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Coupled Fixed Point Results in Banach Spaces with Applications

Mian Bahadur Zada [1], Muhammad Sarwar [1,*], Thabet Abdeljawad [2,3,4,*] and Aiman Mukheimer [2]

1 Department of Mathematics, University of Malakand, Chakdara 18800, Pakistan; mbz.math@gmail.com
2 Department Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia; mukheimer@psu.edu.sa
3 Department of Medical Research, China Medical University, Taichung 40402, Taiwan
4 Department of Computer Science and Information Engineering, Asia University, Taichung 41354, Taiwan
* Correspondence: sarwar@uom.edu.pk (M.S.); tabdeljawad@psu.edu.sa (T.A.)

Abstract: The aim of this work is to discuss the existence of solutions to the system of fractional variable order hybrid differential equations. For this reason, we establish coupled fixed point results in Banach spaces.

Keywords: coupled fixed point theorems; measure of noncompactness; system of variable order hybrid differential equations

MSC: Primary 47H10; Secondary 54H25

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1. Introduction

In recent years, integral and fractional differential operators have been popular in mathematical models. However, it is the last one hundred years in which the majority of the fractional problems in scientific application and engineering have been discovered. For instance, the earthquake's non-linear oscillation can be framed with fractional derivatives [1], and fractional derivatives combined with the fluid dynamic traffic design can eliminate the shortfall originating from the assumption of continuum traffic flow [2]. That is why the differential equation with a fractional order derivative has recently proven to be a strong gadget in the designing of many processes in various areas of engineering and science [3–8].

Many physical phenomena look like they display fractional order behavior that changes with space and time. The integrals and derivatives whose order is a function of specific variables catch the attention because of their applied significance in different fields of research, such as: multifractional Gaussian noises [9], mechanical applications [10], FIR filters [11], anomalous diffusion modeling [12]. Furthermore, a physical study based on experimental data of variable-order fractional operators has been examined in [13]. A study comparing variable-order fractional and constant-order models has been looked analyzed in [14]. The current literature about solutions to the problems of fractional differential equations is pretty vast, only a few articles study the existence of solutions to differential equations with variable-order. Particularly, Limpanukorn and Ngiamsunthorn [15] discussed the existence of solution to the following fractional order hybrid differential equation

$$\begin{cases} {}_0D_t^{\alpha(\mathtt{t})}[\mathtt{u}(\mathtt{t}) - \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t}))] = \mathtt{g}(\mathtt{t},\mathtt{u}(\mathtt{t})), \\ \mathtt{u}(0) = \mathtt{u}_0, \end{cases} \tag{1}$$

where $\mathtt{t} \in [0,\mathtt{T}], \alpha(\mathtt{t}) \in (0,1]$ and the functions $\mathtt{f},\mathtt{g} : [0,\mathtt{T}] \times \mathbb{R} \to \mathbb{R}$ satisfies specific conditions. ${}_0D_t^{\alpha(\mathtt{t})}$ is the Caputo fractional variable order derivative.

To check the existence of solution, we use fixed point theory, because the problem of the existence of solution usually turns into the problem of finding a fixed point of a particular mapping. Due to this fact, the results of fixed point theory could be implemented to the get results of an operator equation. Equation (1) can be expressed in the form of operator equation

$$\mathtt{u} = \mathtt{fu} + \mathtt{gu}, \quad \mathtt{u} \in \mathbb{X} \tag{2}$$

where $\mathbb{X}$ subspace of a linear space, and $\mathtt{f}, \mathtt{g} : \mathbb{X} \to \mathbb{X}$ are self-mappings. A useful result for the existence of solution to Equation (2) is the Krasnosel'skii [16] fixed point theorem. So many generalizations and improvements of the Krasnosel'skii's fixed point theorem have been produced, for instance [17–21]. In particular, Amar et al. [17] stated some new fixed point results for operator Equation (2), where $\mathtt{f}$ is a weakly compact and weakly sequentially continuous mapping and $\mathtt{g}$ is either a weakly sequentially continuous nonlinear contraction or a weakly sequentially continuous separate contraction mapping. Motivated by the work of [15], we will discuss the existence of solution to the following system of fractional variable order hybrid differential equations:

$$\begin{cases} {}_0D_t^{\alpha(\mathtt{t})}[\mathtt{u}(\mathtt{t}) - \mathtt{f}(\mathtt{t}, \mathtt{u}(\mathtt{t}))] = \mathtt{g}(\mathtt{t}, \mathtt{v}(\mathtt{t})), \\ {}_0D_t^{\alpha(\mathtt{t})}[\mathtt{v}(\mathtt{t}) - \mathtt{f}(\mathtt{t}, \mathtt{v}(\mathtt{t}))] = \mathtt{g}(\mathtt{t}, \mathtt{u}(\mathtt{t})), \\ \mathtt{u}(0) = \zeta(\mathtt{u}(\mathtt{t})), \mathtt{v}(0) = \zeta(\mathtt{v}(\mathtt{t})), \end{cases} \tag{3}$$

where $\mathtt{t} \in [0, \mathtt{T}], \alpha(\mathtt{t}) \in (0, 1], \zeta : \mathbb{R} \to \mathbb{R}$ and $\mathtt{f}, \mathtt{g} : [0, \mathtt{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfies specific conditions. ${}_0D_t^{\alpha(\mathtt{t})}$ is the Caputo fractional variable order derivative. The system (3) can be expressed in the form

$$\begin{cases} \mathtt{u} = \mathtt{fu} + \mathtt{gv}, \\ \mathtt{v} = \mathtt{fv} + \mathtt{gu}, \end{cases} \tag{4}$$

where $\mathtt{u}, \mathtt{v} \in \mathbb{X}$ (subspace of a linear space) or $\mathtt{u}, \mathtt{v} \in \mathbb{M} \subset \mathbb{X}$ and $\mathtt{f} : \mathbb{M} \to \mathbb{X}$ and $\mathtt{g} : \mathbb{X} \to \mathbb{X}$.

A useful technique for finding the fixed point of the system (4) is a coupled fixed point theory, which was introduced by Guo and Lakhsmikantham [22]. Bhaskar and Lakhsmikantham [23] were the pioneers who used coupled fixed point theorem for the existence of unique solution to a periodic boundary value problem. Many prominent researchers have taken greater interest regarding the application potential of coupled fixed point theorems.

For the existence of solution to the system (3), we establish coupled fixed point results in Banach spaces by utilizing the results of Amar et al. [17].

2. Preliminaries

We symbolize by $\mathbb{R}$ and $\mathbb{R}_+$ the set of all real numbers and nonnegative real numbers, respectively, by $\mathbb{N}$ the set of all positive integers and by $\overline{\mathbb{A}^{\mathtt{w}}}$ the weak closure $\mathbb{A}$. Additionally, Ξ denote a Banach space, $\mathfrak{B}(\Xi) = \{\Omega \neq \emptyset : \Omega \text{ is a bounded subset of } \Xi\}$, $\ker \mathfrak{M} = \{\Omega \in \mathfrak{B}(\Xi) : \mathfrak{M}(\Omega) = 0\}$ be the kernel of function $\mathfrak{M} : \mathfrak{B}(\Xi) \to \mathbb{R}_+$ and $\underline{\Omega} = \{\Omega : \Omega \neq \emptyset,$ convex, bounded, and closed subset of $\Xi\}$.

