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# Using Wolfram Alpha with Elementary Teacher Candidates: From More Than One Correct Answer to More Than One Correct Solution

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**Abstract:** The paper is a reflection on the author's work with students enrolled in undergraduate and graduate elementary mathematics content and methods courses. Two specific pedagogical issues make up the focus of the paper. The first issue deals with demonstrating to future teachers the diversity of mathematical ideas behind a contextual question asked by a second-grade pupil allowing for multiple solution strategies to be used in addressing the question. The second issue deals with the use of Wolfram Alpha in aiding different mathematical features of this demonstration. The paper is congruous with mathematics teaching standards used across six continents and illustrated by reflective comments of the author's students regarding their mathematics teacher education experiences. Teaching ideas shared in the paper may be of interest to instructors who want to explore elementary mathematics in depth with teacher candidates.

**Keywords:** partition of integers; permutations; generating function; young children; teacher education; action learning; problem posing; Wolfram Alpha

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## 1. Introduction

The goal of this paper is to reflect on the use of Wolfram Alpha—a computational knowledge engine developed by Wolfram Research—with teacher candidates enrolled in elementary mathematics education courses taught by the author. The importance of reflection was emphasized by John Dewey who, as a philosopher of education, saw in this type of thinking "not simply a sequence of ideas, but a *con*-sequence—a consecutive ordering in such a way that each determines the next as its proper outcome, while each outcome in turn leans back on, or refers to, its predecessor" [1] (p. 4, italics in the original). From the position of educational psychology, writing a reflective paper represents an attempt to move "from action to the representation of the action—that is, from just being able to do something to thinking about the doing of that thing" [2] (p. 225). In that sense, reflections on action rooted in personal teaching experience as mentioned by Artigue in [3] (p. 314) "is necessarily subjective". However, in the context of this paper, presumed subjectivity of the reflections by the author is balanced by referring to students' reflective thinking which, due to its diversity, is less subjective and, most importantly, "impels to inquiry" [1] (p. 7). Mathematics education researchers investigate mathematical thinking "through reflection on the status of mathematical constructs" [4] (p. 2), attest that schoolteachers' experience of learning mathematics "becomes grist for reflection" [5] (p. 65) and emphasize the "pivotal role of reflections in mathematising and didactising" [6] (p. 163).

Nowadays, mathematical constructs that affect one's mathematical thinking are provided by the various tools of technology. Both physical and digital tools are essential components of students' mathematics learning experience, assist them in mathematising and enhance didactising by the instructor. That is, the goal of reflections is to improve the quality of teaching "by reconstructing on a higher level what existed on a lower level [because] the higher level is always a wider, more all-embracing field, so that when one reflects onto a higher level, it is incumbent upon one to enrich it with new elements" [2] (p. 225). In that sense, a paper based on the instructor's reflections on mathematics teaching, enhanced by their students' thinking about the learning experience, provides a knowledge base and, with respect to research objective, offers suggestions as one of "the essential functions of reflective activity" [1] (p. 106) for different groups of researchers and practitioners of education interested in comparing their own reflections on the practice of mathematics education with that of their colleagues.

This paper has its origin in a project originally designed for a group of second grade students administered by an elementary teacher candidate as an extension of their student teaching experience. The project was supervised by two education faculty members [7]. Later, the project was administered by several teacher candidates as part of their pre-student teaching. This interdisciplinary project dealt with the context of weather change and included the concepts of range and average (arithmetic mean, introduced as fair sharing) facilitated by the use of integrated spreadsheets [8]. Young children were told that, in a course of five days, the average temperature has increased by one degree, and the temperature increase was recorded during at most two days. A fair sharing of five degrees among five days means the average temperature increased by one degree over five days. Although temperature is commonly considered a continuous variable, second graders only know whole numbers and they are used to seeing such numbers in weather forecasts on TV or online. That is why, in the context of this project, it was tacitly assumed (i.e., without sharing with the second graders) that temperature (just as length, height and weight) is a discrete variable allowing for didactically motivated transition to other discrete concepts discussed in this paper.

Thus, the second graders were supposed to find ways of decomposing the number five in a sum of two numbers from the range [0, 5] in all possible orders. Mathematically, they were expected to represent the number 5 in the following six ways.

$$5 = 5 + 0,\ 5 = 0 + 5,\ 5 = 4 + 1,\ 5 = 1 + 4,\ 5 = 3 + 2,\ 5 = 2 + 3 \quad (1)$$

Note that decomposing a number into a sum of two numbers is an activity which is in accordance with Common Core State Standards, one of the major educational documents in the United States at the time of writing this paper, formulated for kindergarten students as follows, "decompose numbers less than or equal to 10 into pairs in more than one way" [9] (p. 11). Nonetheless, during the project, it was discovered that even second graders had difficulty comprehending the notion of a question (problem) with more than one correct answer. In other words, the children were unable to overcome the concreteness of a single day and to grasp the spread of five-degree temperature increase over several days. They thought that there was only one day when the temperature went up by five degrees. Perhaps the need to coordinate a rather sophisticated real-life context (average temperature increase within a five day range) mapped onto a problem with more than one correct answer and the use of the commutative property of addition were the main reasons why second graders had difficulty with what was expected from kindergarten students; that is, to come up with relations (1). This posed an interesting didactical problem for the author—how can one make an additive decomposition of an integer a natural outgrowth of a tactile activity?

A didactical intervention into the complexity of the situation was designed as follows: *Find all the ways to place five finger rings on the index and the middle fingers*. For this activity, the second grade students were given finger rings bought at a local Dollar store. Experimentally, they were able to find the answer (see the left-hand side of Figure 1). With the appropriate guidance of elementary teacher candidates that were more knowledgeable others [10], they constructed an isomorphic model of the "5 rings-2 fingers" situation and recorded their findings by drawing pictures similar to those shown in the left-hand side of Figure 1. While concrete thinking was found to be a barrier in the way of grasping a rather abstract concept of multiple representations by the second graders, concrete activity was used as a means of grasping the idea of such representations. Using the W$^4$S—we write what we see—principle of transition from visual to symbolic [11], in other words, from the "first-order symbols directly denoting objects or actions [to] the second-order symbolism, which involves the creation of written signs for the spoken symbols of words" [10] (p. 115), they applied their findings to describe possible two day temperature changes in the form of relations (1).

