

Article

Using Tangram as a Manipulative Tool for Transition between 2D and 3D Perception in Geometry

Mária Kmetová ^{1,*}  and Zuzana Nagyová Lehocá ² ¹ Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, 949 74 Nitra, Slovakia² Faculty of Central European Studies, Constantine the Philosopher University in Nitra, 949 74 Nitra, Slovakia; znlehocaka@ukf.sk

* Correspondence: mkmetova@ukf.sk

Abstract: Creating a mental image of our spatial environment is a key process for further abstract geometric thinking. Building a mental representation can be understood as a part of the process of visualisation. From the wide concept of visualisation, in this article, we will focus on the part where the mental representation of spatial relations, mental objects and mental constructions are created, and their manifestations as a 3D physical object and its plane representations arise. Our main goal is to follow the transition between 2D and 3D representations of physical objects and also to observe how and when such a transition happens in students' thinking. For that purpose, we also use Tangram, because manipulation with the Tangram pieces in space and filling out planar figures by them indicates the transition between 3D and 2D. Our research, using an action research methodology, was conducted on the students of three 5th grade primary school classes as a part of a larger long-term project. We pointed out a relationship between spatial abilities and the perception of 2D–3D relationships in students' mind.



Citation: Kmetová, M.; Nagyová Lehocá, Z. Using Tangram as a Manipulative Tool for Transition between 2D and 3D Perception in Geometry. *Mathematics* **2021**, *9*, 2185. <https://doi.org/10.3390/math9182185>

Academic Editors: Jarmila Novotná and Alena Hošpesová

Received: 5 August 2021

Accepted: 3 September 2021

Published: 7 September 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: spatial ability; visualisation; 2D–3D relationship; Tangram

1. Introduction

The great Dutch mathematician, Hans Freudenthal—with a deep interest in mathematics education—said about the importance of learning geometry and developing spatial abilities that “Geometry is grasping space. And since it is about the education of children, it is grasping that space in which the child lives, breathes, and moves. The space that the child must learn to know, explore, and conquer, in order to live, breath and move better in it. Are we so accustomed to this space that we cannot imagine how important it is for us and for those we are educating?” [1] (p. 403). Young children can build simple and substantial models of spatial relationships with different toys, for example, cubic castles and cars, however, according to [2], it is limited until about age 6. Young children can also begin to create mental representations of their spatial environments and detect spatial relationships of them. That environment can differ between boys (they spend more time with construction or sand play) and girls. Typically, boys are engaged in spatial activities more than girls, thus, it supposedly could account for spatial skill advantage for them. However, some studies find no gender differences [2]. On the other hand, some detailed research studies have shown that male superiority is most demonstrative in tasks of mental rotation, with lesser differences evident in orientation and no differences evident in visualisation. “Most researchers also acknowledge that the sex difference does not reliably appear until after puberty, and that, maturation has an effect on spatial development—late maturation is related to high spatial ability” [3] (p. 24). This finding is related to the effect of hormones on spatial ability. The opposing view is that environment plays a primary role in individual development [3].

Considering all those divided views, we have chosen an equal approach to development of spatial abilities for both genders in school education taking into account individual differences between students.

1.1. Students' Difficulties in Acquiring Spatial Skills

The way of creating a mental representation of the spatial environment, mentioned previously, is widely studied and tightly connected with visualisation. However, different authors give slightly different definitions for the concept of visualisation; here, we try to give a unifying understanding of it.

Visualisation generally means visual thinking, i.e., using mental images. According to [4] (p. 10), "Visualisation as a process is a mental or physical action where mental images are involved". In [5], the concept of visualisation is understood as a result of the action mentioned: (1) mental objects—mental images, schemes, constructions, i.e., mental representations, (2) physical objects—geometric illustrations, pictures, diagrams, 3D models, etc., and (3) cognitive processes in which physical or mental visualisations are interpreted—cognitive functions of visual perception, transformation of visual representations in the mind, and abstract ways of thinking. This part is the so-called interpretive visualisation [5]. According to [6], one of the most cited definition of visualisation is from Arcavi [7] (p. 217) that combines and completes different definitions, previously published: "Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". This wide conception includes all features of the concept of visualisation. In this article, we will focus on the part where the mental representation of spatial relations, mental objects and mental constructions are created, and their manifestation or demonstration as a physical object in the form of 3D manipulatives or plane (2D) representations as a paper and pencil geometric illustrations happen. One of the most often used tools for solid geometry teaching is a simple cube. While Rumanová [8] highlighted the 2D representation of a cube by its net, in [6], Vágová traced students' spatial imagination by constructing a cube cross-section on a real 3D model, on a digital 3D model and on a 2D representation of them, and studied the interplay of manipulatives in those three environments. Our main goal is to follow the transition between 2D and 3D representations of physical objects, and also, to observe how and when such a transition happens. We point out a relationship between spatial abilities and the perception of 2D–3D relationships in students' minds.

1.2. Theoretical Framework

The first experience of young children with geometry occurs in 3D space. These experiences are mainly relative positions of objects around them and basic properties of the objects as round, sharp, etc. The concept of a plane is an abstract concept. No existing real object has zero thickness. If we look at Euclid's Elements, we detect a separation between the geometry of a plane (books I–IV) and the geometry of space (books XI–XIII). The separation is not that rigorous, for in the solid geometry books Euclid establishes many results that relate to plane geometry. "While elements of plane geometry are obviously needed in the development of solid geometry, Euclid has no applications of solid geometry in establishing results of plane geometrical statements. Nevertheless, the application of stereometric considerations for solving problems of plane geometry was not unknown to the Greeks" [9] (p. 294). The fact that later on the solid geometry books were less studied than the plane geometry books, had an important effect in keeping spatial considerations separate from plane geometry investigations. "Even at the time of Joseph Gergonne (1771–1859) one bemoaned the separation of plane and solid geometry as counterproductive" [9]. Mathematical practice spoke in favour of mixing considerations of a plane and solid geometry on account of its success [9]. Applying the generally accepted phylogenesis and ontogenesis theory [10,11] to children's understanding of planar—spatial

relation, according to historical conceptual development given above, it is quite normal that they cannot see the neat difference between planar and spatial objects. Perceiving a square as a representation of a cube can individually last a very long time. Moreover, the flat object in space can represent an object in the plane. How thin does a coin have to be to represent a circle in the plane, and how thick does a coin have to be to represent a cylinder? Misconceptions in students' comprehension of the concepts of plane shapes and its properties lead to deeper mistakes in their further studies [12].

1.3. Research Questions

According to the information above, we formulate the following research questions:

1. Is there any connection between spatial and planar geometric skills in students' geometric competencies?
2. How individual students perceive the transition between 2D and 3D representations of geometric shapes?

2. Materials and Methods

2.1. Tangram

Manipulation with a flat 3D object represents the transition from 3D to 2D geometry, so it could help in the creation of the abstract concept of plane figures and their properties. To discover some basic geometric shapes, an ancient Chinese geometric puzzle, the Tangram, is almost ideal. The Tangram is a simple set of seven geometric shapes: three sizes of triangles (two large, one medium, and two small ones), one square, and one rhomboid (see Figure 1).



Figure 1. The Tangram pieces.

The three different-size triangles are similar, right-angled isosceles triangles. All of the Tangram pieces can be completely covered by the smallest type of triangles (see Figure 2).

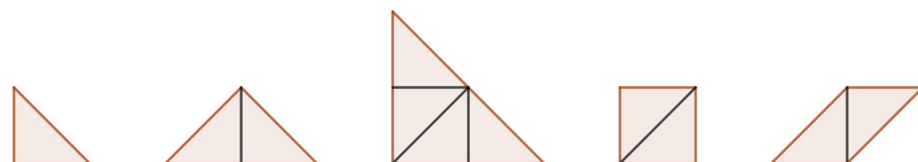


Figure 2. The Tangram pieces covered by the smallest triangle.

“Unlike a jigsaw puzzle where a piece must fit in only one way to complete a picture, the geometric Tangram pieces can be arranged in many different ways to make figures” [13]. The goal of the game is also to use all the pieces in one figure.

The Tangram can be used to provide numerous mathematical experiences for children. Activities introduce children to geometric concepts of planar figures and also help develop spatial perception through manipulation with the Tangram pieces in space. Altogether these activities reinforce the perception of the relationship between plane and space in geometry.