Definition 1 ([24])**.** *The left Riemann-Liouvillle fractional integral of order* $\alpha(\mathtt{t}) \in (0, 1]$ *of a function* $\mathtt{f} : [0, \mathtt{T}] \to \mathbb{R}$ *is*

$${}_{\mathtt{a}}I_{\mathtt{t}}^{\alpha(\mathtt{t})}\mathtt{f}(\mathtt{t}) = \frac{1}{\Gamma(\alpha(\mathtt{t}))} \int_{\mathtt{a}}^{\mathtt{t}} (\mathtt{t} - \mathtt{s})^{\alpha(\mathtt{t})-1} \mathtt{f}(\mathtt{s}) d\mathtt{s}, \quad \mathtt{t} \in [0, \mathtt{T}]. \tag{5}$$

Definition 2 ([25])**.** *The left Caputo fractional derivative of order* $\alpha(\mathtt{t}) \in (0, 1]$ *of a function* $\mathtt{f} : [0, \mathtt{T}] \to \mathbb{R}$ *is*

$${}_{\mathtt{a}}D_{\mathtt{t}}^{\alpha(\mathtt{t})}\mathtt{f}(\mathtt{t}) = \frac{1}{\Gamma(1 - \alpha(\mathtt{t}))} \int_{\mathtt{a}}^{\mathtt{t}} (\mathtt{t} - \mathtt{s})^{-\alpha(\mathtt{t})} \mathtt{f}'(\mathtt{s}) d\mathtt{s}, \quad \mathtt{t} \in [0, \mathtt{T}]. \tag{6}$$

Theorem 1 ([24]). *Let $\alpha : [\mathtt{a}, \mathtt{b}] \to (\mathtt{n}-1, \mathtt{n}]$, where $\mathtt{n} \in \mathbb{N}$. Then*

$$ {}_{\mathtt{a}}I_{\mathtt{t}}^{\alpha(\mathtt{t})}\, {}_{\mathtt{a}}D_{\mathtt{t}}^{\alpha(\mathtt{t})} \mathtt{f}(\mathtt{t}) = \mathtt{f}(\mathtt{t}) - \sum_{k=0}^{n-1} \frac{\mathtt{f}^{(k)}(\mathtt{a})}{k!} (\mathtt{t}-\mathtt{a})^k, \quad \mathtt{t} \in [\mathtt{a}, \mathtt{b}]. \tag{7} $$

The axiomatic definition of an MWNC is as below.

Definition 3 ([26]). *A map $\mathfrak{M}_{\mathtt{w}} : \mathfrak{B}(\Xi) \to \mathbb{R}_+$ is an MWNC in Ξ if for all $\Lambda, \Lambda_1, \Lambda_2 \in \mathfrak{B}(\Xi)$ it satisfies the following axioms:*

(i) $\ker \mathfrak{M}_{\mathtt{w}}$ *is non-empty and relatively weakly compact in Ξ;*
(ii) $\Lambda_1 \subset \Lambda_2 \Rightarrow \mathfrak{M}_{\mathtt{w}}(\Lambda_1) \leqslant \mathfrak{M}_{\mathtt{w}}(\Lambda_2)$;
(iii) $\mathfrak{M}_{\mathtt{w}}(\overline{co}\Lambda) = \mathfrak{M}_{\mathtt{w}}(\Lambda)$;
(iv) $\mathfrak{M}_{\mathtt{w}}(\eta\Lambda_1 + (1-\eta)\Lambda_2) \leqslant \eta\mathfrak{M}_{\mathtt{w}}(\Lambda_1) + (1-\eta)\mathfrak{M}_{\mathtt{w}}(\Lambda_2)$, $\forall \eta \in [0,1]$;
(v) *If $\{\Lambda_n\}$ is a sequence of weakly closed sets in $\mathfrak{B}(\Xi)$ with $\Lambda_{n+1} \subset \Lambda_n$, $\forall n \in \mathbb{N}$ and $\lim_{n\to+\infty} \mathfrak{M}_{\mathtt{w}}(\Lambda_n) = 0$, then $\Lambda_\infty = \bigcap_{n=1}^{+\infty} \Lambda_n \neq \varnothing$.*
In applications, the MWNC satisfies some additional handy conditions.
(vi) $\mathfrak{M}_{\mathtt{w}}(\Lambda) = 0 \iff \Lambda$ *is relatively weakly compact set;*
(vii) $\mathfrak{M}_{\mathtt{w}}(\overline{\Lambda^{\mathtt{w}}}) = \mathfrak{M}_{\mathtt{w}}(\Lambda)$;
(viii) $\mathfrak{M}_{\mathtt{w}}(\lambda\Lambda) = |\lambda|\mathfrak{M}_{\mathtt{w}}(\Lambda), \quad \forall \lambda \in \mathbb{R}$;
(ix) $\mathfrak{M}_{\mathtt{w}}(\Lambda_1 + \Lambda_2) = \mathfrak{M}_{\mathtt{w}}(\Lambda_1) + \mathfrak{M}_{\mathtt{w}}(\Lambda_2)$;
(x) $\mathfrak{M}_{\mathtt{w}}(\Lambda_1 \cup \Lambda_2) = \max\{\mathfrak{M}_{\mathtt{w}}(\Lambda_1), \mathfrak{M}_{\mathtt{w}}(\Lambda_2)\}$.

Remark 1. *Let $\mathfrak{M}_{\mathtt{w}}$ be a measure of noncompactness on a Banach space Ξ, then $\widetilde{\mathfrak{M}_{\mathtt{w}}}(X) = \max\{\mathfrak{M}_{\mathtt{w}}(X_1), \mathfrak{M}_{\mathtt{w}}(X_2)\}$ and $\widetilde{\mathfrak{M}_{\mathtt{w}}}(X) = \mathfrak{M}_{\mathtt{w}}(X_1) + \mathfrak{M}_{\mathtt{w}}(X_2)$ define measures of noncompactness in the space $\Xi \times \Xi$, where $X_i, i = 1, 2$, denotes the natural projections of X.*

Throughout this work, $\rightharpoonup$ will denote the weak convergence and $\to$ will denote the strong convergence, respectively.

Definition 4. *Let $\mathtt{X}$ and $\mathtt{Y}$ be two Banach spaces. A function $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ is called weakly continuous if it is continuous with respect to the weak topologies of $\mathtt{X}$ and $\mathtt{Y}$.*

Definition 5. *Let $\mathtt{X}$ and $\mathtt{Y}$ be two Banach spaces. An operator $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ is said to be weakly sequentially continuous if, for every sequence $(\mathtt{x}_n)_n$ with $\mathtt{x}_n \rightharpoonup \mathtt{x}$, we have $\mathtt{f}\mathtt{x}_n \rightharpoonup \mathtt{f}\mathtt{x}$.*

Theorem 2 ([17]). *Let $\mathit{\Omega} \in \Omega$. If $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$ are two weakly sequentially continuous mappings such that*

(i) $\mathtt{T}$ *is weakly compact;*
(ii) $\mathtt{S}$ *is a nonlinear contraction;*
(iii) $(\mathtt{T} + \mathtt{S})(\mathit{\Omega}) \subset \mathit{\Omega}$.
Then there exists $\mathtt{x} \in \mathit{\Omega}$ such that $\mathtt{x} = \mathtt{T}\mathtt{x} + \mathtt{S}\mathtt{x}$.

Theorem 3 ([17]). *Let $\mathit{\Omega} \in \Omega$. If $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$ are two weakly sequentially continuous mappings such that*

(i) $\mathtt{T}$ *is weakly compact;*
(ii) $\mathtt{S}$ *is a nonlinear contraction;*
(iii) $[\mathtt{x} = \mathtt{S}\mathtt{x} + \mathtt{T}\mathtt{y}, \mathtt{y} \in \mathit{\Omega}] \implies \mathtt{x} \in \mathit{\Omega}$.
Then there exists $\mathtt{x} \in \mathit{\Omega}$ such that $\mathtt{x} = \mathtt{T}\mathtt{x} + \mathtt{S}\mathtt{x}$.

Theorem 4 ([17]). *Let $\mathit{\Omega} \in \Omega$ and $\mathtt{T} : \mathit{\Omega} \to \Xi$ be a weakly sequentially continuous mapping and $\mathtt{S} : \Xi \to \Xi$ such that*

(i) $\mathtt{T}(\mathit{\Omega})$ *is relatively weakly compact;*
(ii) $\mathtt{S}$ *is linear, bounded and there exists $p \in \mathbb{N}^*$ such that $\mathtt{S}^p$ is a nonlinear contraction;*

(iii) $[\mathtt{x} = \mathtt{Sx} + \mathtt{Ty}, \mathtt{y} \in \mathit{\Omega}] \implies \mathtt{x} \in \mathit{\Omega}$.