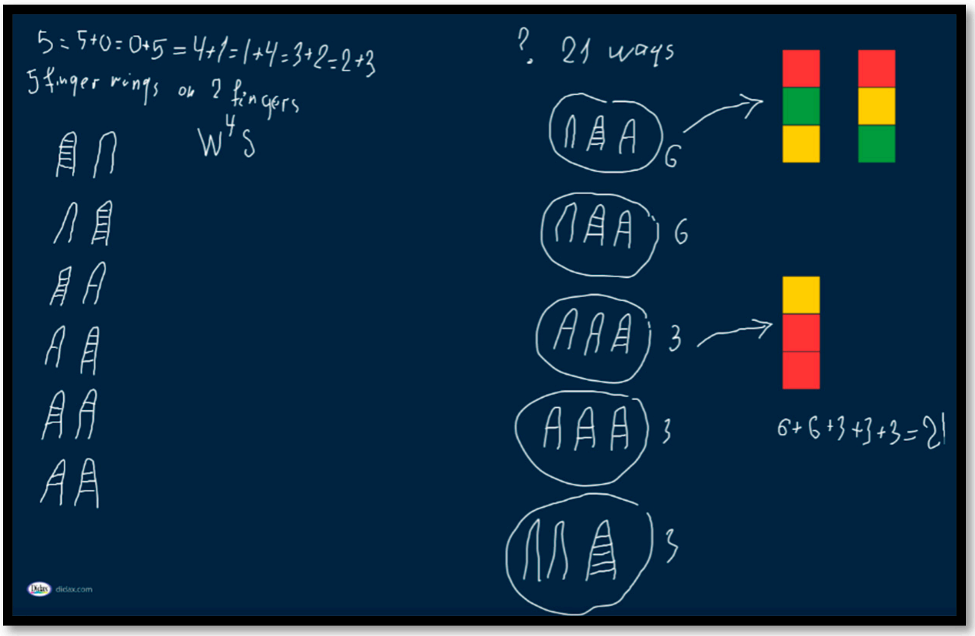


**Figure 1.** A screenshot from a class presentation of the question about three fingers.

## 2. Materials and Methods

Two types of materials have been used by the author when working on this paper. The first type includes teaching and learning mathematics standards from Australia, Canada, Chile, England, Singapore, South Africa, South Korea and the United States. The author's accentuation of the worldwide standards, despite their wide-ranging endorsement being debatable, is at least three-fold. First, writing for an international journal while mentioning only standards of the country in which one teaches is kind of amateurish. Second, not referencing current teaching and learning standards at all in a mathematics education paper would not be tenable. Third, while there exist multiple arguments toward more context-based and authentic mathematics curricula, it appears that regardless of the direction a country takes in the teaching of mathematics, the importance of multiple solution strategies, enabling teachers "to give full attention to alternative possibilities" [1] (p. 30), is not debatable. With this in mind, the author will reference the worldwide standards almost exclusively in the specific context of this paper.

The second type of material is digital—computational knowledge engine Wolfram Alpha, more specifically, its Pro version for educators. The standards uniformly call for the teaching of multiple solution strategies in solving a single problem, for problem solving using technology, for encouraging students to ask questions while expecting teachers not to reject a challenge coming from students' questions, to make mathematical connections through multiple representations and to pose new problems.

Methods specific for mathematics education used in this paper include computer-based mathematics education, standards-based mathematics and problem solving. The university where the author prepares elementary teacher candidates (referred to below as the candidates) to teach mathematics is located in upstate New York in close proximity to Canada, and many of the author's students are Canadians pursuing their master's degrees in education. This diversity of students suggests the importance of aligning mathematics education courses with multiple international perspectives on teaching and learning elementary mathematics. There are two mathematics courses that the candidates have to take: a course to satisfy mathematical prerequisites and a course which is a combination of content and methods of teaching this content. Activities described and analyzed in the paper are aimed at developing conceptual understanding of mathematics [12] in the candidates and may be characterized as action learning [13]. Action learning, sometimes called action research [14], begins with a query into a real problem, and it continually emphasizes conceptual learning, regardless of whether the problem is aimed at pragmatic or epistemic development. The paper is supported by solicited reflections by the candidates in which they describe their experience using Wolfram Alpha when working on various assignments and by helping their own children in elementary grades with homework.

## 3. From A Second Grade Classroom to A Mathematics Teacher Education Course

How can one extend inquiry into the five degree temperature change to three, four or even the entire five day period? In fact, such question (using a captivating context for young children) was immediately asked by one of the second graders—*how many ways can one place five finger rings on three fingers?* This question is an example of what may be called collateral creativity when students become creative due to an unintended outcome of being introduced to a specially designed problem-solving environment. How can this question be answered in a second grade classroom? How can such question be answered in a teacher education classroom using multiple solution strategies? These issues were discussed with the candidates enrolled in a mathematics content and methods course taught both at the undergraduate and graduate levels as well as in a graduate level mathematics course designed as a prerequisite for a childhood teacher education program. The aim of such a discussion was to show how, in the digital era, the problem with more than one correct answer can be used to demonstrate the existence of multiple solutions for obtaining such answers. That is, the concept of multiple representations of a number as a sum of two numbers was chosen to serve as a rudiment for a more complicated concept of multiple solution strategies. While researchers found that traditional school culture of teaching mathematics is to use problems the answers to which are simple, quick and accurate [15–18], the manifold of learning opportunities to be used by teachers in the mathematics classroom includes a variety of approaches to solving a single problem—a topic of worldwide mathematics education research [19–22]. The research is consistent with worldwide mathematics teaching standards. For example, teachers in England "engage students by showing them that there was more than one way of doing things" [23] (p. 18). In the United States, teachers "examine different solution paths for the same problem" [24] (p. 33). In South Africa, teachers "always encourage learners to check their solutions using a variety of methods" [25] (p. 31). A search for an alternative method in solving a problem may result in the development of a new concept and, in the words of Singaporean educators, "to promote greater diversity and creativity in learning" [26] (p. 17). One of the goals of mathematics education programs in Chile is to make sure "that teachers are able to use a wide variety of tools that allow them to address problems from different points of view" [27] (p. 104). In the age of technology, problem solving can be significantly enhanced by the use of computational experiments as "a clear and attractive alternative to traditional paper-and-pencil skills wherever arithmetic computation is required" [28] (p. 241). The candidates can be introduced to such experiments as prompts through which answers to problems (questions) can emerge as means of computational verification and modeling that result in a solution. The following are the words of a candidate: *"Not all students' brains follow the same path to understanding. Often students struggle with solving a problem the way the teacher had done it. Sometimes it is necessary to explore an alternate way of solving a problem. Other times it is just as effective to explain the same method in an alternate manner. Discovering one effective path for a student, helps to clarify the original method. As a teacher, the more ways I have of solving a problem, the more options I have for getting my students to understand."* Another candidate noted, "*I do believe that maybe this type of math class should be something taught to a new teacher like in the case of teaching the many different ways of solving a problem. A teacher may* *only know one way, and if that's all they know, how are they going to teach any other way. And this could hurt the student, later on, placing them even farther behind from a child that had a teacher that was well trained in math.*"

