

New Inequalities of Cusa–Huygens Type

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Abstract: Using the power series expansions of the functions $\cot x$, $1/\sin x$ and $1/\sin^2 x$, and the estimate of the ratio of two adjacent even-indexed Bernoulli numbers, we improve Cusa–Huygens inequality in two directions on $(0, \pi/2)$. Our results are much better than those in the existing literature.

Keywords: sharp the double inequalities of Cusa–Huygens type; circular functions; Bernoulli numbers

1. Introduction

For $x \in (0, \pi/2)$, we know that the functions $\cos x$ and $(\sin x)/x$ are less than 1. In order to confirm the relationship between $(\sin x)/x$ and the weighted arithmetic mean of $\cos x$ and 1, we can examine the Taylor expansion of the following function:

$$\begin{aligned} \frac{\sin x}{x} - [(1 - \beta) + \beta \cos x] &= \sum_{n=1}^{\infty} (-1)^n \frac{1 - \beta(2n+1)}{(2n+1)!} x^{2n} \\ &= x^2 \left(\frac{1}{2} \beta - \frac{1}{6} \right) - x^4 \left(\frac{1}{24} \beta - \frac{1}{120} \right) + \sum_{n=3}^{\infty} (-1)^n \frac{1 - \beta(2n+1)}{(2n+1)!} x^{2n}. \end{aligned}$$

Obviously, when choosing $\beta = 1/3$ we can get the following fact:

$$\frac{\sin x}{x} - \left(\frac{2}{3} + \frac{1}{3} \cos x \right) = -\frac{1}{180} x^4 + \frac{1}{3780} x^6 + O(x^8),$$

which inspires us to prove that for $0 < x < \pi/2$,

$$\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \cos x \quad (1)$$

or

$$\frac{3 \sin x}{2 + \cos x} < x. \quad (2)$$

The existing mathematical historical data (see [1–8]) show that the above inequality (2) was discovered by Nicolaus De Cusa (1401–1464) using a geometrical method in 1451 and was later in 1664 confirmed by CHRISTIAN HUYGENS (1629–1695) when considering the estimation of π . Because of the contribution of Nicolaus De Cusa and Christian Huygens to this inequality (1), we call it *Cusa–Huygens inequality*. Recently, Zhu [9] provided two improvements of (2) as follows.

Proposition 1 ([9]). *The inequalities,*

$$\frac{1}{180} x^5 < x - \frac{3 \sin x}{2 + \cos x} \quad (3)$$

and

$$\frac{1}{2100} x^7 < x - \frac{3 \sin x}{2 + \cos x} \left[1 + \frac{(1 - \cos x)^2}{9(3 + 2 \cos x)} \right] \quad (4)$$



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hold for all $x \in (0, \pi]$, where $1/180$ and $1/2100$ are the best constants in previous inequalities, respectively.

The results of the previous proposition are corrections of Theorem 3.4.20 from monograph Mitrinović [7]. Malešević et al. made a bilateral supplement to the above two inequalities.

This paper focuses on the improvement of (1). Chen and Cheung [10] gave the bounds for $(\sin x)/x$ in terms of $((2 + \cos x)/3)^\delta$ as follows:

$$\left(\frac{2 + \cos x}{3}\right)^{\theta_0} < \frac{\sin x}{x} < \left(\frac{2 + \cos x}{3}\right)^{\vartheta_0} \quad (5)$$

holds for all $0 < x < \pi/2$, where $\vartheta_0 = 1$ and $\theta_0 = (\ln \pi - \ln 2)/(\ln 3 - \ln 2)$ are the best possible constants in (5). The double inequality (5) was proved by Bagul [11] and Zhu [12] in different ways. In Zhu [12] the inequality (1) was found to be true for the broader interval $(-\infty, 0) \cup (0, \infty)$. There are many useful discussions in the literature about the above inequality (1) and its related topics; for interested readers, please refer to [13–46].