Then there exists $\mathtt{x} \in \mathit{\Omega}$ *such that* $\mathtt{x} = \mathtt{Tx} + \mathtt{Sx}$.

Theorem 5 ([17]). *Let* $\mathit{\Omega} \in \mathfrak{\Omega}$. *If* $\mathtt{T} : \mathit{\Omega} \to \Xi$ *and* $\mathtt{S} : \Xi \to \Xi$ *are two weakly sequentially continuous mappings such that*

(i) $\mathtt{T}(\mathit{\Omega})$ *is relatively weakly compact;*
(ii) $\mathtt{S}$ *is a nonlinear contraction such that* $\mathtt{S}(\Xi)$ *is bounded;*
(iii) $[\mathtt{x} = \mathtt{Sx} + \mathtt{Ty}, \mathtt{y} \in \mathit{\Omega}] \implies \mathtt{x} \in \mathit{\Omega}$.

Then there exists $\mathtt{x} \in \mathit{\Omega}$ *such that* $\mathtt{x} = \mathtt{Tx} + \mathtt{Sx}$.

Definition 6. *Let X be a non-empty set. Then the mapping* $\mathtt{F} : X \times X \to X$ *has a coupled fixed point* $(\mathtt{x}, \mathtt{y}) \in X \times X$, *if* $\mathtt{F}(\mathtt{x}, \mathtt{y}) = \mathtt{x}$ *and* $\mathtt{F}(\mathtt{y}, \mathtt{x}) = \mathtt{y}$.

3. Coupled Fixed Point Theorems

Let Ξ be a Banach space and $\mathit{\Omega}$ be a nonempty bounded, convex and closed subset of Ξ. Let $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$. Define $\widetilde{\mathtt{T}} : \mathit{\Omega} \times \mathit{\Omega} \to \Xi \times \Xi$, $\widetilde{\mathtt{S}} : \Xi \times \Xi \to \Xi \times \Xi$ and $\mathtt{G} : \mathit{\Omega} \times \Xi \to \Xi$ by

$$\begin{aligned}\widetilde{\mathtt{T}}(\mathtt{x}, \mathtt{y}) &= (\mathtt{Tx}, \mathtt{Ty}),\\ \widetilde{\mathtt{S}}(\mathtt{x}, \mathtt{y}) &= (\mathtt{Sy}, \mathtt{Sx}),\end{aligned}$$

and

$$\mathtt{G}(\mathtt{x}, \mathtt{y}) = \mathtt{Tx} + \mathtt{Sy}.$$

Now, since

$$(\mathtt{G}(\mathtt{x}, \mathtt{y}), \mathtt{G}(\mathtt{y}, \mathtt{x})) = (\mathtt{Tx} + \mathtt{Sy}, \mathtt{Ty} + \mathtt{Sx}) = (\mathtt{Tx}, \mathtt{Ty}) + (\mathtt{Sy}, \mathtt{Sx}) = \widetilde{\mathtt{T}}(\mathtt{x}, \mathtt{y}) + \widetilde{\mathtt{S}}(\mathtt{x}, \mathtt{y}).$$

Thus, to prove that $\mathtt{G}(\mathtt{x}, \mathtt{y})$ has at least one coupled fixed point in $\mathit{\Omega} \times \mathit{\Omega}$, it is sufficient to prove $\widetilde{\mathtt{T}}(\mathtt{x}, \mathtt{y}) + \widetilde{\mathtt{S}}(\mathtt{x}, \mathtt{y})$ has at least one fixed point in $\mathit{\Omega} \times \mathit{\Omega}$. Now utilizing Theorem 4, we present our first result.

Theorem 6. *Let* $\mathit{\Omega} \in \mathfrak{\Omega}$ *and* $\mathtt{T} : \mathit{\Omega} \to \Xi$ *be a weakly sequentially continuous mapping and* $\mathtt{S} : \Xi \to \Xi$ *such that*

(i) $\mathtt{T}(\mathit{\Omega})$ *is relatively weakly compact;*
(ii) $\mathtt{S}$ *is linear, bounded and there exists* $\lambda \in (0, 1)$ *such that*

$$\|\mathtt{Sx} - \mathtt{Sy}\| \leqslant \lambda \|\mathtt{x} - \mathtt{y}\|,$$

(iii) *If* $\mathtt{x} = \mathtt{Sx}^* + \mathtt{Ty}$, *for some* $\mathtt{x}^*, \mathtt{y} \in \mathit{\Omega}$, *then* $\mathtt{x} \in \mathit{\Omega}$.

Then $\mathtt{G}(\mathtt{x}, \mathtt{y}) = \mathtt{Tx} + \mathtt{Sy}$ *has at least one coupled fixed point in* $\mathit{\Omega} \times \mathit{\Omega}$.

Proof. Let $\{\tilde{\mathtt{u}}_n\} = \{(\mathtt{x}_n, \mathtt{y}_n)\}$ be a sequence in $\mathit{\Omega} \times \mathit{\Omega}$ such that $\mathtt{x}_n \rightharpoonup \mathtt{x}$ and $\mathtt{y}_n \rightharpoonup \mathtt{y}$, where $(\mathtt{x}, \mathtt{y}) = \tilde{\mathtt{u}} \in \mathit{\Omega} \times \mathit{\Omega}$. Then since $\mathtt{T} : \mathit{\Omega} \to \Xi$ is weakly sequentially continuous mapping, we have

$$\widetilde{\mathtt{T}}\tilde{\mathtt{u}}_n = \widetilde{\mathtt{T}}(\mathtt{x}_n, \mathtt{y}_n) = (\mathtt{Tx}_n, \mathtt{Ty}_n) \rightharpoonup (\mathtt{Tx}, \mathtt{Ty}) = \widetilde{\mathtt{T}}(\mathtt{x}, \mathtt{y}) = \widetilde{\mathtt{T}}\tilde{\mathtt{u}}.$$

Thus, $\widetilde{\mathtt{T}} : \mathit{\Omega} \times \mathit{\Omega} \to \Xi \times \Xi$ is weakly sequentially continuous mapping. To show that $\widetilde{\mathtt{T}}(\mathit{\Omega} \times \mathit{\Omega})$ is relatively weakly compact, we have

$$\begin{aligned}\widetilde{\mathtt{T}}(\mathit{\Omega} \times \mathit{\Omega}) &= \{\widetilde{\mathtt{T}}(\mathtt{x}, \mathtt{y}) : (\mathtt{x}, \mathtt{y}) \in \mathit{\Omega} \times \mathit{\Omega}\}\\ &= \{(\mathtt{Tx}, \mathtt{Ty}) : \mathtt{x}, \mathtt{y} \in \mathit{\Omega}\}\\ &= \mathtt{T}(\mathit{\Omega}) \times \mathtt{T}(\mathit{\Omega}).\end{aligned}$$

Since $\mathtt{T}(\mathit{\Omega})$ is relatively weakly compact, so $\mathfrak{M}_{\mathtt{w}}(\mathtt{T}(\mathit{\Omega})) = 0$. Using this fact, we have

$$\begin{aligned}\widetilde{\mathfrak{M}_{w}}\big(\widetilde{T}(\Omega\times\Omega)\big)=&\widetilde{\mathfrak{M}_{w}}(T(\Omega)\times T(\Omega))\\=&\max\{\mathfrak{M}_{w}(T(\Omega)),\mathfrak{M}_{w}(T(\Omega))\}\\=&0.\end{aligned}$$

Hence $\widetilde{T}(\Omega\times\Omega)$ is relatively weakly compact. Next, since S is linear so for $\widetilde{x}=(x_1,x_2)$ and $\widetilde{y}=(y_1,y_2)$ in $\Xi\times\Xi$, we have

$$\begin{aligned}\widetilde{S}(c_1\widetilde{x}+c_2\widetilde{y})=&\widetilde{S}(c_1x_1+c_2y_1,c_1x_2+c_2y_2)\\=&(S(c_1x_2+c_2y_2),S(c_1x_1+c_2y_1))\\=&(c_1Sx_2+c_2Sy_2,c_1Sx_1+c_2Sy_1)\\=&(c_1Sx_2,c_1Sx_1)+(c_2Sy_2,c_2Sy_1)\\=&c_1\widetilde{S}(x_1,x_2)+c_2\widetilde{S}(y_1,y_2)\\=&c_1\widetilde{S}\widetilde{x}+c_2\widetilde{S}\widetilde{y}.\end{aligned}$$

Thus, $\widetilde{S}$ is linear. Furthermore, since S is bounded so there exists $\Omega>0$ such that $\|Sx\|\leqslant\Omega,\ \forall x\in\Xi$. Now, for $\widetilde{x}=(x_1,x_2)\in\Xi\times\Xi$, we have

$$\big\|\widetilde{S}\widetilde{x}\big\|=\big\|\widetilde{S}(x_1,x_2)\big\|=\|(Sx_2,Sx_1)\|=\|Sx_2\|+\|Sx_1\|\leqslant 2\Omega,$$

for all $\widetilde{x}\in\Xi\times\Xi$, that is $\widetilde{S}$ is bounded in $\Xi\times\Xi$.