While multiple software tools can be used as means of computational experiments, the focus is on a single tool in this paper, which provides a variety of representations of a mathematical concept. This tool is Wolfram Alpha, a cutting-edge computational knowledge engine developed by Wolfram Research and is available for free online (https://www.wolframalpha.com/, accessed on 28 August 2021). Wolfram Alpha Pro, available for a relatively inexpensive subscription, provides more information in the cases requiring the program to respond to higher level queries of mathematical complexity. In order to describe this program and its educative usefulness, the following comment from a candidate is worth mentioning: "*My experience in using Wolfram Alpha has been a very positive experience. I have actually used this website to assist my 9-year-old daughter and 10-year-old son with some math problems they have brought home for homework. I also shared this website with their teachers. Their teachers had never heard of it and they were very impressed as well. I liked the website so much that I paid for the subscription*". In what follows, more solicited reflections on mathematics learning experience by the candidates will be shared.

## 4. From Informal to Formal Reasoning

The candidates need to be informed of what young children are capable of when asking mathematical questions and must have experience and knowledge on how "to deal with difficult questions posed by students" [27] (p. 101). Since questions are the major means that drive the learning of mathematics around the world [23–27,29–32], such discussions are an important part of the courses taught by the author. An interesting comment, indicative of the traditional view of a mathematics classroom rooted in the false dichotomy between the teacher's and the student's roles in the learning process, was provided by a candidate: "*Before this class, I never really thought you could have conversations or open-ended questions when it came to math. This made math more enjoyable and fun for me, and I hope to pass this on to my future students*." Figure 1 shows how, through a *conversation* [33], one can explain to the candidates on a picture that there are 21 ways to place five finger rings on three fingers. The right-hand side of Figure 1 includes the use of what is already known to the candidates from the context of kindergarten curriculum, "classify objects and count the number of objects in each category" [9] (p. 12), which is a problem about making different three-cube towers out of three cubes of different colors [34] (p. 19). From a pretty convincing statement that there are two towers categorized by red cube at the top, it follows that there are two towers in each of two other categories, that is, with green and yellow cubes at the top. A similar demonstration relates to the use of only two different colors in creating a three-cube tower. At a more advanced cognitive level, a tree diagram (not taught until grade three) can be used to demonstrate possible permutations of cubes or fingers with rings (Figures 2 and 3). In order to "help students see connections among representations and see affordances of different representations" [35] (p. 16), the candidates themselves need to have experience with different representations of a concept and participate in a classroom discussion of what each representation entails. This discussion is consistent with action learning approach to the teaching of mathematics when pragmatic query into a real problem about weather was resolved by taking an action mediated by finger rings as thinking tools and reflected upon under the umbrella of conceptual learning towards epistemic development of the candidates.

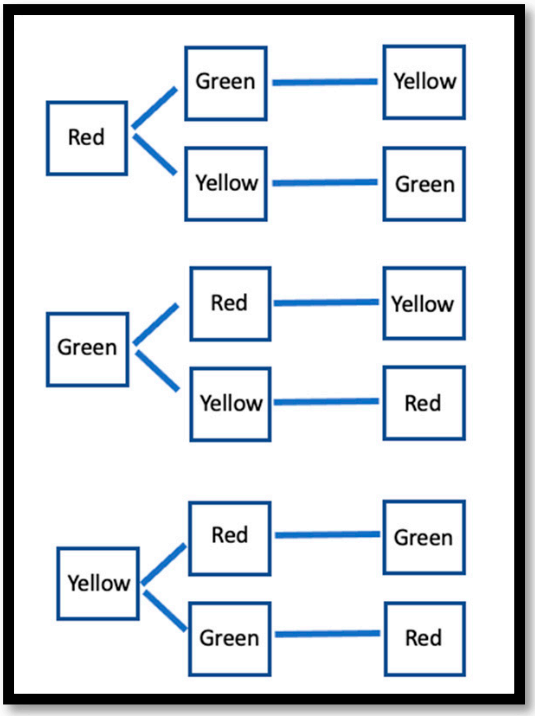


**Figure 2.** A tree with three colors follows the rule of product.

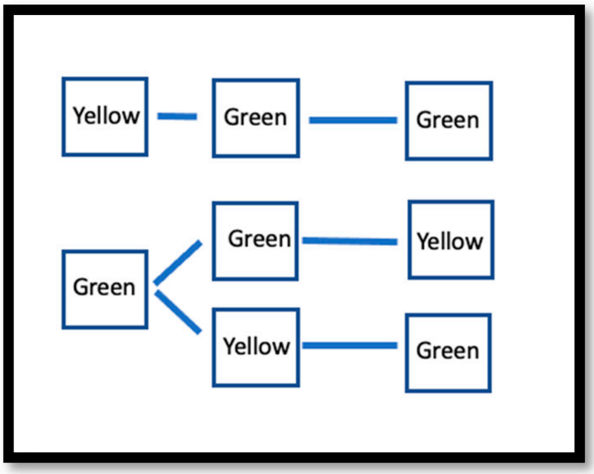


**Figure 3.** A tree with two colors follows the rule of sum of products.

Recall that a tree diagram is a multiplicative structure used to organize counting according to the rule of product: If object A can be selected in $m$ ways and, following this selection, object B can be selected in $n$ ways, then the ordered pair of objects (A, B) can be selected in $m \times n$ ways. As shown in Figure 2, there are three ($m$ = 3) ways to select the bottom cube for a three-cube tower and once such selection is made, there are two ($n$ = 2) ways to place a cube above the first one. Therefore, there are $3 \times 2 = 6$ ways to select the pair (bottom and middle) for a tower. For each such selection, there is only one cube left. Alternatively, a three-word pair selected out of three different words (without their repetition) enables six permutations of the words. A tree diagram is a means of conceptualizing and, already in the case of four colors the diagram, has to be completely redrawn rather than extended.

Figure 3 represents a more complicated tree diagram, which follows what may be called the rule of sum of products. The rule can be described in terms of the selection of an ordered pair (A, B) as follows. If object A can be selected in $m$ ways $k$ of which allow an object B to be selected in $n_1$ ways and the remaining $m - k$ selections of A allow for $n_2$ selections of B, then the ordered pair (A, B) can be selected in $k \times n_1 + (m - k) \times n_2$ ways. In Figure 3, we have $m = 2$, $k = 1$, $n_1 = 1$ and $n_2 = 2$ such that $k \times n_1 + (m - k) \times n_2 = 1 \times 1 + (2 - 1) \times 2 = 3$. Indeed, there are three methods to permute the cubes in a two colored three-cube tower.