In Zhu [12], we can find three new improvements to inequality (1):

$$\left(1 - \frac{x^3}{\pi^3}\right) \frac{2 + \cos x}{3} < \frac{\sin x}{x} < \left(1 - \frac{x^4}{180}\right) \frac{\cos x + 2}{3}, \quad (6)$$

$$\left[1 + \frac{8(\pi - 3)}{\pi^3}x^2\right] \frac{2 + \cos x}{3} - \frac{8(\pi - 3)}{\pi^3}x^2 < \frac{\sin x}{x} < \left(1 + \frac{1}{30}x^2\right) \frac{2 + \cos x}{3} - \frac{1}{30}x^2, \quad (7)$$

and

$$\begin{aligned} & \left[1 + \frac{1}{30}x^2 + \frac{2(240\pi - \pi^3 - 720)}{15\pi^5}x^4\right] \frac{2 + \cos x}{3} - \left[\frac{1}{30}x^2 + \frac{2(240\pi - \pi^3 - 720)}{15\pi^5}x^4\right] \\ & < \frac{\sin x}{x} \\ & < \left(1 + \frac{1}{30}x^2 + \frac{1}{840}x^4\right) \frac{2 + \cos x}{3} - \left(\frac{1}{30}x^2 + \frac{1}{840}x^4\right), \end{aligned} \quad (8)$$

hold for $0 < x < \pi/2$.

Bercu [47] used the truncations of the Fourier cosine series to the inequality (1) and obtained an enhanced form of (1):

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{45}(1 - \cos x)^2, \quad 0 < x < \frac{\pi}{2}. \quad (9)$$

Recently, Bagul et al. [48] drew two conclusions about the improvement of inequality (1):

$$-\left(\frac{2}{3} - \frac{2}{\pi}\right) \frac{1}{(\pi/2 - 1)}(x - \sin x) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi}\right) \frac{1}{(\pi/2 - 1)^2}(x - \sin x)^2 \quad (10)$$

and

$$-\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x \cos x) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x \cos x)^2 \quad (11)$$

hold for $0 < x < \pi/2$.

Inspired by inequalities (9)–(11), this paper intends to improve the famous inequality (1) from two different directions and to draw two results as follows.

Theorem 1. Let $0 < x < \pi/2$. Then

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{180}x^4 \left(\frac{\sin x}{x}\right)^{2/7} \quad (12)$$

holds with the best constant $-1/180$.

Theorem 2. Let $0 < x < \pi/2$. Then

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x \cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)} \quad (13)$$

holds with the best constant $(2/3 - 2/\pi)$.

In this paper, we use the power series expansions of two functions $\cot x$ and $1/\sin x$ and their derivative functions to prove the main conclusions. We know that the Taylor coefficients of these power series expansions are closely related to the Bernoulli number, which is related to the Riemann zeta function through the following identity:

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \quad n \in \mathbb{N}.$$

The latest research information on the Riemann zeta function can be found in Milovanović and Rassias [46].

2. Lemmas

Lemma 1. Let B_{2n} be the even-indexed Bernoulli numbers. Then (see [49–51])

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad (14)$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n-2}, \quad (15)$$

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n-1}, \quad (16)$$

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n}-2)}{(2n)!} |B_{2n}| x^{2n-2} \quad (17)$$

hold for all $x \in (0, \pi)$.

Proof. From

$$\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)' \quad \text{and} \quad \frac{\cos x}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)',$$

the power series expansions (15) and (17) follow. $\square$

Lemma 2 ([52–54]). Let B_{2n} be the even-indexed Bernoulli numbers, $n = 1, 2, \dots$. Then

$$\frac{2^{2n-1}-1}{2^{2n+1}-1} \frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1} \frac{(2n+2)(2n+1)}{\pi^2}. \quad (18)$$

To prove our results, we also need the monotone form of the L'Hospital rule shown in [55–57] and the criterion for the monotonicity of the quotient of power series shown in [58].