2.2. Action Research

The term “action research” was first coined by Kurt Lewin in 1944. In his 1946 paper [14], he described action research as “a comparative research on the conditions and effects of various forms of social action and research leading to social action” that uses “a spiral of steps, each of which is composed of a circle of planning, action and fact-finding about the result of the action”.

Action research is a methodology of research simultaneously combining a process of taking action and conducting research, which are linked to each other by a feedback loop. The scope of action research as a method is impressive. Action research can be used in different areas of the educational environment such as: teaching methods (comparing traditional by discovery methods), learning strategies (adopting an integrated approach to learning), educative procedures, continuing professional development of teachers [15]. “Action research was developed mainly by academics in higher education, who saw it as a useful way of working in professional education, particularly teacher education. They began studying and clarifying the steps involved, and also the principles underpinning action research, such as the need for new democratic practices as well as care and respect for the individual” [16]. Jean McNiff devoted to action research several books. In her last book *Action Research: All You Need to Know* [17], she describes all the circumstances of how and why to conduct this type of research. She claims: “Action research is today prominent not only in teacher professional education but also in management education and organisation studies, social and health care work, and other professional contexts. Action research is open ended. It does not begin with a fixed hypothesis but an idea that we develop. The research process is the developmental process of following through the idea, seeing how it goes, and continually checking whether it is in line with what we wish to happen. Seen in this way, action research is a form of self-evaluation” [16].

In [18] (p. 458), Tripp characterises the practical action research in education this way: “In education the action researcher is looking towards contributing to children’s development, which means that they will be making changes to improve their students’ learning and self-esteem, to increase their interest, autonomy and co-operation, and so on”.

Research results can be achieved through “completing a case study of the action research performed. This means that there are in fact two methodologies to be described and justified in an action research proposal: the action research processes to be used in the field, and the case study method that will be employed to tell the story of the project and its results” [18] (p. 460).

2.2.1. The Action Research Cycle

Action research in education has a dual role. Basically, it is a strategy for the development of teachers as researchers so that they can use their research to improve their teaching. Secondly, this influences the learning of their students. Moreover, in the context of educational action research, various variants have emerged [18]. Atkinson [19] rethought the method of action research especially for the demand of mathematics teachers and researchers. She distinguishes four stages of action research in mathematics education:

- (1) Reconnaissance or observation
- (2) Planning
- (3) Acting or monitoring
- (4) Reflecting or evaluating

These four stages or phases are widely accepted also in other models of action research. Although there are small differences in definition and emphasis, many models have several mutual principles, mainly the cyclical attribute of implementation and phasing. Tripp introduces a four-phase representation of the basic action inquiry cycle [18] as it follows in Figure 3. The Planning and Implementation phases create an ACTION part, and the Observation and Reflection phases create an INQUIRY part of the cycle.

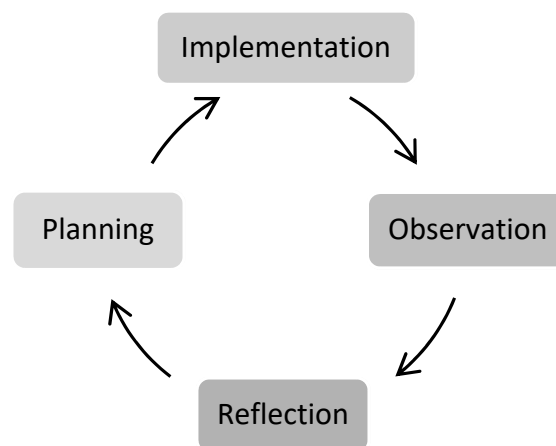


Figure 3. Four phases of action inquiry cycle according to [18].

Atkinson [19] divides all research phases into mini-cycles according to individual sessions or meetings with students and also as a way of conceptualising rapid decisions. Comparing the two approaches, observations during individual sessions can be perceived and described as case studies mentioned above, thus, those create mini-cycles according to Atkinson’s definition.

Tripp stressed the ethics in action research in [18] (p. 456): “No researcher or other participant ever engages in an activity that disadvantages another participant without their knowledge and consent. This usually rules out control group experiments, for instance, because it is disadvantageous to the control group not to have the benefit of changes that the action researcher expects to improve their practice”.

According to what was previously mentioned, we conducted our action research cycle on one group of students with mini-cycles in the form of individual sessions which can be described as mini case studies.

2.3. Participants

The participants of the research were 61 students of three primary schools with teaching languages Slovak (one school, 23 participants, class A) and Hungarian (two schools, 24 and 16 participants, class B and C, respectively) in Slovakia. Our research was a part of a long-term research project on developing an interdisciplinary study program to implement a new, internationally recognised educational model: STEAM (Science, technology, Education, Arts and Mathematics) in the disciplines of Informatics, Mathematics and Visual Arts for the fifth and sixth grades of primary school. The curricular modules were based on practical social requirements of everyday life situations and were targeted towards the development of skills necessary for employment in the labour market. The project was intent to develop subskills as spatial imagination, cognitive processes concerning shifts between two- and three-dimensional, traditional and digital spatial representations, and improve cognitive capacities of primary school students. The central objective of the project was the improvement of the practical skill set of young people.

Characteristics of the Participants

The participants, students of 5th grade (age 10–11 years), received usual education in mathematics according to the National Program of Education (ISCED1) used in Slovakia [20]. As far as spatial geometry skills are concerned, the geometry part of the educational program contains constructions of basic shapes and exercises on building cube bodies according to a plan or picture and creating a plan of a cube building. Students have to display the front view, right view, left view and top view of the object.

The three participating classes were average school classes with no extra teaching hours in mathematics. In class A, the average grade for mathematics was 1.27. All the students had grade 1 (excellent) except for 5 students who received grade 2 (very good).

In class B, the math teacher obviously had a different approach to evaluation, the average grade was 1.92, which means 12 excellent (1) grades, 4 very good (2) grades, 6 good (3) and 2 satisfactory (4) grades. In class C, all the students had excellent (1) grades except for 1 student with grade 2. Because of the different criteria for the award a mark of individual teachers we could not compare the students according to achieved grades.

2.4. Measures

2.4.1. Pre-Test

To establish the level of spatial skills of the participating pupils, we used a reliably evaluated [21,22] spatial ability test. The test (see Appendix A) contains three types of tasks:

1. Creating a desired view of a given spatial object, tasks 1–5 with increasing level of difficulty (see Appendix A). For illustration, we describe the second task here with full text in English.

Task 2 (see Figure 4). I built an object and, standing at point 1, made drawing 1 of it. Imagine that you are standing at point 2. What does the object look like? Colour the squares and triangles in the worksheet accordingly.

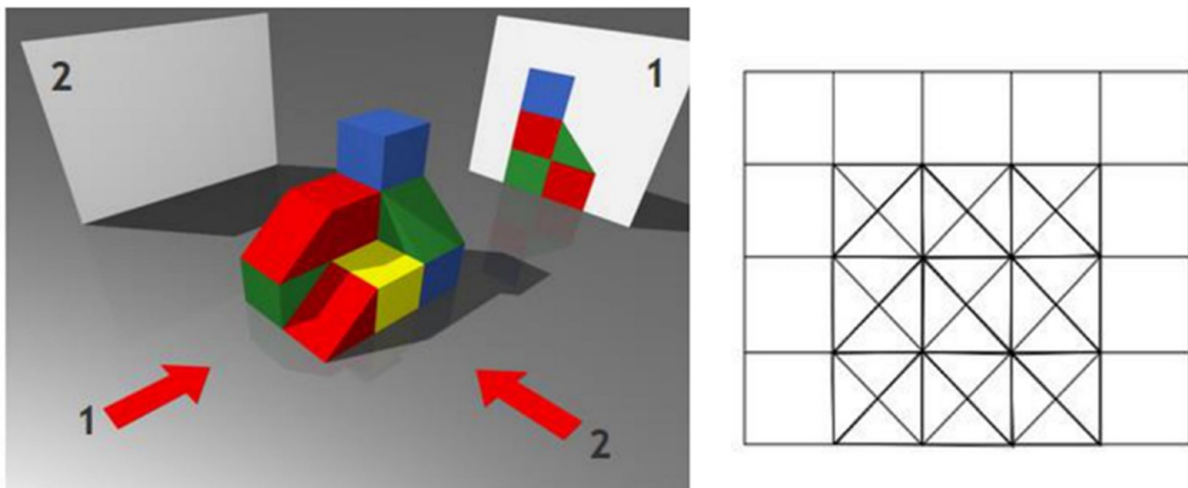


Figure 4. Assignment of Task 2.