## 5. Using a Tree Diagram as a Sign of The Einstellung Effect

The candidates often try to apply a tree diagram as a counting tool in the situations that do not follow the rule of product because, as Pólya [36] (p. 63) put it, "Human nature prompts us to repeat a procedure that has succeeded before in a similar situation". In other words, such behavior evinces the so-called Einstellung effect [37], known also as negative transfer—when successful problem-solving experience of the past results in "habitual rigidity of thinking" [38] (p. 167). In fact, in attempting to answer the question about five rings and three fingers asked by the second grader, a candidate started using a tree diagram (let alone forgetting that second graders are not familiar with this tool) as follows: There are six ways to place a finger ring on the first finger, five ways to place a ring on the second finger and four ways to place a ring on the third finger. Assuming that this is true (in fact, at that point, the candidate was politely paused by the author), multiplying six by five by four yields 120 ways to place five rings on three fingers. Such an increase in the number of ways, from 6 to 120, looks suspicious and a curious student is likely to seek clarification whether this is true or not.

In this regard, in true congruity with action learning when taking an action is followed by reflection [39], that is, by thinking about this action [2], the candidates have to learn not only how to explain why a certain solution is correct but also to unravel why a certain solution is incorrect. This is especially important because, when students are encouraged to solve a problem in more than one way, they expect their teachers to be able to explain why a rejected solution has a flaw or what is the cause of Einstellung effect. When using a tree diagram for the case of three fingers, one explanation is through presenting a counterexample. However, coming up with a working counterexample is not easy. With this in mind, consider the case of putting *two* finger rings on *three* fingers. Experimentally, one can show that there are three ways to have two rings on either of the three fingers and three ways to have one ring on each of the two fingers. That is, as the two cases are independent, there are six (3 + 3) ways to place two rings on three fingers. At the same time, the tree diagram that follows the rule of product would produce the product $3 \times 2 \times 1 = 6$. However, while the rule of the product provides the same answer as the experimental solution, the situation in this case develops through the rule of sum of products, as shown in Figure 4. Thus, the (experimental) counterexample is not well chosen because it is in line with the use of the rule of product. Here, one can use a tree diagram that follows the rule of sum of products. Even when this rule gives the same result as the rule of product, the former rule is not generalizable. This motivates the need for new strategies in solving the problem with rings and fingers posed by the second grader.

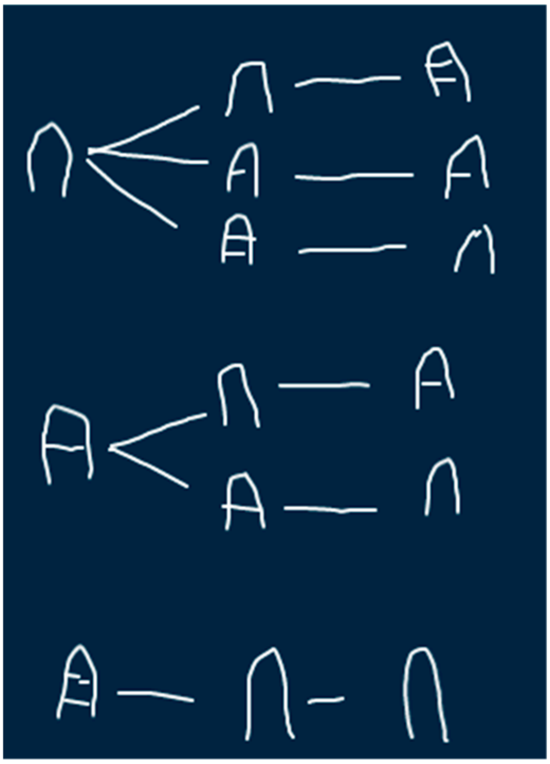

**Figure 4.** Using the rule of sum of products as a counterexample.

## 6. Counting Partitions of Five in Two Parts Using Wolfram Alpha

The candidates were introduced to the use of Wolfram Alpha as a program capable of both numeric and symbolic computations. In particular, the program can solve linear algebraic equations with several unknowns. In traditional algebra, using a grade-appropriate technique, a student moves from an equation in one or several variables to the values of the unknown variables that satisfy the equation. In the early (i.e., informal) algebra, which is a part of the courses taught to the candidates, one moves from unknowns to equations which are symbolic descriptions of drawings that enabled finding the values of unknowns. This transition from visual to symbolic provides the foundation for observing the picture in a more general manner as the description of the variety of situations relative to the situation with the finger rings and fingers is only one of the many. This interplay between concrete and abstract was described by Vygotsky [40] (Section 3) as follows.

> "Let us compare the direct image of a nine, for example, the figures on playing cards, and the number 9. The group of nine on playing cards is richer and more concrete than our concept "9", but the concept "9" involves a number of judgments which are not in the nine on the playing card; "9" is not divisible by even numbers, is divisible by 3, is $3^2$, and the square root of 81; we connect "9" with the series of whole numbers, etc. Hence it is clear that psychologically speaking the process of concept formation resides in the discovery of the connections of the given object with a number of others, in finding the real whole. That is why a mature concept involves the whole totality of its relations, its place in the world, so to speak. "9" is a specific point in the whole theory of numbers with the possibility of infinite development and infinite combination which are always subject to a general law".

Whereas a child is not mature enough to observe, in the image of the number 9 on a playing card, the variety of abstractions described by Vygotsky, the candidates are capable of comprehending the meaning of the word equation as a concept. This comprehension opens a window to a new solution strategy of solving problems relative to placing finger rings on fingers using Wolfram Alpha as a computer algebra system. Of course, such use requires certain programming skills. In the words of a candidate, "*The Wolfram Alpha was very confusing to use this semester, but once I got the hang of it, it was very easy and very helpful. Some of the math homework that I did this semester, this tool came in very handy when working on them. This will help other students with knowing how to do a problem about math.*" With this in mind, the original problem of partitioning the number five in two addends can be formulated as solving the following equation:

$$a + b = 5 \tag{2}$$

over the non-negative integers. As shown in Figure 5, the command "solve over the integers $a + b = 5$, $a \geq 0$, $b \geq 0$" entered into the input box of the program yields six solutions (or, better and partitions). These are the exact six cases shown in the left-hand side of Figure 1. The candidates are encouraged to describe the difference in the systems used in the pencil-and-paper environment of Figure 1 and in the symbolic computation environment of Figure 5 provided by Wolfram Alpha. In the former case one can recognize the use the commutative property of addition and, in the latter case, the move from the smallest to the largest non-negative integer values of the variable $a$ or, alternatively, from the largest to the smallest values of the variable $b$.