Now, to show that $\widetilde{S}^{\mathfrak{p}}$ is a nonlinear contraction, we use induction. Let $\widetilde{x}=(x_1,x_2)$, $\widetilde{y}=(y_1,y_2)\in\Xi\times\Xi$, then for $\mathfrak{p}=1$ and using condition (ii), we have

$$\begin{aligned}\big\|\widetilde{S}\widetilde{x}-\widetilde{S}\widetilde{y}\big\|=&\big\|\widetilde{S}(x_1,x_2)-\widetilde{S}(y_1,y_2)\big\|\\=&\|(Sx_2,Sx_1)-(Sy_2,Sy_1)\|\\=&\|(Sx_2-Sy_2,Sx_1-Sy_1)\|\\=&\|Sx_2-Sy_2\|+\|Sx_1-Sy_1\|\\\leqslant&\lambda\|x_2-y_2\|+\lambda\|x_1-y_1\|\\=&\lambda\|(x_1-y_1,x_2-y_2)\|\\=&\lambda\|(x_1,x_2)-(y_1,y_2)\|\\=&\lambda\|\widetilde{x}-\widetilde{y}\|\\=&\varphi(\|\widetilde{x}-\widetilde{y}\|).\end{aligned}$$

Assume that $\widetilde{S}^{\mathfrak{q}}$ is a nonlinear contraction for $\mathfrak{q}>1$, that is

$$\big\|\widetilde{S}^{\mathfrak{q}}\widetilde{x}-\widetilde{S}^{\mathfrak{q}}\widetilde{y}\big\|\leqslant\lambda^{\mathfrak{q}}\|\widetilde{x}-\widetilde{y}\|=\varphi(\|\widetilde{x}-\widetilde{y}\|). \tag{8}$$

Thus, using inequality (8) for $\mathfrak{q}+1$, we have

$$\begin{aligned}\Big\|\widetilde{S}^{\mathfrak{q}+1}\widetilde{x}-\widetilde{S}^{\mathfrak{q}+1}\widetilde{y}\Big\|=&\big\|\widetilde{S}\big(\widetilde{S}^{\mathfrak{q}}\widetilde{x}\big)-\widetilde{S}\big(\widetilde{S}^{\mathfrak{q}}\widetilde{y}\big)\big\|\\\leqslant&\lambda\big\|\widetilde{S}^{\mathfrak{q}}\widetilde{x}-\widetilde{S}^{\mathfrak{q}}\widetilde{y}\big\|\\\leqslant&\lambda^{\mathfrak{q}+1}\|\widetilde{x}-\widetilde{y}\|.\end{aligned}$$

That is $\widetilde{S}^{\mathfrak{q}+1}$ is a nonlinear contraction. In general, for any $\mathfrak{p}\in\mathbb{N}$ we can write

$$\big\|\widetilde{S}^{\mathfrak{p}}\widetilde{x}-\widetilde{S}^{\mathfrak{p}}\widetilde{y}\big\|\leqslant\lambda^{\mathfrak{p}}\|\widetilde{x}-\widetilde{y}\|=\varphi(\|\widetilde{x}-\widetilde{y}\|),$$

where $\varphi(x) = \lambda^{\mathfrak{p}} x$ with $\lambda \in (0,1)$, that is $\widetilde{\mathtt{S}}$ is a nonlinear contraction. Hence for all $\mathfrak{p} \in \mathbb{N}^*$, $\widetilde{\mathtt{S}}^{\mathfrak{p}}$ is a nonlinear contraction. Consequently, there exists $\mathfrak{p} \in \mathbb{N}^*$ such that $\widetilde{\mathtt{S}}^{\mathfrak{p}}$ is a nonlinear contraction.

Finally, if $\widetilde{\mathtt{x}} = \widetilde{\mathtt{S}}\widetilde{\mathtt{x}} + \widetilde{\mathtt{T}}\widetilde{\mathtt{y}}$, for some $\widetilde{\mathtt{y}} = (\mathtt{y}_1, \mathtt{y}_2) \in \mathit{\Omega} \times \mathit{\Omega}$, then we have to show that $\widetilde{\mathtt{x}} = (\mathtt{x}_1, \mathtt{x}_2) \in \mathit{\Omega} \times \mathit{\Omega}$. For this, we have

$$(\mathtt{x}_1, \mathtt{x}_2) = \widetilde{\mathtt{S}}(\mathtt{x}_1, \mathtt{x}_2) + \widetilde{\mathtt{T}}(\mathtt{y}_1, \mathtt{y}_2) = (\mathtt{Sx}_2, \mathtt{Sx}_1) + (\mathtt{Ty}_1, \mathtt{Ty}_2) = (\mathtt{Sx}_2 + \mathtt{Ty}_1, \mathtt{Sx}_1 + \mathtt{Ty}_2),$$

which implies that $\mathtt{x}_1 = \mathtt{Sx}_2 + \mathtt{Ty}_1$ and $\mathtt{x}_2 = \mathtt{Sx}_1 + \mathtt{Ty}_2$, by condition (iii), $\mathtt{x}_1, \mathtt{x}_2 \in \mathit{\Omega}$ and hence $\widetilde{\mathtt{x}} \in \mathit{\Omega} \times \mathit{\Omega}$. Thus, by Theorem 4, there exists at least one fixed point of $\widetilde{\mathtt{S}} + \widetilde{\mathtt{T}}$ in $\mathit{\Omega} \times \mathit{\Omega}$ and hence there exists at least one coupled fixed point of $\mathtt{G}(\mathtt{x}, \mathtt{y})$ in $\mathit{\Omega} \times \mathit{\Omega}$. □

Utilizing Theorem 5, we establish the following result:

Theorem 7. *Let $\mathit{\Omega} \in \underline{\Omega}$. If $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$ are two weakly sequentially continuous mappings such that*

(i) $\mathtt{T}(\mathit{\Omega})$ *is relatively weakly compact;*

(ii) *There exists $\lambda \in (0,1)$ such that*

$$\|\mathtt{Sx} - \mathtt{Sy}\| \leqslant \lambda \|\mathtt{x} - \mathtt{y}\|;$$

(iii) *If $\mathtt{x} = \mathtt{Sx}^* + \mathtt{Ty}$, for some $\mathtt{x}^*, \mathtt{y} \in \mathit{\Omega}$, then $\mathtt{x} \in \mathit{\Omega}$.*

If $\mathtt{S}(\Xi)$ is bounded, then $\mathtt{G}(\mathtt{x}, \mathtt{y}) = \mathtt{Tx} + \mathtt{Sy}$ has at least one coupled fixed point in $\mathit{\Omega} \times \mathit{\Omega}$.