The solution of this task requires creating a mental image of 3D object, imagine the front view in 2D, then changing the point of view and creating a new front view in detail.

2. Orientation in space, identification of the object in the given 2D map, tasks 6–9 with increasing level of difficulty. In the sixth task, the pupils have to find the church, school and the ice-cream shop on the map according to the picture of a village and mark them. In the seventh, eighth, and ninth tasks they have to find the coloured buildings on the map and colour them accordingly. Students have to imagine a top view of the village, apply a mental rotation, compare the result of it with the given map and identify the desired buildings. The English text of the seventh task in Figure 5 is the following:

Task 7 (Figure 5). Find and colour the two yellow houses to the same colour on the map. Pay attention because the map has been rotated.

3. Mental rotation task, the tenth assignment. The students have to create a detailed image of the given object and identify the steps of the rotation process in the pictures according to details (the mutual positions of parts of the object in the 2D view). The text of the task in English is the following:

Task 10 (Figure 6). BIP, the robot, is rotating around. Set the rotation of the robot in chronological order. Picture A is the first, E is the second. Write the appropriate numbers (3, 4, 5, 6) to the letters.

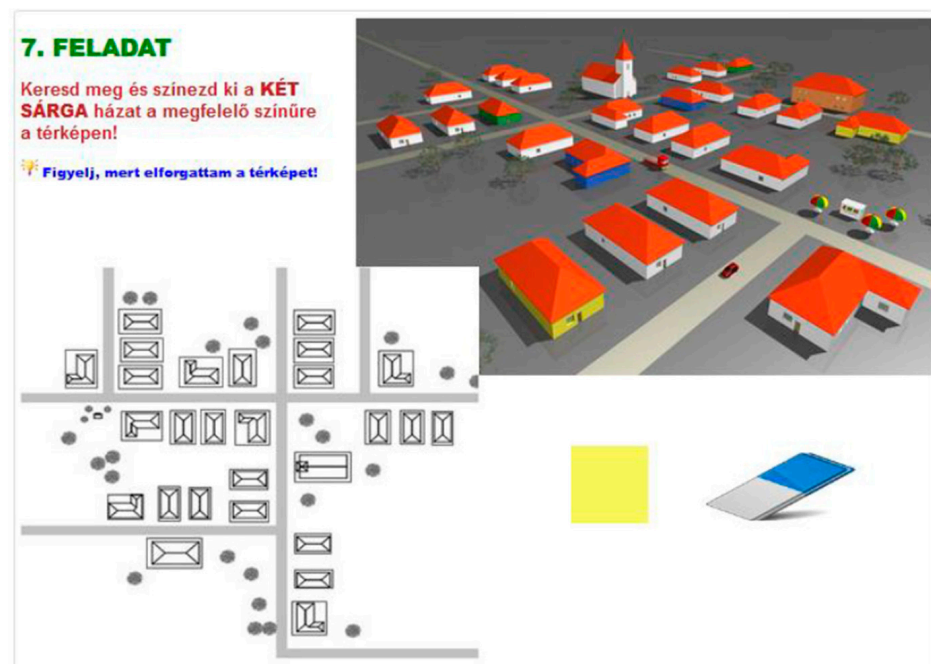


Figure 5. The assignment of Task 7.

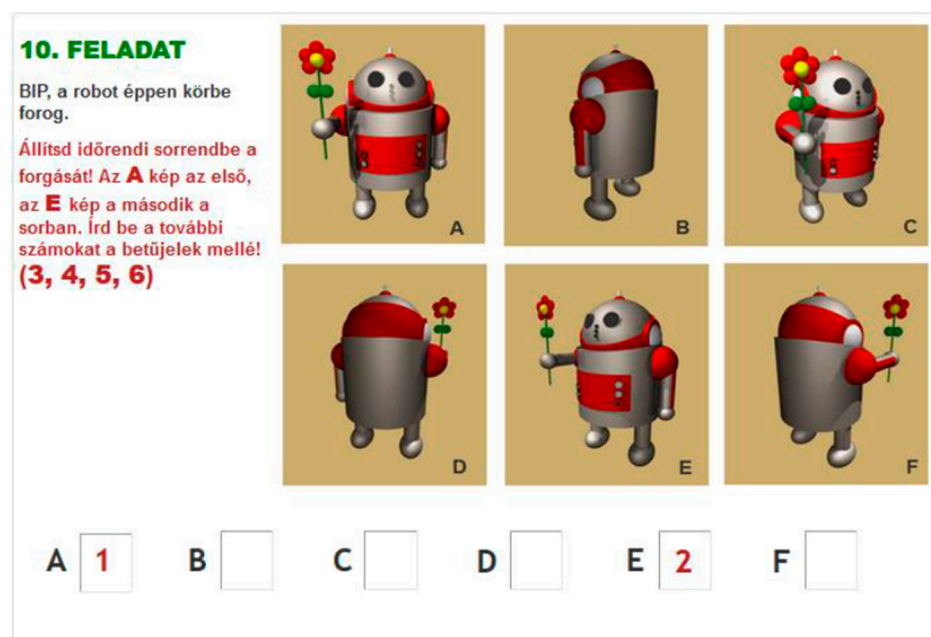


Figure 6. The assignment of Task 10.

The three types of tasks mentioned are not typical for school geometry education, rather could be characterised as non-routine assignments. In all of them though, the 2D – 3D relation is dominating but in an unusual context and in combination with mental rotation.

2.4.2. The Tangram Tasks

In accordance with the research questions, we decided on the observation of pupils during the problem-solving with Tangram. Because the Tangram was not known to the students, we needed to introduce it to them during an introductory lesson. Preparatory tasks served as an introduction to the work with the new geometric puzzle. The students were asked to solve simple tasks to see the possibilities offered by the Tangram.

Preparatory tasks:

1. Draw a house, e.g., a square with a triangle on it. Divide the figure into two or more pieces (like puzzle pieces) and give them to a schoolmate to figure out what shape was cut out. The puzzle pieces can be put together individually and then exchanged between the children.
2. Measure the sizes of a sample Tangram and then try to copy it on a sheet of paper.
3. Discuss, what are the important data to look out for, when constructing on the paper.
4. Make a Tangram out of cardboard, put it in an envelope, it will serve the class. Not all parts have to be used what enables free creation. Use items also from multiple sets at once. Make photographs of the results.
5. Lay out new pictures, draw them around, and add lines to divide them into pieces.

After the preparatory activities were completed, these tasks were solved:

Task 1: Observe the following picture carefully (see Figure 7), memorise it, then cover it and put it together from the Tangram pieces.

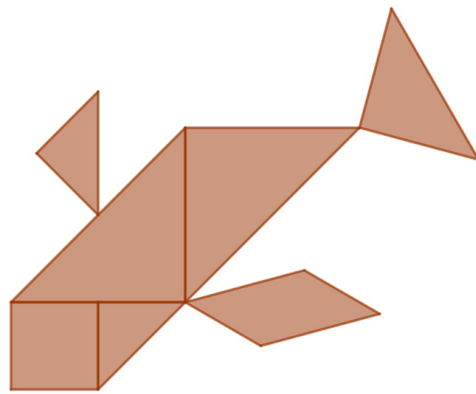


Figure 7. Assignment of Task 1.

Task 2: Use the square and the two small triangles to create shapes where you fit sides of equal length to each other. Find all the solutions and determine their properties: by perimeter, area, and parallel pages. Classify them according to these criteria! Line up the shapes of the Tangram according to the size of their area. Express exactly how many times the area of the shape is larger than the area of the small triangle.

Task 3: Create a picture freely, it is not necessary to use each piece of the Tangram.

Task 4: Using the two largest triangles, as well as the medium and the two small triangles, lay out two rectangles of different perimeters. Are their areas equal?

Task 5: What quadrilaterals can you create from one medium and two small triangles?