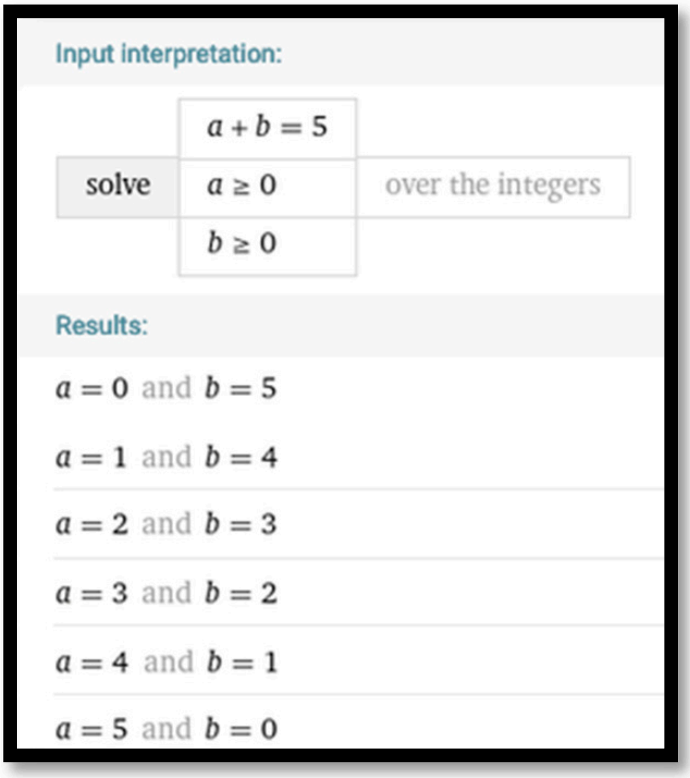


**Figure 5.** Approaching the problem algebraically.

### 7. Counting Partitions of Five in Three Parts Using Wolfram Alpha

Success in using Wolfram Alpha for the problem with two fingers (or days during which the temperature increased by five degrees) prompts solving a problem with three fingers (or days) and five rings (or degrees). The problem can be formulated as solving the following equation:

$$a + b + c = 5 \tag{3}$$

over non-negative integers. This transition from Equation (2) to Equation (3) in the context of Wolfram Alpha is pretty straightforward, and the computational power of the program takes care of preventing a possible Einstellung effect. The solution, resulting from the command "solve over the integers $a + b + c = 5$, $a \geq b \geq c \geq 0$", is shown in Figure 6. It matches (to the order of the triples displayed) the right-hand side of Figure 1. This enables one to have, in terms of Vygotsky [10], a situational reference in the form of the first-order symbols for the second-order symbolism of numeric triples. It appears that, due to the concreteness of the first-order symbols, permuting fingers with rings is easier to imagine and perform (Figure 1) compared to permuting elements of numeric triples in order to have the total of 21 triples.

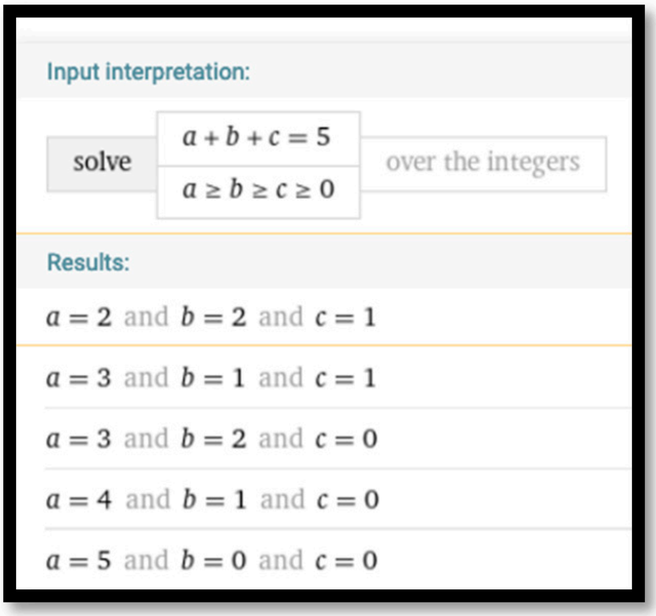


**Figure 6.** Using a Wolfram Alpha solution for posing a problem.

The candidates were also introduced to the idea of using the results of solving linear algebraic equations for posing problems. It was demonstrated that the inequalities $a \geq 0$, $b \geq 0$ and $c \geq 0$, when used instead of the inequalities $a \geq b \geq c \geq 0$ in the command that resulted in five triples, yield all 21 triples. The rationale for such demonstration was twofold: to show the operational sensitivity of Wolfram Alpha to a slight change in a mathematical condition and to explain in the context of computations the difference between two types of inequalities. The candidates recognized that the inequalities with certain restrictions *among* the unknowns are responsible for fewer solutions, and vice versa the inequalities with only basic restrictions (non-negativity) on each of the unknowns are responsible for more solutions. It is useful to discuss contextualization of the inequalities communicated to Wolfram Alpha in terms of finger rings and fingers. This discussion opens a window to posing contextual problems. For example, by observing the entire set of 21 triples, a candidate noted that the second and the third elements in each of the triples (1, 1, 3), (1, 2, 2) and (1, 3, 1), with the number one as the first element, follow from partitioning the number four in two non-zero parts and posed a problem: *How can one place five finger rings on three fingers with the index finger having one ring?* In other words, the candidates learned that by adding an additional condition to a question with more than one correct answer, the number of correct answers can be significantly reduced. In the words of a candidate, "*Technology such as the Wolfram Alpha allows students to use this as a resource and check their work. This is a great tool for teachers as well because it allows them to check themselves and make sure the work, they are showing the students is accurate.*" The issue of problem posing with technology is discussed in the next section.

## 8. Technology and Problem Posing

The use of computers in formulating mathematical problems provides both pragmatic and epistemic support for teachers in designing new curriculum materials for their students. Pragmatically, by using technology for problem posing, a teacher saves time for other professional activities including planning for teaching. The importance of providing teachers with more time for taking care of the intellectual advancement of their students was recognized at the outset of the teaching machine movement by Pressey [41] (p. 376), who argued that "mechanical aids are possible which would leave the teacher more free for her most important work, for developing in her pupils fine enthusiasms, clear thinking and high ideas". Such work occurs in the classroom and a teacher wants to be confident that problems students are asked to solve are pedagogically appropriate and satisfy the modern-day worldwide standards [9,25–27,30–32,35,42,43].