Lemma 3 ([55–57]). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous functions, which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If f'/g' is increasing (or decreasing) on (a, b) , then

the functions $(f(x) - f(b^-))/(g(x) - g(b^-))$ and $(f(x) - f(a^+))/(g(x) - g(a^+))$ are also increasing (or decreasing) on (a, b) .

Lemma 4 ([58]). Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$ ($R \leq +\infty$). If $b_n > 0$ for $n = 0, 1, 2, \dots$, and if $\epsilon_n = a_n/b_n$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $(0, R)$ ($R \leq +\infty$).

Lemma 5. Let $0 < x < \pi/2$, and

$$\varphi = \frac{\ln(2/\pi)}{\ln(2^{3/2}/\pi)} = 4.3004\dots, \quad \phi = 4.$$

Then

$$\left(\frac{\sin(x/2)}{(x/2)}\right)^{\varphi} < \frac{\sin x}{x} < \left(\frac{\sin(x/2)}{(x/2)}\right)^{\phi}$$

holds with the optimal exponents ϕ and φ .

Proof. Let

$$Q(x) = \frac{\ln(\frac{\sin x}{x})}{\ln \frac{\sin \frac{x}{2}}{\frac{x}{2}}} =: \frac{a(x)}{b(x)}, \quad 0 < x < \frac{\pi}{2}.$$

Then

$$\begin{aligned} a'(x) &= -\frac{1}{x \sin x} (\sin x - x \cos x), \\ b'(x) &= -\frac{1}{2x \sin \frac{x}{2}} \left(2 \sin \frac{x}{2} - x \cos \frac{x}{2}\right) = -\frac{1}{2x \sin \frac{x}{2} \cos \frac{x}{2}} \left(2 \sin \frac{x}{2} \cos \frac{x}{2} - x \cos^2 \frac{x}{2}\right) \\ &= -\frac{1}{x \sin x} \left(\sin x - x \cos^2 \frac{x}{2}\right) = -\frac{1}{2x \sin x} \left(2 \sin x - 2x \cos^2 \frac{x}{2}\right) \\ &= -\frac{1}{2x \sin x} [2 \sin x - x(\cos x + 1)], \end{aligned}$$

and

$$\begin{aligned} \frac{a'(x)}{b'(x)} &= \frac{-\frac{1}{x \sin x} (\sin x - x \cos x)}{-\frac{1}{2x \sin x} [2 \sin x - x(\cos x + 1)]} = 2 \frac{\sin x - x \cos x}{2 \sin x - x(\cos x + 1)} \\ &= 2 \frac{\frac{\sin x - x \cos x}{\sin x}}{\frac{2 \sin x - x(\cos x + 1)}{\sin x}} =: 2 \frac{A(x)}{B(x)}, \end{aligned}$$

where

$$\begin{aligned} A(x) &= \frac{\sin x - x \cos x}{\sin x} = 1 - x \cot x = 1 - x \left[\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \right] \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n}, \\ B(x) &= \frac{2 \sin x - x(\cos x + 1)}{\sin x} = 2 - x \cot x - \frac{x}{\sin x} \\ &= 2 - x \left[\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \right] - x \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \right] \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} - \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} \frac{2}{(2n)!} |B_{2n}| x^{2n}. \end{aligned}$$

Since

$$\frac{\frac{2^{2n}}{(2n)!} |B_{2n}|}{\frac{2}{(2n)!} |B_{2n}|} = \frac{\frac{2^{2n}}{(2n)!}}{\frac{2}{(2n)!}} = 2^{2n-1}$$

is increasing for $n \geq 1$, by Lemma 4 we have that $a'(x)/b'(x)$ is increasing on $(0, \pi/2)$. By Lemma 3 we get that the function $Q(x) = a(x)/b(x) = [a(x) - a(0^+)]/[b(x) - b(0^+)]$ is increasing on $(0, \pi/2)$. At the same time, we find that

$$\lim_{x \rightarrow 0^+} Q(x) = 4, \quad \lim_{x \rightarrow (\pi/2)^-} Q(x) = \frac{\ln \frac{2}{\pi}}{\ln \frac{2^{3/2}}{\pi}} = 4.3004\dots$$