Task 6: Create triangles using two, three, four, five, six, and finally all seven Tangram shapes. Draw the solutions.

Task 7: Use each element to cover the predefined shapes in Figure 8.

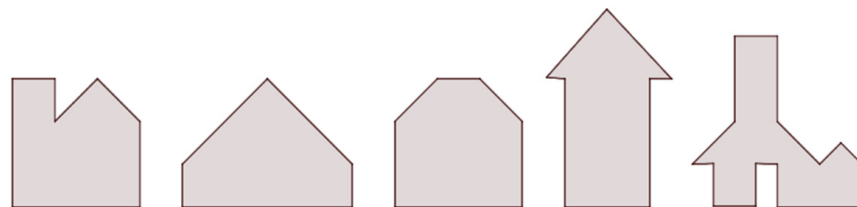


Figure 8. Assignment of Task 7.

Task 8: In our street, we drew the pediment of some houses. Draw what you imagine the street to be like. Colour the houses, you can also draw trees.

3. Results

3.1. Results of the Pre-Test

The spatial ability test (see Appendix A) results showed the students' ability to apply their spatial skills in a new, unknown context. The correct solution of a task was coded as 1, the wrong one as 0. Thus, one student could obtain a maximum of 10 points.

The distribution of the results is shown in Figure 9.

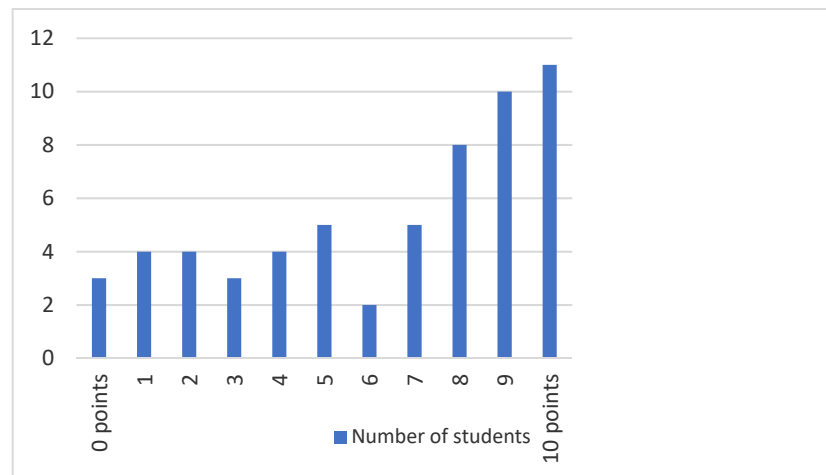


Figure 9. Results of the pre-test.

According to the results reached, the fifth task turned out to be the most difficult, and the sixth one the least difficult, in which 47 out of 61 students answered correctly (see Figure 10).

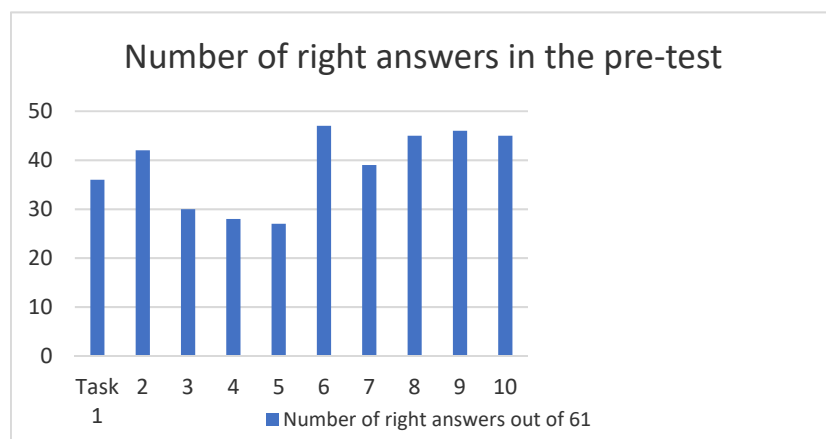


Figure 10. Results according to the tasks.

We can conclude that orientation according to a map is more familiar to students than creating the front view of an unfamiliar object. It also means that if the children have to apply their knowledge in a completely new circumstance, they are more prosperous in a context that has practical meaning. The abstract context causes additional difficulties. On the other hand, skills trained in abstract contexts can be applied in different practical contexts. This aim of geometry teaching was confirmed by many authors [23–25]. In addition, the authors of [25] also stressed the role of manipulation and the creation of geometrical images in developing the so-called spatial intuitive skills. One of the examples introduced in [25] is creating new figures by using triangles, which is, at its core, very similar to our tasks 2, 5 and 6 regarding the Tangram pieces. We evaluate the Tangram tasks in the following subchapter.

3.2. Evaluation of the Tangram Tasks

The intention of such tasks is to stimulate pupils' spatial imagination despite the fact that it happens in the plane through manipulation with plane figures as triangles and quadrangles. In line with the authors of [25] we also claim that through the activity of manipulation, students may be able to imagine figures of all combinations or create them in the mind, which we consider as important elements of spatial imagination.

Task 1 also requires spatial imagination by memorising a shape, supporting this way the creation of a visual mental representation [26]. The students were able to recall the image of the fish without significant differences.

Task 2 required creating different combinations of a square and the two small triangles into one geometrical shape according to the given rule. The solutions are shown in Figure 11. The students were not able to find all the satisfying shapes immediately, they could only find the shapes from the first row of Figure 11, and then they tried until they finally found them all. This assignment was one of the more difficult for the pupils, but also the most important from the geometric creativity point of view, also stressed in [25], where it was illustrated by Figure 12. In the next part of this assignment, the students have to decide about the properties of the shapes. The children had no problem fitting the sides of equal length next to each other, but they were already having difficulties establishing the properties. It also took a while until they realised that if the shapes were laid out in equal parts, then their area should also be equal. The connection between the small triangles and the area of the square was quickly seen, but there were already problems with the whole area. Thus, sorting by area and perimeter was a more difficult task. They realised that the area of the two triangles was the same as the area of the square, and then they discussed that they had the same shapes so the area matched. Interestingly, the perimeter of the shapes was less understood.

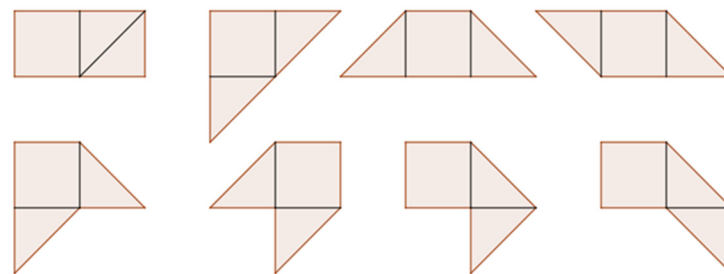


Figure 11. Result shapes of Task 2.

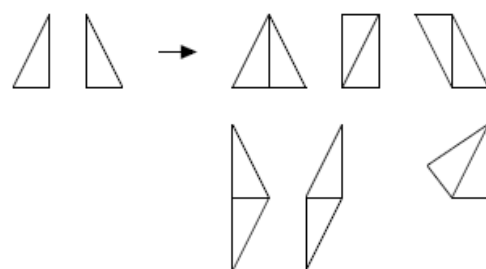


Figure 12. Example of 'Making new figures' from [25].

Task 3 is a typical Tangram task with no restriction to support the creativity of students. This was the easiest task, the students could concentrate on the geometric properties of the shapes, they did not need to keep in mind any other requirements. They created very different pictures, enjoying the free creativity (Figure 13).

Most students found the required rectangles in Task 4 with different perimeters, but at first glance, it was not clear for some of them that the areas of these shapes (Figure 14) are equal. After comparing the equal parts in the two shapes, they accepted this fact easily.