Epistemically, by using technology for problem posing, a teacher develops new skills and learns the art of problem posing by paying attention to the didactical coherence of a problem [44,45]. This art includes one's ability to navigate among different digital tools and selecting the most appropriate one for posing and solving a problem in order to support the learning of a specific topic. In doing so, the candidates "need to develop the ability to critically evaluate the affordances and limitations of a given tool, both for their own learning and to support the learning of their students" [24] (p. 34). Problem posing may include reformulation of existing problems [46] or of the newly formulated problems for which the solution is displayed by a problem-posing tool. Such a display enables a teacher to correctly interpret the information displayed and think of the possible modifications of the problem.

For example, in order to have first-hand experience with formulating a worthwhile task, upper elementary students in South Korea "are provided with 'problem-posing' activities where they create new problems by changing conditions of the problem given" [42] (p. 209). At the same time, as was already demonstrated above, the changing conditions of a problem are not always easy, and the outcome must be skillfully examined from different directions. Indeed, one of the teaching principles formulated by educators in Singapore points directly at mathematical problem posing as a skill; in order to build on students' knowledge, teachers need to know how "to develop learning tasks that are stimulating and challenging" [26] (p. 21). Viewing problem posing as a skill infers fair evidence of logical reasoning because, as noted by South African mathematics educators, "logic forms an integral part of mathematical thinking, however even adults sometimes find it difficult to reason in a formal logical way" [25] (p. 18). The use of Wolfram Alpha can provide students with support to develop reasoning skills based on logic and "solve problems by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication" [31] (p. 1). The next section will show how such sophistication can be achieved.

## 9. Productive Thinking Leads to A New Solution Strategy

A transition to a new problem-solving strategy is not obvious. In order to motivate this transition, one has to demonstrate productive thinking [47], a problem-solving strategy based on *insight*—a sudden recognition of the relationship between two seemingly unrelated ideas— or, alternatively, *sagacity* defined by Aristotle as "a hitting by guess upon the essential connection in an inappreciable time" [48] (p. 58). Such sudden recognition or accidental guessing can be stimulated by questioning. The candidates were asked the following: Where do people usually keep finger rings? A possible answer is a partitioned box. Two words, partitioned (related to box) and partition (related to number), may prompt the idea of a new representation of the problem with rings on fingers through rings in a partitioned box. That is how productive thinking works. However, what does it mean mathematically to partition an integer in two or three parts? It means placing a plus sign (or signs) between two (or three) parts. Thus, the partitions of the number five shown in Figures 5 and 6 can prompt the idea of viewing Equations (2) and (3) as permutations of symbols in the strings "rrrrr+" and "rrrrr++", respectively. Thus, one can view the case "rrrrr+" (or "+rrrrr") as all five finger rings kept in the far-left (or far-right) box with a plus sign meaning a partition in the box. Likewise, one can view the case "rrrrr++" (or "++rrrrr") as all five finger rings kept in the far-left (or the far-right) box with the other two empty boxes separated from the first one by two plus signs (Figure 7).

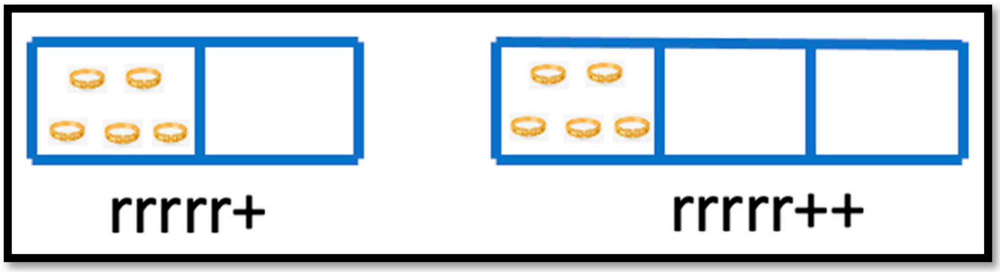


**Figure 7.** Finger rings in partitioned boxes and their symbolic representations.

Thus, one has to know how the number of permutations of elements of a string with different elements can be computed. In the case of two boxes, the permutations can be developed by moving the plus sign as follows: rrrrr+, rrrr+r, rrr+rr, rr+rrr, r+rrrr and +rrrrr. Thus, once again, we have six permutations or six ways to place five finger rings in two boxes (or on two fingers). The permutations of symbols in the string "rrrrrpp" can be carried out by Wolfram Alpha (Figure 8) through the command "permutations (r, r, r, r, r, p, p)", where the letter p replaces the "+" sign as the latter means an operation for the program. Once again, we have 21 as the answer. In general, the number of ways $n$ finger rings can be placed in a partitioned box with $m$ partitions is equal to the number of permutations of letters in the word $\underbrace{rr\ldots r}_{n\ times}\underbrace{pp\ldots p}_{m\ times}$. The latter number can be found through the following formula (this formula is taught to elementary teacher candidates in the context of mathematics prerequisites required for the Childhood Education Master of Science in Teaching diploma):

$$\frac{(n+m)!}{n! \times m!} \tag{4}$$

whereas the computational capabilities of Wolfram Alpha are limited in terms of the number of symbols to be included in the command "permutations (...)". The program is capable of dealing with Equation (4) for relatively large values of $n$ and $m$. For example, Figure 9 shows how Wolfram Alpha computes the case $n = 100$ and $m = 99$. The result is a 59 digit number. This is an example of the dual nature of Wolfram Alpha in terms of the program's affordances and limitations. Whereas Equation (4) significantly recovers the limitation of the command "permutations (...)" in the context of Wolfram Alpha, making use of this computational affordance of the program requires the candidates to possess conceptual understanding of how the permutations of letters in a word with repeating letters are computed.