This completes the proof of Lemma 5. $\square$

Lemma 6 ([59,60]). Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$, and

$$S(t) = - \sum_{k=0}^m a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0, r)$ ($r > 0$). Then the following statements are true:

- (1) If $S(r^-) \leq 0$, then $S(t) < 0$ for all $t \in (0, r)$;
- (2) If $S(r^-) > 0$, then there exists $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

3. Proofs of Main Results

3.1. Proof of Theorem 1

The desired conclusion is equivalent to

$$\begin{aligned} & \frac{2 + \cos x}{3} - \frac{\sin x}{x} > \frac{1}{180} x^4 \left(\frac{\sin x}{x} \right)^{2/7} \\ \Leftrightarrow & \left(\frac{2 + \cos x}{3} - \frac{\sin x}{x} \right)^7 > \left(\frac{1}{180} x^4 \left(\frac{\sin x}{x} \right)^{2/7} \right)^7 = \frac{x^{26} \sin^2 x}{6122\,200\,320\,000\,000} \\ \Leftrightarrow & 7 \ln \left(\frac{2 + \cos x}{3} - \frac{\sin x}{x} \right) > \ln \left(\frac{x^{26} \sin^2 x}{6122\,200\,320\,000\,000} \right). \end{aligned}$$

Let

$$F(x) = 7 \ln \left(\frac{2 + \cos x}{3} - \frac{\sin x}{x} \right) - \ln \left(\frac{x^{26} \sin^2 x}{6122\,200\,320\,000\,000} \right),$$

where $0 < x < \pi/2$. Then

$$F'(x) = \frac{\sin x}{3x^2 \left(\frac{2 + \cos x}{3} - \frac{\sin x}{x} \right)} f(x),$$

where

$$\begin{aligned} f(x) &= \frac{(99 - 7x^2) \sin^2 x - 2x^2 \cos^2 x - 41x \cos x \sin x - 4x^2 \cos x - 52x \sin x}{\sin^2 x} \\ &= 99 - 5x^2 - 2x^2 \frac{1}{\sin^2 x} - 41x \frac{\cos x}{\sin x} - 4x^2 \frac{\cos x}{\sin^2 x} - 52 \frac{x}{\sin x}. \end{aligned}$$

By substituting the power series expansions of all functions involved in Lemma 1 into

$f(x)$, we obtain that

$$f(x) = \sum_{n=4}^{\infty} \frac{(4n-13)2^{2n} - (16n-112)}{(2n)!} |B_{2n}| x^{2n} > 0$$

due to $(4n-13)2^{2n} - (16n-112) > 0$ for $n \geq 4$, which can be proved by mathematical induction. So $F'(x) > 0$ for all $x \in (0, \pi/2)$. Then $F(x)$ increases on $(0, \pi/2)$. Therefore, $F(x) > F(0^+) = 0$. At the same time, we find that

$$\lim_{x \rightarrow 0^+} \frac{\frac{2+\cos x}{3} - \frac{\sin x}{x}}{x^4 \left(\frac{\sin x}{x}\right)^{2/7}} = \frac{1}{180}.$$

Then the proof of Theorem 1 is complete.