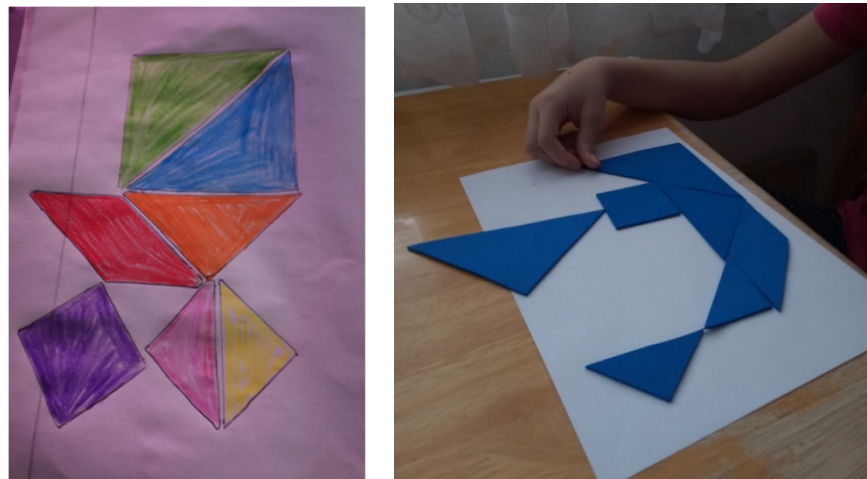


Figure 13. Manifestation of free creativity.

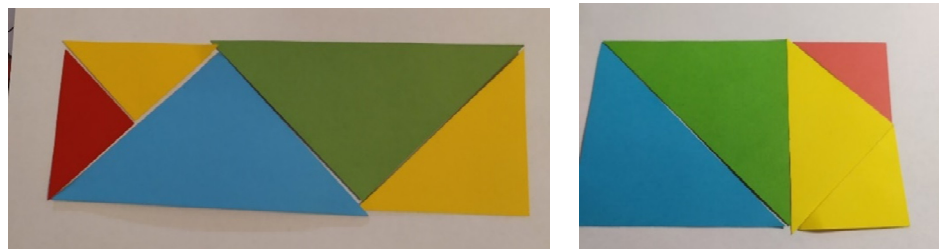


Figure 14. Rectangles with different perimeters.

Task 5 served as a propaedeutic question to learn about more geometric shapes, such as trapezoid and parallelogram. The students put together square and rectangular shapes and recognised the new quadrilaterals and learned their names (Figure 15).

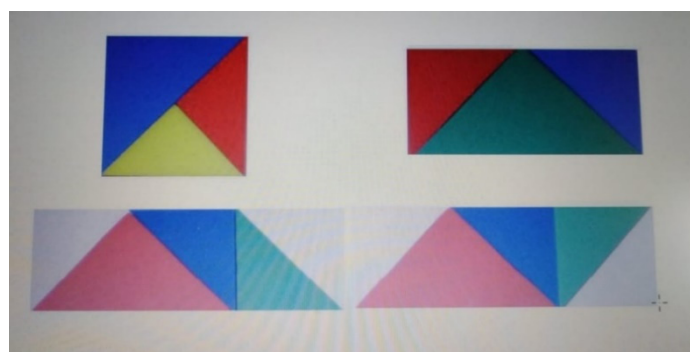


Figure 15. Quadrilaterals.

Task 6 was similar to Task 2, but here all the shapes had to form triangles. From two and three pieces it was easy, but from four and more pieces it became much harder (Figure 16). Only a few children gave a solution for how to make a triangle out of all the pieces.

Task 7 was demanding but the students enjoyed it. They devoted a long time to fulfil all the Tangram pieces into the given pictures (Figure 17).

Task 8 had the aim of connecting the Tangram planar shapes with the planar projection of the 3D scene. Pupils understood the task differently, some of them tried to use shapes from previous tasks (Figure 18). Using the middle line of the road as an axis to combine two front views of the two sides of the road is an interesting approach (Figure 19).



Figure 16. Creating triangles.

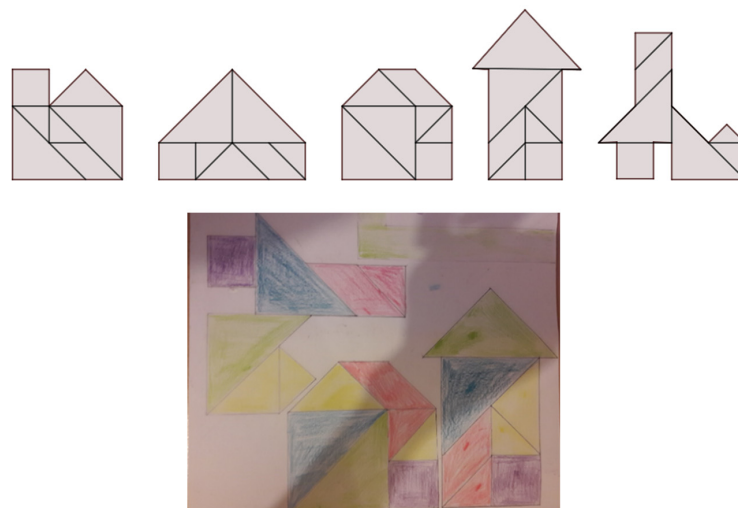


Figure 17. Solution of Task 7 (top), and one of the student's solution (bottom).



Figure 18. One solution of Task 8.

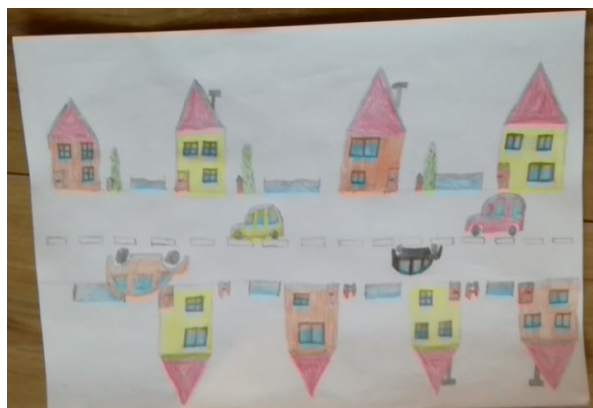


Figure 19. Combination of two front views.

4. Discussion

In the preparing and planning phase of the action research conducted, firstly, we started out from the generally accepted fact that is stressed among the others and also by J. Piaget in his theory of cognitive development. Children's thinking is concrete until they are 10 years old due to their age characteristics, and only then do they begin to think abstractly [27,28] (p. 251). The use of manipulation activities contributes most to the development of this process. Secondly, we decided to apply non-standard tasks contrary to standard school tasks of the fifth grade, which are obviously about a cube, cube net, and different views of the cube structure. In accordance with [22,25], we decided on a larger understanding of the concept of spatial skills. According to [22], spatial skills are present in

1. Spatial perception such as: orientation in space, experiencing space, understanding spatial relationships.
2. Spatial representation (mostly in 2D) such as: representing mutual positions in space or mutual positions in a mental visual image, representation of motion in space through time, reconstruction of space elements.
3. Creation of spatial objects in 2D or 3D such as: modelling and construction.

According to [9,25], we could also conclude that the complex development of spatial skills also requires diverse experience in plane geometry. The Tangram conundrum offers exactly such types of problems that the mean connection of 2D and 3D problem solving using real object manipulation. The Tangram pieces are 3D objects, needing manipulation in 3D space which represent a solution of a 2D problem—mainly covering a given shape with the Tangram pieces. With tangram tasks, we strengthen the pupils' manipulation skills and in-plane experience. If plane geometric concepts are not understood and lacking sufficient experience, the development of spatial vision may be more difficult or incomplete.

The implementation phase of our action research was conducted in classes A, B, and C described above. In the first part of this phase, the students solved the tasks of the spatial ability test given in Appendix A. In the second part, they solved the Tangram puzzle problems while we were observing their work method. Individual observations (mini case studies) were conducted as micro-cycles of this phase of action research according to [19] mentioned in Section 2.2.1.

The observations show that there are significant differences in spatial skills and also in Tangram problem-solving skills of the students. Interestingly, the points reached in the spatial ability test by individual students did not correspond with their grades from mathematics as a subject. Presumably, the reason for this was that the questions were non-standard and did not correspond with the subject matter of fifth-grade geometry. They probably needed a wider range of repeated long-term experience with different kinds of geometric objects, or other practical, manipulative spatial and visual experiences [29]. The most noticeable difference between the mathematics grades and the performance in the spatial ability test occurred in class C, where excellent students from mathematics (marked 1) reached only 2 to 9 points (7 on average) from the maximum possible 10 points. On the contrary, excellent students in class A reached 6 to 10 points, 7.8 on average (in class B they reached 2 to 10 points, 7.1 on average). From individual sessions in the form of mini case studies, we can conclude that the students with low performance in the spatial ability test (0 to 3 points) needed more help with the Tangram problems. Figure 20 shows some of their attempts.