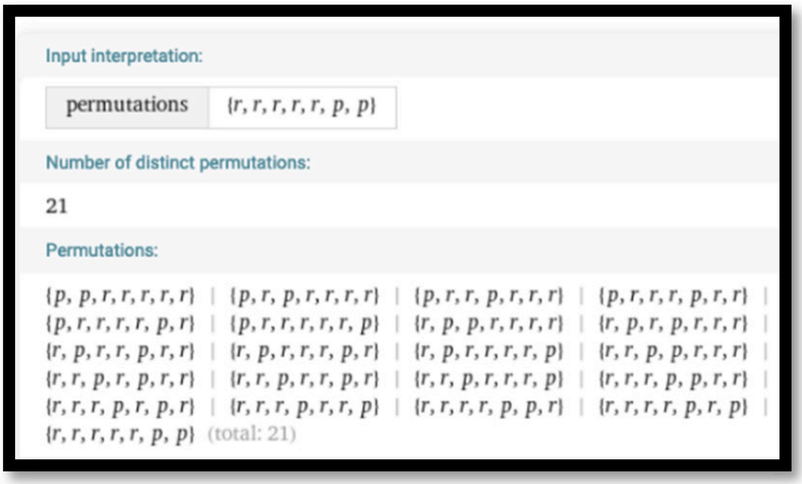


**Figure 8.** Answering the question asked by the second grader by permuting letters in a word.


**Figure 9.** Computing the number of ways to place 100 rings into a 100-section box.

## 10. Using a Generating Function Method

In order to demonstrate to the candidates another method of answering the question asked by the second grader, they can be reminded that Equations (2) and (3) are mathematical models of placing five finger rings on two and three fingers, respectively. This suggests that the right-hand side of each of the two equations points at the number of finger rings and the number of unknowns in the left-hand side points at the number of fingers. By using Wolfram Alpha and changing $n$ in the equation:

$$a + b = n \tag{5}$$

in the range $0 \leq n \leq 5$, one arrives at the sequence 1, 2, 3, 4, 5, 6, the terms of which stand for the number of ways $n$ finger rings can be placed on two fingers, $0 \leq n \leq 5$.

In the case of one finger (when in Equation (5), $b = 0$), regardless of the number of rings available (that is, regardless of $n$), there is only one way to place all the rings on a single finger. This brings about the sequence 1, 1, 1, 1, . . . , the terms of which stand for the number of ways $n$ finger rings can be placed on a single finger, $n = 0, 1, 2, \ldots$. At the same time, this sequence serves as the coefficients in the powers of $x$ in the infinite geometric series $1 + x + x^2 + x^3 + \ldots$, the sum of which is the fraction $\frac{1}{1-x}$ . In other words, because the function $f(x) = \frac{1}{1-x}$ generates the sequence of ones when expanded into an infinite series, it is called the generating function of this sequence [49]. Likewise, the natural number sequence 1, 2, 3, ..., can be generated by its generating function. Since the sum of the expression $(1 + x + x^2 + \ldots)^2$ is the fraction $\frac{1}{(1-x)^2}$, expanding the latter with the help of Wolfram Alpha yields the series $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$, where the coefficients in the powers of $x$ are consecutive natural numbers, and the coefficient in $x^5$ is equal to 6. That is, the function $f(x) = \frac{1}{(1-x)^2}$ is the generating function of the sequence of natural numbers. Due to the following formula:

$$x^a \cdot x^b = x^{a+b} \tag{6}$$

the values of $a$ and $b$ are solutions to Equation (2) shown in Figure 5. Note that knowing Equation (6) and similar equations is required at the certification exam that the candidates have to take as part of their degree requirements. The recollection of Equation (6) in the context of discussion of multiple solution strategies for the rings and fingers problem provides a perfect opportunity for the candidates' learning to become certified teachers. This makes it possible to introduce the generating function method as yet another method to answer the question asked by the second grader. More specifically, the number of solutions of a linear equation similar to Equations (2) and (3) can be found through this method. Figure 10 shows how expanding the function $\frac{1}{(1-x)^3}$ yields the number 21 as the coefficient in $x^5$. As was already found by using different problem-solving strategies, the number of integer partitions of five in three parts is equal to 21. Alternatively, five rings can be placed on three fingers in 21 ways.

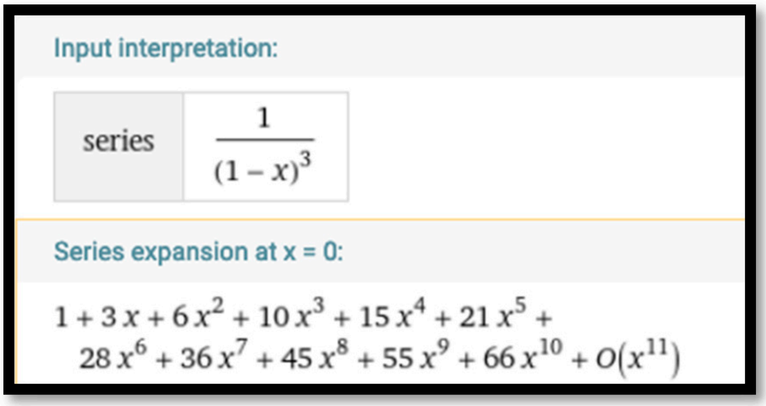


**Figure 10.** Expanding $\frac{1}{(1-x)^3}$ as answering the second grader's question.

Note that other generating functions can be considered and other problems can be formulated. For example, by using Wolfram Alpha, one can observe that the number of non-negative integer solutions to the equation $a + 2b + 3c = 5$ coincides with the coefficient in $x^5$ (Figure 11) generated by the function $f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}$ or, alternatively, by the product of three geometric series $(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)$. Contextually, the following problem can be formulated as follows: *How many ways can one place five finger rings on three fingers so that the index finger has at most one ring, the middle finger has at most two rings and the ring finger has at most three rings*? The coefficient in $x^4$ shown in Figure 11 suggests that there are four ways to arrange four finger rings under such conditions. The transition from data generated by Wolfram Alpha to its possible contextual interpretation is not straightforward, and the candidates need training in such a transition from formal to informal. Whereas the generating function method may be used by the candidates to pose problems, technology-based problem posing requires conceptual understanding of modeling data. For example, the largest exponent in a partial sum of the geometric series is the largest number of rings that can be used in posing a problem. Consider Figure 10. The meaning of the term 3x is that there are three ways to place one ring on three fingers—a pretty obvious statement. Less obvious is the statement, expressed through the term $6x^2$, that there are six ways to place two rings on three fingers—this statement requires a tactile explanation. Finally, the statement, expressed through the term $21x^5$, that there are 21 ways to place five rings on three fingers requires mathematical explanation which, as was demonstrated in this paper, can be performed with a variety of methods. Depending on the level of students, a teacher can pose questions of that type by using the results of modeling shown in Figure 10 (or Figure 11).