Proof. Since $\mathtt{S}(\Xi)$ is bounded, so there exists $\mathfrak{N} > 0$ such that $\|\mathtt{x}\| \leqslant \mathfrak{N}$, $\forall \mathtt{x} \in \mathtt{S}(\Xi)$. Let $\widetilde{\mathtt{x}} = (\mathtt{x}_1, \mathtt{x}_2) \in \widetilde{\mathtt{S}}(\Xi \times \Xi)$, then since

$$\begin{aligned} \widetilde{\mathtt{S}}(\Xi \times \Xi) &= \{\widetilde{\mathtt{S}}(\mathtt{x}, \mathtt{y}) : (\mathtt{x}, \mathtt{y}) \in \Xi \times \Xi\} \\ &= \{(\mathtt{Sy}, \mathtt{Sx}) : \mathtt{x}, \mathtt{y} \in \Xi\} \\ &= \mathtt{S}(\Xi) \times \mathtt{S}(\Xi), \end{aligned}$$

so $\mathtt{x}_1, \mathtt{x}_2 \in \mathtt{S}(\Xi)$ and $\|\widetilde{\mathtt{x}}\| = \|(\mathtt{x}_1, \mathtt{x}_2)\| = \|\mathtt{x}_1\| + \|\mathtt{x}_2\| \leqslant 2\mathfrak{N}$, that is $\widetilde{\mathtt{S}}(\Xi \times \Xi)$ is bounded. For the rest of the proof see Theorem 6. □

Utilizing Theorem 3, we present the following coupled fixed point result:

Theorem 8. *Let $\mathit{\Omega} \in \underline{\Omega}$. If $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$ are two weakly sequentially continuous mappings such that*

(i) $\mathtt{T}$ *is weakly compact such that $\mathtt{T}(\mathbb{D}) \subset \mathtt{T}(\mathit{\Omega})$ for every bounded subset $\mathbb{D} \subset \mathit{\Omega}$;*

(ii) $\mathtt{S}$ *is a nonlinear contraction;*

(iii) *If $\mathtt{x} = \mathtt{Sx}^* + \mathtt{Ty}$, for some $\mathtt{x}^*, \mathtt{y} \in \mathit{\Omega}$, then $\mathtt{x} \in \mathit{\Omega}$.*

Then $\mathtt{G}(\mathtt{x}, \mathtt{y}) = \mathtt{Tx} + \mathtt{Sy}$ has at least one coupled fixed point in $\mathit{\Omega} \times \mathit{\Omega}$.

Proof. Since $\mathtt{T} : \mathit{\Omega} \to \Xi$ and $\mathtt{S} : \Xi \to \Xi$ are two weakly sequentially continuous mappings, so using the same arguments as in Theorem 6, we can easily show that $\widetilde{\mathtt{T}} : \mathit{\Omega} \times \mathit{\Omega} \to \Xi \times \Xi$ and $\widetilde{\mathtt{S}} : \Xi \times \Xi \to \Xi \times \Xi$ are two weakly sequentially continuous mappings. To show that $\widetilde{\mathtt{T}}$ is weakly compact, we have to show that $\widetilde{\mathtt{T}}$ is bounded and $\widetilde{\mathtt{T}}(\mathbb{D} \times \mathbb{D})$ is relatively weakly compact for every bounded subset $\mathbb{D} \times \mathbb{D} \subset \mathit{\Omega} \times \mathit{\Omega}$. For this, since $\mathtt{T}$ is bounded, so there exists $\mathfrak{N} > 0$ such that $\|\mathtt{Tx}\| \leqslant \mathfrak{N}$, $\forall \mathtt{x} \in \mathit{\Omega}$. Now, for $\widetilde{\mathtt{x}} = (\mathtt{x}_1, \mathtt{x}_2) \in \mathit{\Omega} \times \mathit{\Omega}$, we have

$$\left\|\widetilde{\mathtt{T}}\widetilde{\mathtt{x}}\right\| = \left\|\widetilde{\mathtt{T}}(\mathtt{x}_1, \mathtt{x}_2)\right\| = \|(\mathtt{Tx}_1, \mathtt{Tx}_2)\| = \|\mathtt{Tx}_1\| + \|\mathtt{Tx}_2\| \leqslant 2\mathfrak{N},$$

for all $\widetilde{\mathtt{x}} \in \mathit{\Omega} \times \mathit{\Omega}$, that is $\widetilde{\mathtt{T}}$ is bounded in $\mathit{\Omega} \times \mathit{\Omega}$. Following the same steps as in Theorem 6 we obtain that $\widetilde{\mathtt{T}}(\mathbb{D} \times \mathbb{D})$ is relatively weakly compact. Hence $\widetilde{\mathtt{T}}$ is weakly compact.

Next we show that $\widetilde{\mathtt{S}}$ is a nonlinear contraction. For this, using condition (ii), for every $\widetilde{\mathtt{x}} = (\mathtt{x}_1, \mathtt{x}_2), \widetilde{\mathtt{y}} = (\mathtt{y}_1, \mathtt{y}_2) \in \Xi \times \Xi$ and for $\varphi(\mathfrak{r}) = \lambda\mathfrak{r}$ with $\lambda \in (0,1)$, we have

$$\begin{aligned}
\|\widetilde{\mathtt{S}}\widetilde{\mathtt{x}} - \widetilde{\mathtt{S}}\widetilde{\mathtt{y}}\| &= \|\widetilde{\mathtt{S}}(\mathtt{x}_1, \mathtt{x}_2) - \widetilde{\mathtt{S}}(\mathtt{y}_1, \mathtt{y}_2)\| \\
&= \|(\mathtt{S}\mathtt{x}_2, \mathtt{S}\mathtt{x}_1) - (\mathtt{S}\mathtt{y}_2, \mathtt{S}\mathtt{y}_1)\| \\
&= \|(\mathtt{S}\mathtt{x}_2 - \mathtt{S}\mathtt{y}_2, \mathtt{S}\mathtt{x}_1 - \mathtt{S}\mathtt{y}_1)\| \\
&= \|\mathtt{S}\mathtt{x}_2 - \mathtt{S}\mathtt{y}_2\| + \|\mathtt{S}\mathtt{x}_1 - \mathtt{S}\mathtt{y}_1\| \\
&\leqslant \varphi(\|\mathtt{x}_2 - \mathtt{y}_2\|) + \varphi(\|\mathtt{x}_1 - \mathtt{y}_1\|) \\
&= \lambda\|\mathtt{x}_2 - \mathtt{y}_2\| + \lambda\|\mathtt{x}_1 - \mathtt{y}_1\| \\
&= \lambda\|(\mathtt{x}_1 - \mathtt{y}_1, \mathtt{x}_2 - \mathtt{y}_2)\| \\
&= \lambda\|(\mathtt{x}_1, \mathtt{x}_2) - (\mathtt{y}_1, \mathtt{y}_2)\| \\
&= \lambda\|\widetilde{\mathtt{x}} - \widetilde{\mathtt{y}}\| \\
&= \varphi(\|\widetilde{\mathtt{x}} - \widetilde{\mathtt{y}}\|),
\end{aligned}$$

that is $\widetilde{\mathtt{S}}$ is a nonlinear contraction. Finally, if $\widetilde{\mathtt{x}} = \widetilde{\mathtt{S}}\widetilde{\mathtt{x}} + \widetilde{\mathtt{T}}\widetilde{\mathtt{y}}$, for some $\widetilde{\mathtt{y}} = (\mathtt{y}_1, \mathtt{y}_2) \in \Omega \times \Omega$, then following the same steps as in Theorem 6, one can get $\widetilde{\mathtt{x}} \in \Omega \times \Omega$. Thus, by Theorem 3 there exists at least one fixed point of $\widetilde{\mathtt{S}} + \widetilde{\mathtt{T}}$ in $\Omega \times \Omega$ and hence there exists at least one coupled fixed point of $\mathtt{G}(\mathtt{x}, \mathtt{y})$ in $\Omega \times \Omega$. □