We also recognised that during the problem-solving, the children worked alternately in space and in the plane in a manner that first they manipulated, rotated the Tangram pieces in space then summarised the result in the plane as it is seen in Figure 20. This way, the pupils could discover how the plane shapes created a part of spatial shapes. Comparing the solution of Tangram problems with the results of the spatial ability test, it can be said that students who solved the spatial orientation problems correctly, performed better in solving the plane problems too, so they have more advanced plane manipulation skills and are able to notice relationships between basic geometrical shapes in the plane.

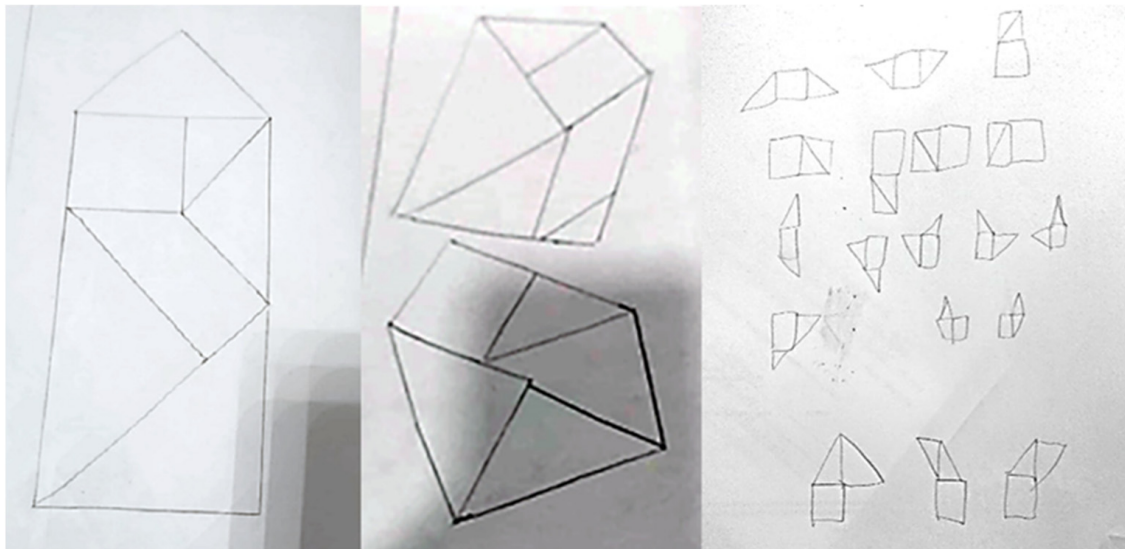


Figure 20. Attempts to fill out a pentagon shape and to find combinations for Task 2.

5. Conclusions

Based on what has been said, we can answer the research questions posed in Section 1.3.

1. There is a noticeable connection between spatial and planar geometric skills in student's geometric competencies. In the next cycles of our action research, we would like to support the development of these skills using different types of 2D and 3D geometric puzzles, manipulatives in a physical and virtual environment, a combination of different views (projections) of objects in the plane, and a wide range of various problems requiring the application of interplay of spatial and planar viewing and imaging.

2. Only a few students recognised the transition between 2D and 3D representations of geometric shapes. It turned out that 10-year-old children are on the way to making a real difference between plane and spatial shapes. One of the typical mistakes characterising this understanding is to call a cube both a cube and a square, too. The pupils naturally drew the Tangram pieces into their paper notes, without taking care that this is the 2D result of 3D spatial manipulation. A different situation occurred in the solution of tasks where they needed to apply a mental rotation of a spatial object to determine the left and back views. These questions proved to be the most difficult. On the contrary, establishing the top view (on the map) proved to be the easiest. It shows—together with the Tangram paper notes—that the creation of the top view is a natural transition between 3D and 2D representations of spatial objects for 10-year-old children. This indicates that for further development of spatial skills, we have to create more opportunities for building mental representations [29] of the spatial environment and its imaging by projection into the plane.

According to our research results described above, we can recommend the use of the Tangram (besides other similar geometric puzzles) as one of the supportive activities for 10-year-old pupils to acquire a better understanding of not only plane geometry problems, but also 2D–3D geometric relations. We recommend including similar activities in the teaching of both planar and spatial geometry for better comprehension of the transition between 2D and 3D geometry.

Author Contributions: Conceptualization, M.K.; methodology, M.K. and Z.N.L.; validation, M.K.; formal analysis, M.K.; investigation, M.K. and Z.N.L.; resources, M.K.; data curation, Z.N.L.; writing—original draft preparation, M.K.; writing—review and editing, M.K.; visualization, M.K. and Z.N.L.; supervision, M.K.; project administration, Z.N.L. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are available on request from the corresponding author. The data are not publicly available due to the privacy of participating students.

Acknowledgments: The paper is supported by projects KEGA 015 UKF-4/2020 Development of spatial abilities of 10–12-year-old students and KEGA 015UKF-4/2021 Collaboration as a means of professional growth of mathematics teachers.

Conflicts of Interest: The authors declare no conflict of interest.

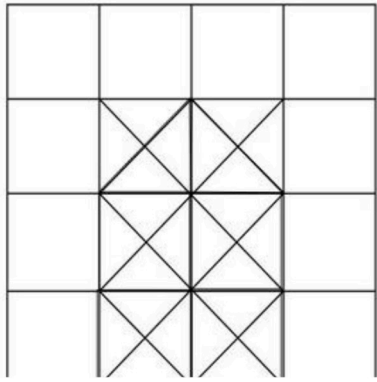
Appendix A


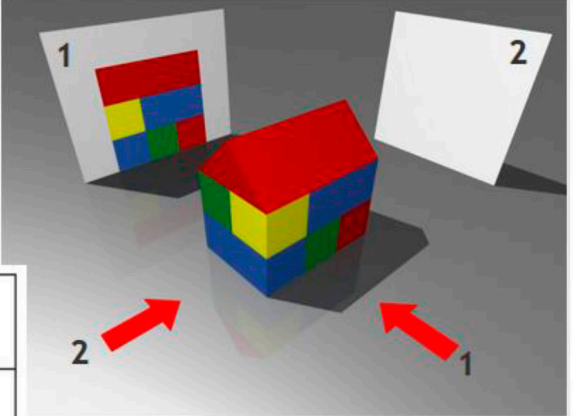
1. FELADAT

Építettem egy házat, és az 1. ponton állva az 1. rajzot készítettem.

Képzeld el, hogy a 2. ponton állsz! Milyenek látod a házat? Színezd ki ennek megfelelően a feladatlapon a négyzeteket és a háromszögeket!

2



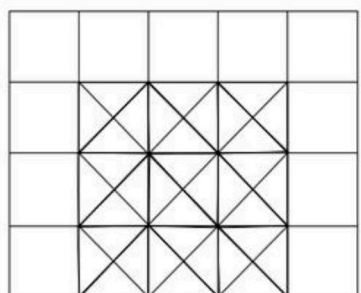



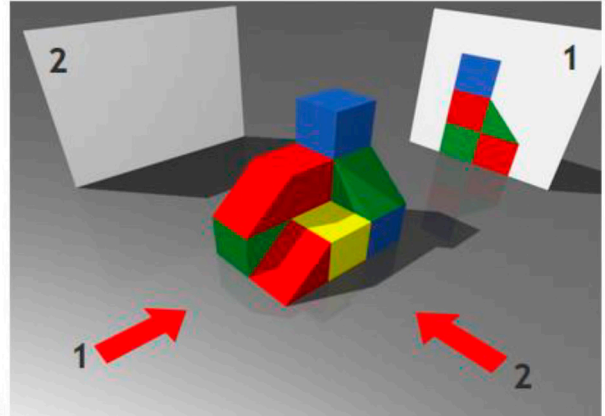
2. FELADAT

Építettem egy formát, és az 1. ponton állva az 1. rajzot készítettem.

Képzeld el, hogy a 2. ponton állsz! Milyenek látod a formát? Színezd ki ennek megfelelően a feladatlapon a négyzeteket és a háromszögeket!