3.2. Proof of Theorem 2

The desired conclusion is equivalent to

$$\frac{2+\cos x}{3} - \frac{\sin x}{x} > \left(\frac{2}{3} - \frac{2}{\pi}\right) (\sin x - x \cos x)^{\frac{\pi^2-12}{3\pi^2-\pi^3}}, \quad 0 < x < \frac{\pi}{2}.$$

Let

$$G(x) = \ln\left(\frac{2+\cos x}{3} - \frac{\sin x}{x}\right) - \ln\left(\frac{2}{3} - \frac{2}{\pi}\right) - \frac{\pi^2-12}{3\pi^2-\pi^3} \ln(\sin x - x \cos x), \quad 0 < x < \frac{\pi}{2}.$$

Then

$$G(0^+) = \infty, \quad G\left(\left(\frac{\pi}{2}\right)^-\right) = 0,$$

and

$$G'(x) = -\frac{(\sin^2 x)g(x)}{\pi^2 x(2x - 3 \sin x + x \cos x)(\sin x - x \cos x)(\pi - 3)},$$

where

$$\begin{aligned} (\sin^2 x)g(x) &= 2(12 - \pi^2)x^3(\sin x) + x(\sin x \cos x) \left[(2\pi^2 - \pi^3 + 12)x^2 - 18\pi^2 + 6\pi^3 \right] \\ &\quad + (9\pi^2 - 3\pi^3)x^2 + (\sin^2 x) \left[(4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \right]. \end{aligned}$$

The proof of $G'(x) < 0$ on $(0, \pi/2)$ is complete when proving $g(x) > 0$ on $(0, \pi/2)$. In fact, by Lemma 1 we have

$$\begin{aligned} g(x) &= 2(12 - \pi^2)x^3 \frac{1}{\sin x} + x(\cot x) \left[(2\pi^2 - \pi^3 + 12)x^2 - 18\pi^2 + 6\pi^3 \right] \\ &\quad + (9\pi^2 - 3\pi^3) \frac{x^2}{\sin^2 x} + \left[(4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \right] \\ &= 2(12 - \pi^2)x^3 \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \right] \\ &\quad + x \left[\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \right] \left[(2\pi^2 - \pi^3 + 12)x^2 - 18\pi^2 + 6\pi^3 \right] \\ &\quad + (9\pi^2 - 3\pi^3)x^2 \left[\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n-2} \right] \\ &\quad + (4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \end{aligned}$$

$$\begin{aligned}
&= 2x^2(12 - \pi^2) + 2x^3(12 - \pi^2) \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \\
&\quad + \left[6\pi^3 - 18\pi^2 + (2\pi^2 - \pi^3 + 12)x^2 \right] \left[1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \right] \\
&\quad - 3\pi^2(\pi - 3) \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} \right] \\
&\quad + (4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \\
&= 2x^2(12 - \pi^2) + 2(12 - \pi^2) \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n+2} \\
&\quad + (6\pi^3 - 18\pi^2 + (2\pi^2 - \pi^3 + 12)x^2) \\
&\quad - (6\pi^3 - 18\pi^2 + (2\pi^2 - \pi^3 + 12)x^2) \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\
&\quad - 3\pi^2(\pi - 3) \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} \right] \\
&\quad + (4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \\
&= 2x^2(12 - \pi^2) + 2(12 - \pi^2) \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n+2} + (6\pi^3 - 18\pi^2 + (2\pi^2 - \pi^3 + 12)x^2) \\
&\quad - (6\pi^3 - 18\pi^2) \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} - (2\pi^2 - \pi^3 + 12) \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n+2} - 3\pi^2(\pi - 3) \\
&\quad - 3\pi^2(\pi - 3) \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} + (4\pi^3 - 9\pi^2 - 36)x^2 + 9\pi^2 - 3\pi^3 \\
&= 2(12 - \pi^2) \sum_{n=2}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n+2} - (6\pi^3 - 18\pi^2) \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\
&\quad - (2\pi^2 - \pi^3 + 12) \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n+2} - 3\pi^2(\pi - 3) \sum_{n=3}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} \\
&= 2(12 - \pi^2) \sum_{n=2}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n+2} - (2\pi^2 - \pi^3 + 12) \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n+2} \\
&\quad - 3\pi^2(\pi - 3) \sum_{n=3}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} - (6\pi^3 - 18\pi^2) \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\
&= 2(12 - \pi^2) \sum_{n=3}^{\infty} \frac{2^{2n-2} - 2}{(2n-2)!} |B_{2n-2}| x^{2n} - (2\pi^2 - \pi^3 + 12) \sum_{n=3}^{\infty} \frac{2^{2n-2}}{(2n-2)!} |B_{2n-2}| x^{2n} \\
&\quad - 3\pi^2(\pi - 3) \sum_{n=3}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| x^{2n} - (6\pi^3 - 18\pi^2) \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\
&=: \sum_{n=3}^{\infty} a_n x^{2n},
\end{aligned}$$