4. Applications

In this section, we discuss the existence of solution to the system (3) of fractional variable order hybrid differential equations. First we recall the definition of $\alpha(\mathtt{t})$ over the interval $[0, \mathtt{T}]$. Let $\mathtt{P} = \{[0, \mathtt{T}_1], (\mathtt{T}_1, \mathtt{T}_2], (\mathtt{T}_2, \mathtt{T}_3], \ldots, (\mathtt{T}_{\mathtt{N}-1}, \mathtt{T}_{\mathtt{N}}]\}$ be a partition of the finite interval $[0, \mathtt{T}]$, where $\mathtt{N}$ is a positive integer. Then a piecewise constant function $\alpha : [0, \mathtt{T}] \to (0, 1]$ with respect to $\mathtt{P}$ is defined by

$$\alpha(\mathtt{t}) = \sum_{k=1}^{\mathtt{N}} \alpha_k \mathbb{I}_k(\mathtt{t}) = \begin{cases} \alpha_1, & \mathtt{t} \in [0, \mathtt{T}_1], \\ \alpha_2, & \mathtt{t} \in (\mathtt{T}_1, \mathtt{T}_2], \\ \alpha_3, & \mathtt{t} \in (\mathtt{T}_2, \mathtt{T}_3], \\ \quad\vdots \\ \alpha_{\mathtt{N}}, & \mathtt{t} \in (\mathtt{T}_{\mathtt{N}-1}, \mathtt{T}_{\mathtt{N}}], \end{cases} \tag{9}$$

where $\alpha_k \in (0, 1], k = 1, 2, \ldots, \mathtt{N}$ and $\mathbb{I}_k$ is the indicator of the interval $[\mathtt{T}_{k-1}, \mathtt{T}_k]$ with $\mathtt{T}_0 = 0$ and $\mathtt{T}_{\mathtt{N}} = \mathtt{T}$, that is

$$\mathbb{I}_k(\mathtt{t}) = \begin{cases} 1, & \mathtt{t} \in (\mathtt{T}_{k-1}, \mathtt{T}_k], \\ 0, & \text{otherwise}. \end{cases}$$

First we establish the following lemma:

Lemma 1. *A solution of the fractional variable order differential equation*

$$_0D_t^{\alpha(\mathtt{t})}[\mathtt{u}(\mathtt{t}) - \mathtt{f}(\mathtt{t}, \mathtt{u}(\mathtt{t}))] = \mathtt{g}(\mathtt{t}, \mathtt{v}(\mathtt{t})), \tag{10}$$

with initial condition $\mathtt{u}(0) = \zeta(\mathtt{u}(\mathtt{t}))$ *on the interval* $[\mathtt{T}_{k-1}, \mathtt{T}_k]$ *is*

$$\mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0, \zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t}, \mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_i)} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{s})^{\alpha_i - 1} \mathtt{g}(\mathtt{s}, \mathtt{v}(\mathtt{s})) d\mathtt{s}. \tag{11}$$

Proof. First of all apply the left Riemman–Liouville fractional integral operator $_0I_{\mathtt{t}}^{\alpha(\mathtt{t})}$ of order $\alpha(\mathtt{t})$ to Equation (10) and using Theorem 1, we can easily deduce that

$$\mathtt{u}(\mathtt{t}) = \mathtt{u}(0) - \mathtt{f}(0, \mathtt{u}(0)) + \mathtt{f}(\mathtt{t}, \mathtt{u}(\mathtt{t})) +_0 I_{\mathtt{t}}^{\alpha(\mathtt{t})} \mathtt{g}(\mathtt{t}, \mathtt{v}(\mathtt{t})), \quad \mathtt{t} \in [0, \mathtt{T}].$$

Apply initial condition, we get

$$\mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha(\mathtt{t}))}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha(\mathtt{t})-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))\mathtt{ds}, \quad \mathtt{t}\in[0,\mathtt{T}]. \tag{12}$$

Using (9) the Equation (12) on $[0,\mathtt{T}_1]$ becomes

$$\mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_1)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_1-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))\mathtt{ds}. \tag{13}$$

Again, using (9) the Equation (12) on $(\mathtt{T}_1,\mathtt{T}_2]$ becomes

$$\mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_2)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_2-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))\mathtt{ds}. \tag{14}$$

Proceeding the same way the Equation (12) on $(\mathtt{T}_{i-1},\mathtt{T}_i]$ becomes

$$\mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))\mathtt{ds}. \tag{15}$$

□

With the help of Lemma 10, the initial value problem (3) can be reformulated as the system of integral equations:

$$\begin{cases} \mathtt{u}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))\mathtt{ds}, \\ \mathtt{v}(\mathtt{t}) = \zeta(\mathtt{v}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{v}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{v}(\mathtt{t})) + \frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}, \end{cases} \tag{16}$$

where $\mathtt{t}\in(\mathtt{T}_{i-1},\mathtt{T}_i]$ and $i = 1,2,3,\ldots,\mathtt{N}$.

Theorem 9. *Assume that the following hypotheses hold.*
(A_1) *There exists positive constants* $\daleth'_{\mathtt{f}}$ *and* $\daleth_{\mathtt{f}}$ *such that*

$$|\mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})) - \mathtt{f}(\mathtt{t},\mathtt{v}(\mathtt{t}))| \leqslant \frac{\daleth_{\mathtt{f}}\|\mathtt{u}-\mathtt{v}\|}{\daleth'_{\mathtt{f}} + \|\mathtt{u}-\mathtt{v}\|}, \quad \forall \mathtt{t}\in[0,\mathtt{T}_i];$$

(A_2) $\zeta:\mathbb{R}\to\mathbb{R}$ *is a continuous function such that*

$$|\zeta(\mathtt{u}(\mathtt{t})) - \zeta(\mathtt{v}(\mathtt{t}))| \leqslant \frac{\|\mathtt{u}-\mathtt{v}\|}{\daleth'_{\mathtt{f}} + \|\mathtt{u}-\mathtt{v}\|}, \quad \forall \mathtt{t}\in[0,\mathtt{T}_i];$$

(A_3) *There exists a continuous function* $\mathtt{h}\in\mathtt{L}^{\gamma}([0,\mathtt{T}_i],\mathbb{R})$ *such that*

$$\mathtt{g}(\mathtt{t},\mathtt{u}(\mathtt{t})) \leqslant \mathtt{h}(\mathtt{t}), \quad \forall \mathtt{t}\in[0,\mathtt{T}_i];$$

(A_4) *There exist positive constants* $\daleth_{\mathtt{g}}$ *such that*

$$|\mathtt{g}(\mathtt{t},\mathtt{u}(\mathtt{t})) - \mathtt{g}(\mathtt{t},\mathtt{v}(\mathtt{t}))| \leqslant \daleth_{\mathtt{g}}\|\mathtt{u}-\mathtt{v}\|, \quad \forall \mathtt{t}\in[0,\mathtt{T}_i].$$

In addition, if $\daleth'_{\mathtt{f}} > 2\daleth_{\mathtt{f}} + 1$, *then the system* (3) *has a solution.*

Proof. Let $\mathtt{X} = C([0,\mathtt{T}_i],\mathbb{R})$. Define $\Omega\subset\mathtt{X}$ by

$$\Omega = \{\mathtt{x}\in\mathtt{X} : \|\mathtt{x}\| \leqslant \Xi\}, \tag{17}$$

where $\Xi \geqslant 2 + \Delta_0 + \frac{\|\mathtt{h}\|_{L^1}\mathtt{T}^{\alpha_i}}{\Gamma(\alpha_i+1)}$ with $\Delta_0 = |\zeta(0)| + \max_{\mathtt{t}\in[0,\mathtt{T}_i]} |\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))| + \max_{\mathtt{t}\in[0,\mathtt{T}_i]} |\mathtt{f}(\mathtt{t},0)|$. Then, clearly Ω is a nonempty convex, bounded and closed subset of X. Now, $\mathtt{u}(\mathtt{t})$ is a solution of the system (3) if and only if $\mathtt{u}(\mathtt{t})$ satisfies the system (16). Thus, finding the existence of solution to the system (3) is equivalent to finding the existence of solution to system (16). For this, define the operators $\mathtt{S} : \mathtt{X} \to \mathtt{X}$ and $\mathtt{T} : \Omega \to \mathtt{X}$ by

$$\begin{cases} \mathtt{Su}(\mathtt{t}) = \zeta(\mathtt{u}(\mathtt{t})) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t}))) + \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t})), \\ \mathtt{Tu}(\mathtt{t}) = \dfrac{1}{\Gamma(\alpha_i)} \displaystyle\int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\alpha_i-1} \mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}. \end{cases}$$