2



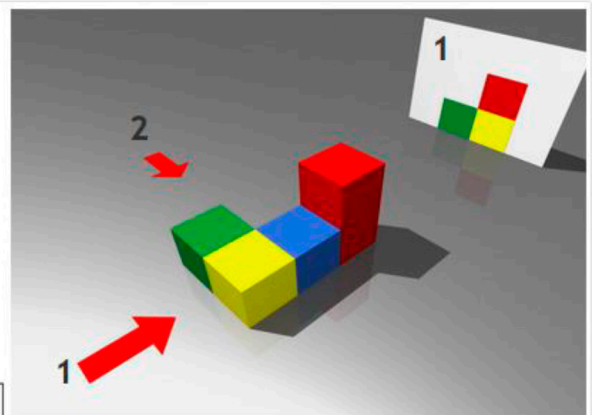
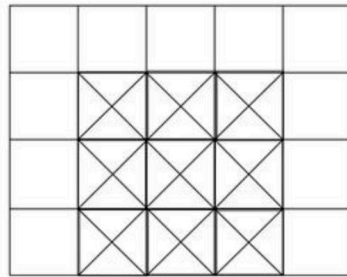


3. FELADAT

Építettem egy formát, és az 1. ponton állva az 1. rajzot készítettem.

Képzeld el, hogy a 2. ponton állsz! Milyenek látod a formát? Szinezd ki ennek megfelelően a feladatlapon a négyzeteket!

2

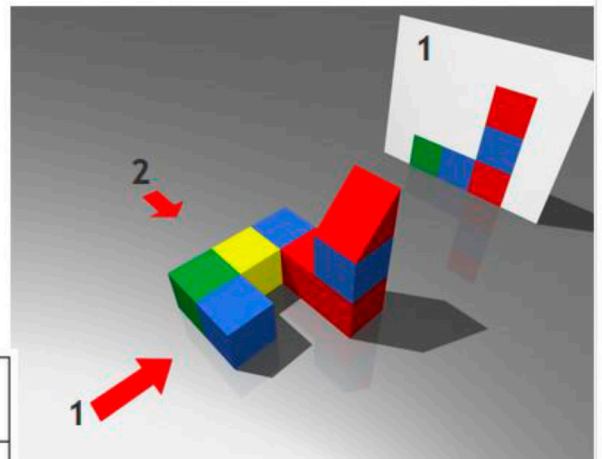
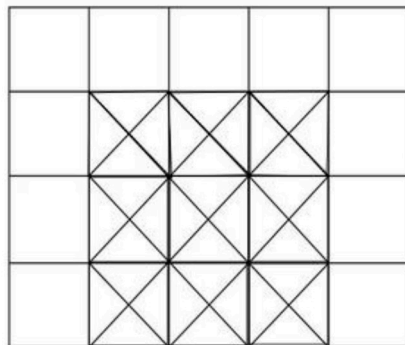


4. FELADAT

Építettem egy formát, és az 1. ponton állva az 1. rajzot készítettem.

Képzeld el, hogy a 2. ponton állsz! Milyenek látod a formát? Szinezd ki ennek megfelelően a feladatlapon a négyzeteket és a háromszögeket!

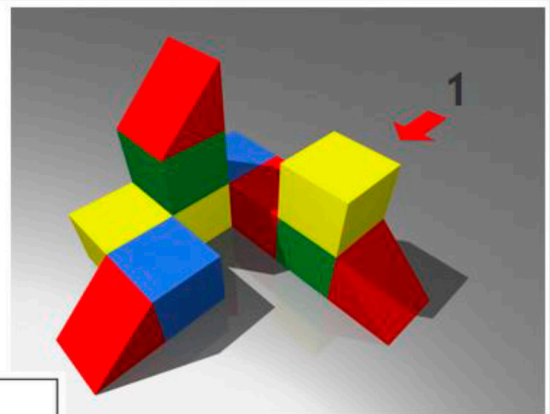
2



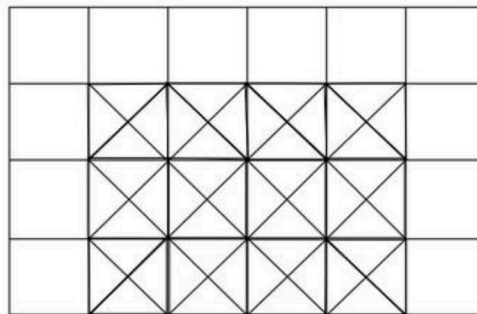
5. FELADAT

Építettem egy formát.

Képzeld el, hogy az 1. ponton állsz!
Milyennek látod a formát? Színezd ki
ennek megfelelően a feladatlapon a
négyzeteket és a háromszögeket!

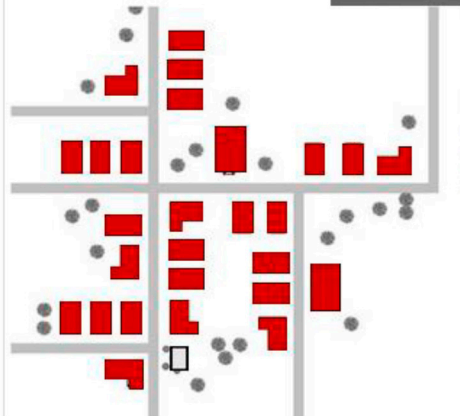


1



6. FELADAT

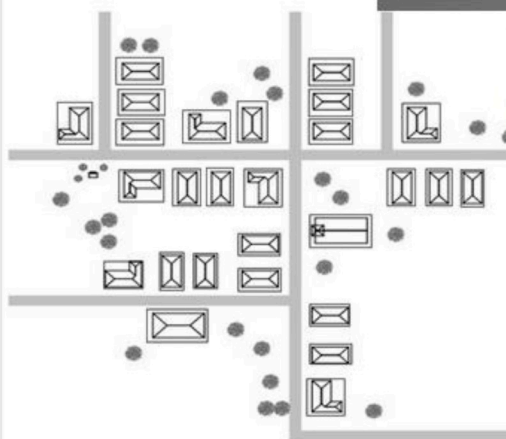
Keresd meg a térképen a
TEMPLOMOT, az ISKOLÁT, a
FAGYIZÓT és kattints rájuk!



7. FELADAT

Keress meg és színezd ki a **KÉT SÁRGA** házat a megfelelő színűre a térképen!

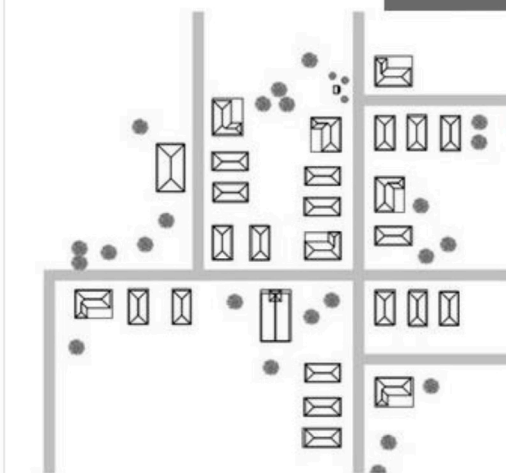
 **Figyelj, mert elforgattam a térképet!**



8. FELADAT

Keress meg és színezd ki a **KÉT KÉK** házat a megfelelő színűre a térképen!

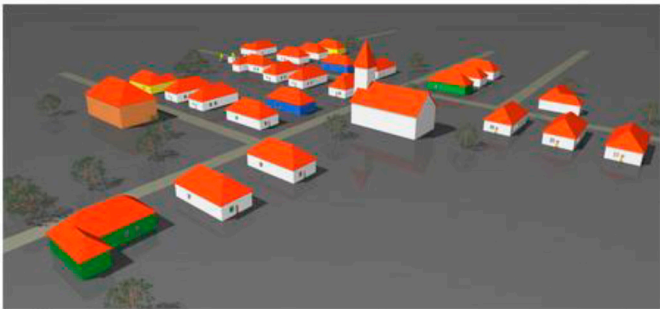
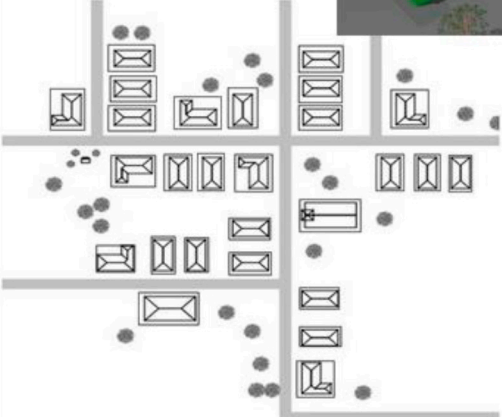

 **Figyelj, mert elforgattam a térképet!**



9. FELADAT

Keress meg és színezd ki a **KÉT ZÖLD** házat a megfelelő színűre a térképen!