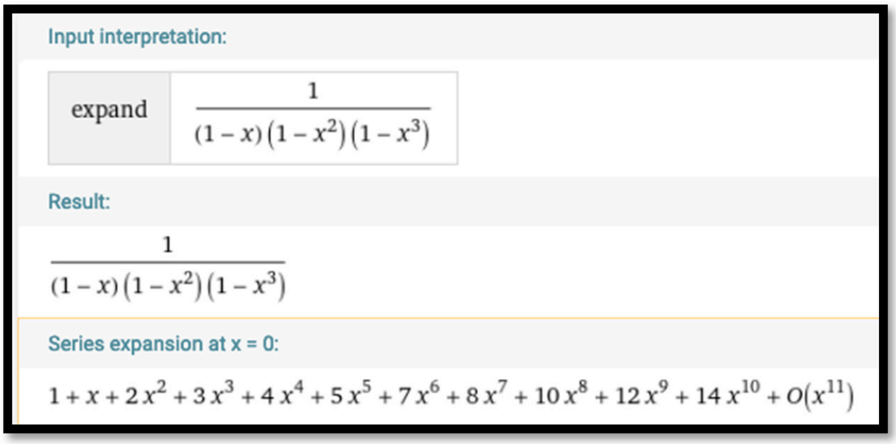


**Figure 11.** The generating function method as a concept-laden tool for posing a problem.

## 11. Conclusions

Three major issues debated nowadays in mathematics education (in addition to alternative views on pre-college curricula mentioned in Section 2) concern mathematical knowledge for teaching [50,51], the relationship between conceptual and procedural knowledge [52,53] and the use of digital technology [54,55]. This paper, written as the author's reflection on the use of Wolfram Alpha with elementary teacher candidates that found it conducive in uplifting their conceptual understanding of mathematics and in fostering skills in numeric and symbolic computations, contributes to this debate. It should be noted that reflection in teacher education as a field of disciplined inquiry aimed at the improvement of teacher quality [56–58] has been occurring mostly outside of the specific context of mathematics education and, to follow suit, the paper was not written to contribute to this field.

The general focus of this paper was on the author's experience working with the candidates, both in the classroom and in the field (pre-student and student teaching). An episode observed in a second grade classroom was used as background for the paper. Theoretically, the paper was structured along the lines of action research [14], by starting with a probe into a real problem followed, through continuous reflection, by conceptual learning of both pragmatic and epistemic aspects of mathematics involved. Pedagogically, the discussion in the mathematics teacher education classroom was about using a single question as a demonstration of the existence of multiple solution strategies varying on the spectrum from a hands-on solution to the method of generating functions. The following comment by a candidate is indicative of the general value and usefulness of this pedagogy, *"Having students work on a grade level problem that have many solutions would be interesting to see what they are able to do with that. I think it would be beneficial for the students to present their thinking and problem solving that went into the problem, why they got that answer, etc."*

More specific focus of the paper was on the use of a single digital tool—Wolfram Alpha—capable of a variety of numeric and symbolic computations needed to assist the candidates in dealing with mathematically distinct ways of answering the question asked by a second grader. As it followed from solicited comments by the candidates related to their use of Wolfram Alpha, the tool was found to be very useful in many aspects, including its use with their own children who were elementary school students. As noted by one of the candidates, *"I love the idea of using technology in the classroom. By writing the equations and then physically doing the equation you are appealing to both logical and visual thinkers allowing for more students to grasp the concept in different ways. Using Wolfram Alpha could lead to more experimenting in the class, more engage students, and a successful classroom as a whole."*

The paper discussed the use of Wolfram Alpha (and technology, in general) as a tool for posing mathematical problems for young children to solve, keeping in mind that schoolchildren are not expected to use Wolfram Alpha for solving a problem posed by a teacher with the help of the tool. At the same time, the idea of teaching schoolchildren to use Wolfram Alpha (or other technological tools) is consistent with the expectations of the Common Core State Standards [9] (p. 7) for students who "are able to identify relevant external mathematical resources and use them to pose or solve problems to explore and deepen their understanding of concepts". This issue, however, was beyond the scope of this paper. A point was made that the candidates need good conceptual understanding of mathematics in order to be able to correctly interpret the results of digital modeling provided by Wolfram Alpha. In general, the use of technology in posing mathematical problems saves time for a teacher as problem posing goes hand-in-hand with problem solving. The paper was supported by mathematics teaching and learning standards from the countries of all six continents emphasizing multiple solution strategies for a single problem.

Starting from a hands-on solution, formal methods and tools for answering a question asked by a second grader were drawn from algebra (solving equations), number theory (partition of integers), combinatorics (permutations) and discrete mathematics (generating functions). These formal methods were facilitated by Wolfram Alpha without which mathematics involved would not be manageable by the candidates. The method of generating functions was connected to the rule of multiplying exponents with the same base, something that the candidates need to know when taking a teacher certification exam mandated by the state of New York in which the author works. The complexity of ideas included in the courses taught by the author supports the belief of the Conference Board of the Mathematical Sciences—an umbrella organization comprising 19 professional societies in the United States concerned (in particular) with the mathematical preparation of schoolteachers—that "mathematics courses that explore elementary school mathematics in depth can be interesting for instructors to teach and for teachers to take" [24] (p. 31).

The pedagogy of multiple solution strategies works against the false dichotomy between the teacher's and the student's roles in the learning process within which there exists only one, which is the teacher's method of solving a mathematical problem. It provides opportunities for the candidates to learn how the diversity of teaching methods stems from the deep knowledge of mathematical content. The more diverse the methods are, the more eager the students are in asking questions and the more prepared the candidates are in answering the questions. The duality of technology in motivating students' queries and assisting the candidates' learning to address the queries suggests that mathematics education courses should demonstrate the diverse uses of commonly available digital tools of which Wolfram Alpha appears to be one of the prime examples. We conclude this paper with the following comment by a candidate who looks forward using Wolfram Alpha and other technological tools in their own elementary classroom: *"When using a tool such as Wolfram Alpha, students are able to see basic mathematical concepts and facts being broken up and represented in different ways. Teachers can have their students utilize such a resource to have students have a bigger foundational understanding of how such mathematical tools can help aid and guide them while working on and solving various math questions and concepts. Wolfram Alpha is also a great tool for students to see step-by-step solutions to various arithmetical and algebraic questions and problems they may be working on. Teachers can use Wolfram Alpha as a tool to help guide students who may need a more in-depth conceptual understanding of various mathematical concepts they may be learning in the classroom".* The candidate's appreciation of different methods of representation and a bigger foundational in-depth understanding of mathematical concepts in the context of using Wolfram Alpha is a testament that the author's efforts to use problems with more than one correct answer as a rudiment of promoting the idea of multiple solution strategies are conducive to the diversification of the methods of teaching elementary school mathematics.

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