Thus, the system of integral Equation (16) is transformed into the system of the following operator equations:

$$\begin{cases} \mathtt{u}(\mathtt{t}) = \mathtt{Su}(\mathtt{t}) + \mathtt{Tv}(\mathtt{t}), \\ \mathtt{v}(\mathtt{t})) = \mathtt{Sv}(\mathtt{t}) + \mathtt{Tu}(\mathtt{t}), \quad \mathtt{t} \in [0,\mathtt{T}_i]. \end{cases} \tag{18}$$

We have to show that the system (18) satisfies all the conditions of Theorem 7. First we show that $\mathtt{T} : \Omega \to \mathtt{X}$ and $\mathtt{S} : \mathtt{X} \to \mathtt{X}$ are two weakly sequentially continuous mappings. For this, let $(\mathtt{x}_n)_n \subset \Omega$ be a sequence with $\mathtt{x}_n \rightharpoonup \mathtt{x}$ for some $\mathtt{x} \in \mathtt{X}$, we have to show that $\mathtt{Tx}_n \rightharpoonup \mathtt{Tx}$. For this, consider

$$\begin{aligned} |\mathtt{Tx}_n(\mathtt{t}) - \mathtt{Tx}(\mathtt{t})| &= \left| \frac{1}{\Gamma(\alpha_i)} \int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\alpha_i-1} \mathtt{g}(\mathtt{s},\mathtt{x}_n(\mathtt{s}))\mathtt{ds} - \frac{1}{\Gamma(\alpha_i)} \int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\alpha_i-1} \mathtt{g}(\mathtt{s},\mathtt{x}(\mathtt{s}))\mathtt{ds} \right| \\ &\leqslant \frac{1}{\Gamma(\alpha_i)} \int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\alpha_i-1} |\mathtt{g}(\mathtt{s},\mathtt{x}_n(\mathtt{s})) - \mathtt{g}(\mathtt{s},\mathtt{x}(\mathtt{s}))|\mathtt{ds} \\ &\leqslant \frac{1}{\Gamma(\alpha_i)} \int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{s})^{\alpha_i-1} \|\mathtt{x}_n - \mathtt{x}\|\mathtt{ds}. \end{aligned}$$

However, $\mathtt{x}_n \rightharpoonup \mathtt{x}$, that is $\|\mathtt{x}_n - \mathtt{x}\| \rightharpoonup 0$ and hence $|\mathtt{Tx}_n(\mathtt{t}) - \mathtt{Tx}(\mathtt{t})| \rightharpoonup 0$. Thus, $\mathtt{Tx}_n(\mathtt{t}) \rightharpoonup \mathtt{Tx}(\mathtt{t})$, that is $\mathtt{T} : \Omega \to \mathtt{X}$ is a weakly sequentially continuous mapping. Similarly,

$$\begin{aligned} |\mathtt{Su}_n(\mathtt{t}) - \mathtt{Su}(\mathtt{t})| &\leqslant |\zeta(\mathtt{u}_n(\mathtt{t})) - \zeta(\mathtt{u}(\mathtt{t}))| + |\mathtt{f}(0,\zeta(\mathtt{u}_n(\mathtt{t}))) - \mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))| \\ &\quad + |\mathtt{f}(\mathtt{t},\mathtt{u}_n(\mathtt{t})) - \mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t}))| \\ &\leqslant \frac{\|\mathtt{u}_n - \mathtt{u}\|}{\daleth'_{\mathtt{f}} + \|\mathtt{u}_n - \mathtt{u}\|} + \frac{\daleth_{\mathtt{f}}\|\mathtt{u}_n - \mathtt{u}\|}{\daleth'_{\mathtt{f}} + \|\mathtt{u}_n - \mathtt{u}\|} + \frac{\daleth_{\mathtt{f}}\|\mathtt{u}_n - \mathtt{u}\|}{\daleth'_{\mathtt{f}} + \|\mathtt{u}_n - \mathtt{u}\|} \\ &= \left(\frac{1 + 2\daleth_{\mathtt{f}}}{\daleth'_{\mathtt{f}} + \|\mathtt{u}_n - \mathtt{u}\|} \right) \|\mathtt{u}_n - \mathtt{u}\|, \end{aligned}$$

However, $\mathtt{x}_n \rightharpoonup \mathtt{x}$, that is $\|\mathtt{x}_n - \mathtt{x}\| \rightharpoonup 0$ and hence $|\mathtt{Sx}_n(\mathtt{t}) - \mathtt{Sx}(\mathtt{t})| \rightharpoonup 0$. Thus, $\mathtt{Sx}_n(\mathtt{t}) \rightharpoonup \mathtt{Sx}(\mathtt{t})$, that is $\mathtt{S} : \mathtt{X} \to \mathtt{X}$ is a weakly sequentially continuous mapping.

Now, we need to show that $\mathtt{T}(\Omega)$ is relatively weakly compact. By definitions of $\mathtt{T}$ and (17), we write $\mathtt{T}(\Omega) = \{\mathtt{Tx} \in \mathtt{X} : \mathtt{x} \in \Omega\}$. For all $\mathtt{t} \in [0,\mathtt{T}_i]$, we have $\mathtt{T}(\Omega)(\mathtt{t}) = \{\mathtt{Tx}(\mathtt{t}) \in \mathtt{X} : \mathtt{x} \in \Omega\}$.

We need to show that $\mathtt{T}(\Omega)$ is bounded and equicontinuous. For $\mathtt{Tu} \in \mathtt{T}(\Omega)$ and $\mathtt{t} \in [0,\mathtt{T}_i]$, we have

$$\begin{aligned}|\mathtt{Tu}(\mathtt{t})| =& \left|\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ \leqslant& \frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}|\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))|\mathtt{ds} \\ \leqslant& \frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}|\mathtt{h}(\mathtt{s})|\mathtt{ds} \\ \leqslant& \frac{\|\mathtt{h}\|_{L^1}}{\Gamma(\alpha_i+1)}\mathtt{t}^{\alpha_i} \\ \leqslant& \frac{\|\mathtt{h}\|_{L^1}\mathtt{T}^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ \leqslant& \Xi.\end{aligned}$$

It follows that $\mathtt{T}(\Omega)$ is bounded. For equicontinuity of $\mathtt{T}(\Omega)$, let $\mathtt{u} \in \Omega$ and $\mathtt{t}_1, \mathtt{t}_2 \in [0, \mathtt{T}_i]$, we have