where

$$\begin{aligned} a_n &= 2\left(12 - \pi^2\right) \frac{2^{2n-2} - 2}{(2n-2)!} |B_{2n-2}| - \left(2\pi^2 - \pi^3 + 12\right) \frac{2^{2n-2}}{(2n-2)!} |B_{2n-2}| \\ &\quad - 3\pi^2(\pi - 3) \frac{2^{2n}(2n-1)}{(2n)!} |B_{2n}| - \left(6\pi^3 - 18\pi^2\right) \frac{2^{2n}}{(2n)!} |B_{2n}| \\ &= \frac{(\pi^3 + 12 - 4\pi^2)2^{2n} + 16\pi^2 - 192}{4(2n-2)!} |B_{2n-2}| - \frac{3(\pi - 3)\pi^2(2n+1)2^{2n}}{(2n)!} |B_{2n}|. \end{aligned}$$

We can compute to get

$$a_3 = \frac{1}{20}\pi^2 - \frac{1}{45}\pi^3 + \frac{1}{5} \approx 4.4518 \times 10^{-3} > 0,$$

and prove $a_n < 0$ for $n \geq 4$. The latter is equivalent to

$$\frac{|B_{2n}|}{|B_{2n-2}|} > \frac{[(\pi^3 + 12 - 4\pi^2)2^{2n} + 16\pi^2 - 192](2n)(2n-1)}{12(\pi - 3)\pi^2 2^{2n}(2n+1)}. \quad (19)$$

By Lemma 2, we have

$$\frac{|B_{2n}|}{|B_{2n-2}|} > \frac{2^{2n-3} - 1}{2^{2n-1} - 1} \frac{(2n)(2n-1)}{\pi^2},$$

so the proof of (19) is complete when proving that for $n \geq 4$,

$$\frac{2^{2n-3} - 1}{2^{2n-1} - 1} \frac{(2n)(2n-1)}{\pi^2} > \frac{[(\pi^3 + 12 - 4\pi^2)2^{2n} + 16\pi^2 - 192](2n)(2n-1)}{12(\pi - 3)\pi^2 2^{2n}(2n+1)},$$

that is,

$$\frac{2^{2n-3} - 1}{2^{2n-1} - 1} > \frac{(\pi^3 + 12 - 4\pi^2)2^{2n} + 16\pi^2 - 192}{12(\pi - 3)(2n+1)2^{2n}},$$

or

$$\begin{aligned} & \left(2^{2n-3} - 1\right)12(\pi - 3)(2n+1)2^{2n} - \left(2^{2n-1} - 1\right)[(\pi^3 + 12 - 4\pi^2)2^{2n} + 16\pi^2 - 192] \\ &= \frac{1}{2}h(n) > 0, \end{aligned}$$

where

$$\begin{aligned} h(n) &= 2^{4n} - 2^{2n} \frac{2[24n(\pi - 3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6n(\pi - 3) + 3\pi + 4\pi^2 - \pi^3 - 21} \\ &\quad - \frac{2(192 - 16\pi^2)}{6n(\pi - 3) + 3\pi + 4\pi^2 - \pi^3 - 21} \end{aligned} \quad (20)$$

