can state a theorem with applications in the understanding the theory of the Yang–Baxter equation.

Theorem 3. *The set of equations of operators of type $R : V \otimes_B V \rightarrow V \otimes_B V$ and $R^{ij} : V \otimes_B V \otimes_B V \rightarrow V \otimes_B V \otimes_B V$, with $i, j \in \{1, 2, 3\}$, has a natural distributive lattice structure. In this lattice, one can define a partial order, a minimum $\perp$, a maximum $\top$, and operations $\neg$ and $\rightarrow$.*

The Yang–Baxter equation represents some kind of compatibility condition in logic. Let us consider three logical sentences p ; q ; r , and let us suppose that if all of them are true, then the conclusion A could be drawn, and if p ; q ; r are all false, then the conclusion C can be drawn.

Modeling this situation by the map R , defined by $(p; q) \rightarrow (p \text{ or } q; p \text{ and } q)$, helps us to comprise our analysis. The Yang–Baxter equation guarantees that the order in which we start this analysis is not important (see [56]). How to obtain solutions for set-theoretic Yang–Baxter equations in a context of Boole or Heyting Algebras is also indicated in [52,57,58], or even in [59].

9. Conclusions and Further Considerations

To conclude, in this paper we obtain three kinds of contributions. One is related to pure mathematics and consist in constructing the non-standard analysis in **SET**-type topoi (Theorem 1); another one is related to foundations of mathematics; arguments from physics, neuroscience, cybernetics are used in order to justify the importance of introduction of the **IIST** axiomatics and to provide models for it (based on Theorem 1 again); and, last but not least, the introduction and justification of the infons, energons, receptons: how they should look like and which is their usefulness (Sections 2–4). We should also refer to non-mathematical original contributions: building a cybernetic model of the global brain (Section 4), inspired from original interpretations of the human brain (Section 2).

We will continue our work with part II, where we will build non-standard analysis in general exponential topoi, with applications to infons and receptons in their natural form. Then, these notions should be used in concrete problems of physics, neuroscience, cybernetics, artificial intelligence, economics, etc. In particular, we will study the (quantum) neural networks and (quantum) Turing machines in the context of topoi, related to the interplay of brain and mind (genetic and epigenetic) and in connection with the global brain. Braid groups will be also considered in the context.

Example 8. *We end this paper noting that the theory from Section 7 (**ZF** from Section 5 in the classical case) represents an infon, containing the empty set. From it (so, from nothingness), one built all the mathematics, by receiving the theory through various receptons, parts of the global brain. By Tegmark’s axiom for multiverses of rank 4 (see Section 3.1) all the physical entities progressively appear, through consciousness (and subconsciousness).*

Remark 23. *We might put the problem in which way the information from infon is received by recepton. We propose as a model of receiving the (quantum) neural networks, as defined in [60], modeled in the **IIST** axiomatics, in a non-standard extension of a special case of temporal topoi (of exponential type) as in [61]. These are not **SET**-type topoi, so we have to develop non-standard analysis in such topoi also, in the second part of the paper. The dendrites are connected to infons, and the axons to receptons.*

In our brains, the neural networks (synapses) help the concrete understanding of the existing entities, and in the global brain the artificial neural networks also plays an important role. Due to the work of R. Penrose [62–64] and S. Hameroff, we must consider

the quantum phenomena from the brain, at least partially, although there are discussions on their existence (see [65]); however, it seems that some phosphorus compounds extend the period of superposition in the worm and humid environment from the brain, with possible applications to quantum computers, which might work at the room temperature also, not necessarily close to 0 degree Kelvin).

Remark 24. *In the literature there are two more notions belonging to the family of words related to consciousness, namely superconsciousness and unconsciousness.*

(A) The word superconsciousness has several meanings: religious, related to meditation etc. (in general, belonging to the spiritual world). We are not interested now in these meanings, the definition which we use is that superconsciousness is the consciousness of the global brain, topic already considered in Section 4. We give three classes of examples of informations comming to us through superconsciousness, namely:

(i) the Human Genome Project (aiming to make a map of the information from a human ADN: genomics): <https://www.genome.gov/human-genome-project>;

(ii) the Human Brain Project (aiming to make a map of the neurons and synapses from a brain: connectomics): <https://www.humanbrainproject.eu/en/>, and <https://www.epfl.ch/research/domains/bluebrain/>;

(iii) to make the map of the known univers: <https://www.nature.com/articles/d41586-021-01466-1>.

In all these examples there is no scientist (or human being) to know all the information, but the specialists can see the maps (which are not yet ready, it is work in progress in all three cases). In order to produce these maps, networks of supercomputers are necessary (situated, preferably, in cloud, so with unknown locations, to avoid the hacking). So, the information comes from many scientists from many arias of research, and uses high technology, most probably without the help of quantum computers the works cannot be done. All these statements related with many human brains and technology are parts of the global brain. So, we become aware of these things with the help of global brain.

(B) The word unconsciousness has also several meanings, including in a medical sense. Here we consider it as denoting that information which exists, both in our brain (in neurons, synapses) and in ADN, but we become—maybe—aware of it only through the interplay between genetics and epigenetics (see Section 2). It can be mathematically modeled using both (quantum) neural networks and (quantum) Turing mashines. It can be specific both to a human being, and to the global brain (archetypes, cultural customs, etc.). For further information, see also [66,67].

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