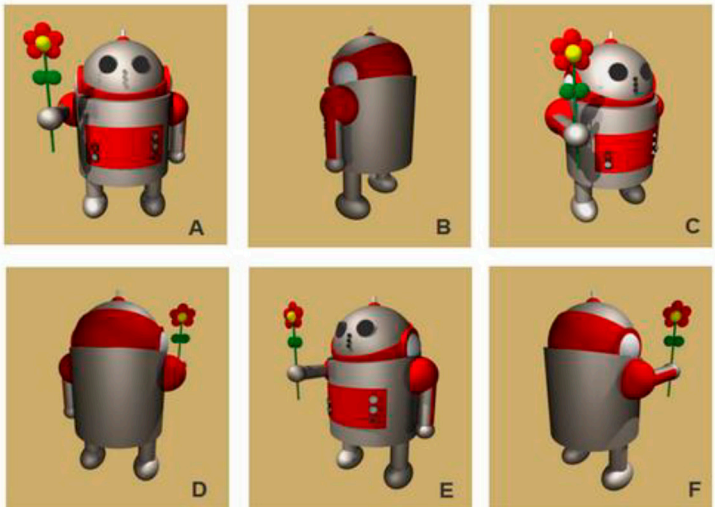
☀ **Figyelj, mert elforgattam a térképet!**

10. FELADAT

BIP, a robot éppen körbe forog.

Állítsd időrendi sorrendbe a forgását! Az **A** kép az első, az **E** kép a második a sorban. Írd be a további számokat a betűjelek mellé! (**3, 4, 5, 6**)



A B C D E F

References

- Freudenthal, H. *Mathematics as an Educational Task*; Reidel: Dordrecht, The Netherlands, 1973.
- National Research Council. *Mathematics Learning in Early Childhood: Paths toward Excellence and Equity*; The National Academies Press: Washington, DC, USA, 2009. [[CrossRef](#)]
- Mohler, J.L. A Review of Spatial Ability Research. *Eng. Des. Graph. J.* **2008**, *72*, 19–30.
- Gutiérrez, A. Visualization in 3-dimensional geometry. In *Search of a Framework, Proceedings of the 20th International Conference for the Psychology of Mathematics Education, Valencia, Spain, 8–12 July 1996*; Puig, L., Gutiérrez, A., Eds.; University of Valencia: Valencia, Spain, 1996; Volume 1, pp. 3–19.
- Phillips, L.M.; Norris, S.R.; Macnab, J.S. *Visualization in Mathematics, Reading and Science Education*; Springer: New York, NY, USA, 2010; Volume 5.
- Vágová, R. *Exploring the Interplay of Physical and Digital Manipulatives to Develop Students' Visualisation in Solid Geometry*; UKF Nitra: Nitra, Slovakia, 2021.

7. Arcavi, A. The role of visual representations in the learning of mathematics. *Educ. Stud. Math.* **2003**, *52*, 215–241. [[CrossRef](#)]
8. Rumanová, L.; Vallo, D.; Smiešková, E. Blueprint as a Motivation Factor for the Development of Spatial Ability in Teaching of Mathematics: An Action Research Study. *TEM J.* **2021**, *10*, 732–742. [[CrossRef](#)]
9. Arana, A.; Moncosu, P. On the relationship between plane and solid geometry. *Rev. Symb. Log.* **2012**, *5*, 294–353. [[CrossRef](#)]
10. Furinghetti, F.; Radford, L. Historical Conceptual Developments and the Teaching of Mathematics: From Phylogenesis and Ontogenesis Theory to Classroom Practice. In *Handbook of International Research Mathematics Education*; English, L.D., Ed.; Rutledge: New York, NY, USA, 2002.
11. Hejný, M.; Benešová, M.; Bereková, H.; Bero, P.; Hrdina, L.; Repáš, V.; Vantuch, J. *Teória Vyučovania matematiky 2*; SPN: Bratislava, Slovakia, 1990.
12. Zilkova, K.; Gucaga, J.; Kopacova, J. (Mis)Conceptions about Geometric Shapes in Pre-Service Primary Teachers. *Acta Didact. Napoc.* **2015**, *8*, 27–35.
13. Bohning, G.; Althouse, J.K. Using Tangrams to Teach Geometry to Young Children. *Early Child. Educ. J.* **1997**, *24*, 239–242. [[CrossRef](#)]
14. Lewin, K. Action research and minority problems. *J. Soc. Issues* **1946**, *2*, 34–46. [[CrossRef](#)]
15. Cohen, L.; Manion, L.; Morrison, K. *Research Methods in Education*, 6th ed.; Routledge: London, UK; New York, NY, USA, 2007.
16. McNiff, J. A Place for Learning, Sharing and Creating New Knowledge, Booklet. Available online: <https://www.jeanmcniff.com/ar-booklet.asp> (accessed on 17 July 2021).
17. McNiff, J. *Action Research: All You Need to Know*, 1st ed.; SAGE Publication Ltd.: Los Angeles, CA, USA, 2017.
18. Tripp, D. Action research: A methodological introduction. *Educação e Pesquisa* **2005**, *31*, 444–467.
19. Atkinson, S. Rethinking the Principles and Practice of Action Research: The tensions for the teacher-researcher. *Educ. Action Res.* **1994**, *2*, 383–401. [[CrossRef](#)]
20. Mathematics, National Educational Program (in Slovak: Matematika, Štátny Vzdelávací Program ISCED 1). Available online: https://www.statpedu.sk/files/articles/dokumenty/statny-vzdelavaci-program/matematika_isced1.pdf (accessed on 18 July 2021).
21. Babály, B.; Kárpáti, A. A téri képességek vizsgálata papír alapú és online tesztekkel. *Magy. Pedagógia* **2015**, *115*, 67–92. [[CrossRef](#)]
22. Kárpáti, A.; Babály, B.; Budai, L. Developmental Assessment of Spatial Abilities through Interactive, Online 2D and Virtual 3D Task. *Int. J. Arts Educ.* **2014**, *12*, 94–124.
23. Serin, H. Perspectives on the Teaching of Geometry: Teaching and Learning Methods. *J. Educ. Train.* **2018**, *5*, 1–7. [[CrossRef](#)]
24. Sinclair, N.; Bussi, M.G.; de Villiers, M.; Jones, K.; Kortenkamp, U.; Leung, A.; Owens, K. Geometry Education, Including the Use of New Technologies: A Survey of Recent Research. In Proceedings of the 13th International Congress on Mathematical Education, Hamburg, Germany, 24–31 July 2016; Kaiser, G., Ed.; Springer: Cham, Switzerland, 2017. [[CrossRef](#)]
25. Fujita, T.; Jones, K.; Yamamoto, S. Geometrical Intuition and the Learning and Teaching of Geometry. In Proceedings of the 10th International Congress on Mathematical Education (ICME-10), Copenhagen, Denmark, 4–11 July 2004.
26. Paivio, A. *Mental Representations: A Dual Coding Approach*; Oxford University Press: Oxford, UK, 1990.
27. Piaget's Theory of Cognitive Development. Available online: https://en.wikipedia.org/wiki/Piaget%27s_theory_of_cognitive_development (accessed on 27 July 2021).
28. Kollár, K.N.; Szabó, É. *Pszichológia Pedagógusoknak*; Orisis: Budapest, Hungary, 2004.
29. Presmeg, N. Research on visualization in learning and teaching mathematics: Emergence from psychology. In *Handbook of Research on the Psychology of Mathematics Education: Past, Present, and Future*; Gutierrez, A., Boero, P., Eds.; Sense Publishers: Rotterdam, The Netherlands, 2006; pp. 205–235.