with

$$\begin{aligned} h(4) &= \frac{32(-53568\pi - 8001\pi^2 + 2032\pi^3 + 183564)}{-27\pi - 4\pi^2 + \pi^3 + 93} \approx 74463 > 0, \\ h(5) &= \frac{32(-1072896\pi - 130305\pi^2 + 32704\pi^3 + 3605004)}{-33\pi - 4\pi^2 + \pi^3 + 111} \approx 1.0519 \times 10^6 > 0, \\ h(6) &= \frac{32(-20407296\pi - 2094081\pi^2 + 524032\pi^3 + 67485708)}{-39\pi - 4\pi^2 + \pi^3 + 129} \approx 1.6771 \times 10^7 > 0. \end{aligned}$$

Since

$$2^{2n} \frac{2[24n(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6n(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21} > \frac{2(192 - 16\pi^2)}{6n(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21},$$

we complete the proof of $h(n) > 0$ when proving

$$2^{2n} > \frac{4[24n(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6n(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21} \quad (21)$$

for $n \geq 6$. We proved (21) by mathematical induction. When $n = 6$, this inequality (21) is obviously true. Now let us say that (21) is true for $n = m$, so

$$2^{2m} > \frac{4[24m(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6m(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21}. \quad (22)$$

Since

$$2^{2m+2} = 4 \cdot 2^{2m} > \frac{16[24m(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{[6m(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21]},$$

we just proved that (21) is true for $n = m + 1$ when proving

$$\begin{aligned} & \frac{16[24m(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6m(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21} \\ & > \frac{4[24(m+1)(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6(m+1)(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21}, \end{aligned}$$

or

$$\begin{aligned} \frac{A}{B} & : = \frac{4[24m(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6m(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21} \\ & > \frac{[24(m+1)(\pi-3) + 12\pi + 12\pi^2 - \pi^3 - 144]}{6(m+1)(\pi-3) + 3\pi + 4\pi^2 - \pi^3 - 21} := \frac{C}{D}, \end{aligned}$$

which is equivalent to

$$AD - BC$$

$$\begin{aligned} & = 939\pi^3 - 2736\pi^2 - 5652\pi + 99\pi^4 - 48\pi^5 + 3\pi^6 + 17,928 \\ & \quad - (7992\pi + 648\pi^2 - 774\pi^3 + 90\pi^4 - 16,200)m + (432\pi^2 - 2592\pi + 3888)m^2 \\ & = (8928\pi^2 - 146,916\pi + 5583\pi^3 - 441\pi^4 - 48\pi^5 + 3\pi^6 + 255,096) \\ & \quad + (4536\pi^2 - 39,096\pi + 774\pi^3 - 90\pi^4 + 62,856)(m-6) \\ & \quad + (432\pi^2 - 2592\pi + 3888)(m-6)^2 \\ & > 0 \end{aligned}$$

for $m \geq 6$. So those coefficients in the power series of $-g(x)$ satisfy the conditions of Lemma 6, and $-g((\pi/2)^-) = 0$. From Lemma 6, it follows that $-g(x) < 0$ on $(0, \pi/2)$. Therefore, $g(x) > 0$ on $(0, \pi/2)$. So $G'(x) < 0$ for all $x \in (0, \pi/2)$. Then $G(x)$ decreases on $(0, \pi/2)$. Then $G(x) > G((\pi/2)^-) = 0$ for all $x \in (0, \pi/2)$. At the same time, we find that

$$\lim_{x \rightarrow 0^+} \frac{\frac{2+\cos x}{3} - \frac{\sin x}{x}}{(\sin x - x \cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)}} = \frac{2}{3} - \frac{2}{\pi}.$$

Then the proof of Theorem 2 is complete.

4. Remarks

In this section, we compare new conclusions (12) and (13) with (9)–(11).

Remark 1. The inequality (12) is better than the one (9) because

$$\begin{aligned} -\frac{1}{180}x^4\left(\frac{\sin x}{x}\right)^{2/7} &< -\frac{1}{45}(1-\cos x)^2 \\ \iff x^4\left(\frac{\sin x}{x}\right)^{2/7} &> 4(1-\cos x)^2 = [2(1-\cos x)]^2 \\ \iff x^2\left(\frac{\sin x}{x}\right)^{1/7} &> 2(1-\cos x) = \left(2\sin\frac{x}{2}\right)^2 \\ \iff \left(\frac{\sin x}{x}\right)^{1/7} &> \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \\ \iff \left(\frac{\sin x}{x}\right) &> \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^{14}. \end{aligned}$$

The last inequality follows from Lemma 5 due to

$$\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^\varphi > \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^{14} \iff \frac{\sin\frac{x}{2}}{\frac{x}{2}} < 1,$$

where

$$\varphi = \frac{\ln(2/\pi)}{\ln(2^{3/2}/\pi)} = 4.3004\dots$$

Remark 2. It is pointed out in [48] that

$$\begin{aligned} \frac{\sin x}{x} - \frac{2+\cos x}{3} &< -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x\cos x)^2 \\ &< -\left(\frac{2}{3} - \frac{2}{\pi}\right)\frac{1}{(\pi/2-1)^2}(x-\sin x)^2. \end{aligned}$$

Now we can obtain that the inequality (13) is better than the right-hand side ones of (10) and (11), that is,

$$\begin{aligned} \frac{\sin x}{x} - \frac{2+\cos x}{3} &< -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x\cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)} \\ &< -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x\cos x)^2 \end{aligned}$$

holds for all $x \in (0, \pi/2)$ due to

$$\begin{aligned} -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x\cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)} &< -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x\cos x)^2 \\ \iff (\sin x - x\cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)} &> (\sin x - x\cos x)^2 \\ \iff 0 < \sin x - x\cos x &< 1. \end{aligned}$$

Remark 3. We can find that the inequality (12) is better than the right-hand side one of (13) on $(0, 1.4117)$ while (13) is better than the one (12) on $(1.4117, \pi/2)$. So the inequality (12) has the advantage on the left-hand side of the interval $(0, \pi/2)$ and the advantage of the inequality (13) lies near this point $\pi/2$.

Remark 4. We also note that the following inequality conclusion appears in [29]:

$$\frac{\sin x}{x} < \frac{2 + \cos(x/2^n)}{3} \prod_{k=1}^n \cos(x/2^k), \quad n = 0, 1, \dots \quad (23)$$

holds for all $x \in (0, \pi/2)$. From (23) we have that the inequality

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < \frac{2 + \cos(x/2^n)}{3} \prod_{k=1}^n \cos(x/2^k) - \frac{2 + \cos x}{3}, \quad n = 0, 1, \dots \quad (24)$$

holds for all $x \in (0, \pi/2)$. In particular, letting $n = 3$ in the above inequality gives that for all $x \in (0, \pi/2)$,

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < \frac{2 + \cos(x/2^3)}{3} \cos(x/2^3) \cos(x/2^2) \cos(x/2) - \frac{2 + \cos x}{3}. \quad (25)$$

It is not hard to find that the inequality (25) is better than the one (13) on $(0, 1.5446)$ but the inequality (13) is stronger than the one (25) on $(1.5446, \pi/2)$. In other words, the advantage of (25) even (24) is on the left-hand side of the interval $(0, \pi/2)$ while the advantage of (13) is near the right endpoint $\pi/2$.

5. Conclusions

In this paper, using the power series expansions of the functions $\cot x$, $1/\sin x$, and $1/\sin^2 x$, and the estimate of the ratio of two adjacent even-indexed Bernoulli numbers, we obtained some new Cusa–Huygens type inequalities, which greatly improve known results.

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