$$\begin{aligned}|\mathtt{Tu}(\mathtt{t}_1)-\mathtt{Tu}(\mathtt{t}_2)| =& \left|\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}_1}(\mathtt{t}_1-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}-\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ \leqslant& \frac{1}{\Gamma(\alpha_i)}\left|\int_0^{\mathtt{t}_1}(\mathtt{t}_1-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}-\int_0^{\mathtt{t}_1}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ &+\frac{1}{\Gamma(\alpha_i)}\left|\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ =& \frac{1}{\Gamma(\alpha_i)}\left|\int_0^{\mathtt{t}_1}(\mathtt{t}_1-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}+\int_{\mathtt{t}_1}^{0}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ &+\frac{1}{\Gamma(\alpha_i)}\left|\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{u}(\mathtt{s}))\mathtt{ds}\right| \\ \leqslant& \frac{1}{\Gamma(\alpha_i)}\left|\int_0^{\mathtt{t}_1}(\mathtt{t}_1-\mathtt{s})^{\alpha_i-1}\mathtt{h}(\mathtt{s})\mathtt{ds}+\int_{\mathtt{t}_1}^{0}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{h}(\mathtt{s})\mathtt{ds}\right| \\ &+\frac{1}{\Gamma(\alpha_i)}\left|\int_{\mathtt{t}_1}^{\mathtt{t}_2}(\mathtt{t}_2-\mathtt{s})^{\alpha_i-1}\mathtt{h}(\mathtt{s})\mathtt{ds}\right| \\ =& \frac{\|\mathtt{h}\|_{L^1}}{\Gamma(\alpha_i)}\left(\left|\frac{(\mathtt{t}_2-\mathtt{t}_1)^{\alpha_i}}{\alpha_i}+\frac{\mathtt{t}_1^{\alpha_i}}{\alpha_i}-\frac{\mathtt{t}_2^{\alpha_i}}{\alpha_i}\right|+\left|\frac{(\mathtt{t}_2-\mathtt{t}_1)^{\alpha_i}}{\alpha_i}\right|\right) \\ \leqslant& \frac{\|\mathtt{h}\|_{L^1}}{\Gamma(\alpha_i+1)}\left(|\mathtt{t}_2-\mathtt{t}_1|^{\alpha_i}+\left|\mathtt{t}_1^{\alpha_i}-\mathtt{t}_2^{\alpha_i}\right|+|\mathtt{t}_2-\mathtt{t}_1|^{\alpha_i}\right).\end{aligned}$$

Since $\mathtt{t}^\alpha$ is uniformly continuous on $[0, \mathtt{T}_i]$, so for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left|\mathtt{t}_1^{\alpha_i}-\mathtt{t}_2^{\alpha_i}\right| < \frac{\Gamma(\alpha_i+1)}{3\|h\|_{L^1}}\varepsilon, \text{ whenever } |\mathtt{t}_1-\mathtt{t}_2| < \delta_1.$$

Let $\delta = \min\left\{\delta_1, \left(\frac{\Gamma(\alpha_i+1)}{3\|h\|_{L^1}}\right)^{\frac{1}{\alpha_i}}\right\}$, then whenever $|\mathtt{t}_1 - \mathtt{t}_2| < \delta$, we have

$$|\mathtt{Tu}(\mathtt{t}_1)-\mathtt{Tu}(\mathtt{t}_2)| \leqslant \frac{\|h\|_{L^1}}{\Gamma(\alpha_i+1)}\left(\frac{\Gamma(\alpha_i+1)}{3\|h\|_{L^1}}\varepsilon+\frac{\Gamma(\alpha_i+1)}{3\|h\|_{L^1}}\varepsilon+\frac{\Gamma(\alpha_i+1)}{3\|h\|_{L^1}}\varepsilon\right) = \varepsilon.$$

That is $\mathtt{T}(\Omega)$ is equicontinuous. Hence by Arzelà–Ascoli's theorem for any sequence $(\mathtt{x}_n)$ in $\mathtt{T}(\Omega)$ there is a subsequence $(\mathtt{x}_{n_k})$ such that $\mathtt{x}_{n_k} \rightharpoonup \mathtt{x} \in \mathtt{T}(\Omega)$. Consequently, $\mathtt{T}(\Omega)$ is relatively weakly sequentially compact. Thus, by Eberlein-Smulian theorem $\mathtt{T}(\Omega)$ is relatively weakly compact.

Next, we have to verify condition (ii) of Theorem 7. To do this, consider

$$\begin{aligned}|\mathtt{Su}(\mathtt{t})-\mathtt{Sv}(\mathtt{t}))| \leqslant&|\zeta(\mathtt{u}(\mathtt{t}))-\zeta(\mathtt{v}(\mathtt{t}))|+|\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))-\mathtt{f}(0,\zeta(\mathtt{v}(\mathtt{t})))|\\ &+|\mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t}))-\mathtt{f}(\mathtt{t},\mathtt{v}(\mathtt{t}))|\\ \leqslant&\frac{\|\mathtt{u}-\mathtt{v}\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}-\mathtt{v}\|}+\frac{\daleth_{\mathtt{f}}\|\mathtt{u}-\mathtt{v}\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}-\mathtt{v}\|}+\frac{\daleth_{\mathtt{f}}\|\mathtt{u}-\mathtt{v}\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}-\mathtt{v}\|}\\ =&\left(\frac{1+2\daleth_{\mathtt{f}}}{\daleth'_{\mathtt{f}}+\|\mathtt{u}-\mathtt{v}\|}\right)\|\mathtt{u}-\mathtt{v}\|,\end{aligned}$$

which implies that

$$\|\mathtt{Su}-\mathtt{Sv}\| \leqslant \varpi\|\mathtt{u}-\mathtt{v}\|,$$

where $\varpi = \frac{1+2\daleth_{\mathtt{f}}}{\daleth'_{\mathtt{f}}+\|\mathtt{u}-\mathtt{v}\|} < 1$.

Furthermore, we have to prove condition (iii) of Theorem 7, let $\mathtt{u}^*, \mathtt{v} \in M$ such that $\mathtt{u} = \mathtt{Su}^* + \mathtt{Tv}$, by assumptions (A_1) and (A_2), we have

$$\begin{aligned}|\mathtt{u}(\mathtt{t})| =&|\mathtt{Su}^*(\mathtt{t})+\mathtt{Tv}(\mathtt{t})|\\ =&\left|\zeta(\mathtt{u}^*(\mathtt{t}))-\mathtt{f}(0,\zeta(\mathtt{u}^*(\mathtt{t})))+\mathtt{f}(\mathtt{t},\mathtt{u}^*(\mathtt{t}))+\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))d\mathtt{s}\right|\\ \leqslant&|\zeta(\mathtt{u}^*(\mathtt{t}))-\zeta(0)|+|\zeta(0)|+|\mathtt{f}(0,\zeta(\mathtt{u}^*(\mathtt{t})))|+|\mathtt{f}(\mathtt{t},\mathtt{u}^*(\mathtt{t}))-\mathtt{f}(\mathtt{t},0)|+|\mathtt{f}(\mathtt{t},0)|\\ &+\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}|\mathtt{g}(\mathtt{s},\mathtt{v}(\mathtt{s}))|d\mathtt{s}\\ \leqslant&\frac{\|\mathtt{u}^*\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}^*\|}+\frac{\daleth_{\mathtt{f}}\|\mathtt{u}^*\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}^*\|}+|\zeta(0)|+\max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(0,\zeta(\mathtt{u}^*(\mathtt{t})))|+\max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(\mathtt{t},0)|\\ &+\frac{1}{\Gamma(\alpha_i)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{s})^{\alpha_i-1}|\mathtt{h}(\mathtt{s})|d\mathtt{s}\\ \leqslant&2+\Delta_0+\frac{\|\mathtt{h}\|_{L^1}}{\Gamma(\alpha_i+1)}\mathtt{t}^{\alpha_i}\\ \leqslant&2+\Delta_0+\frac{\|\mathtt{h}\|_{L^1}\mathtt{T}^{\alpha_i}}{\Gamma(\alpha_i+1)}\\ \leqslant&\Xi.\end{aligned}$$

That is $\|\mathtt{u}\| \leqslant \Xi$ and hence $\mathtt{u} \in \Omega$. Thus, condition (ii) of Theorem 7 holds. Finally, since $\mathtt{S}(\mathtt{X}) = \{\mathtt{S}(\mathtt{x}) : \mathtt{x} \in \mathtt{X}\}$, so for $\mathtt{u} \in \mathtt{X}$ and $\mathtt{t} \in [0, \mathtt{T}_i]$, we have

$$\begin{aligned}|\mathtt{Su}(\mathtt{t})| =&|\zeta(\mathtt{u}(\mathtt{t}))-\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))+\mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t}))|\\ \leqslant&|\zeta(\mathtt{u}(\mathtt{t}))-\zeta(0)|+|\zeta(0)|+|\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))|+|\mathtt{f}(\mathtt{t},\mathtt{u}(\mathtt{t}))-\mathtt{f}(\mathtt{t},0)|+|\mathtt{f}(\mathtt{t},0)|\\ \leqslant&\frac{\|\mathtt{u}\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}\|}+\frac{\daleth_{\mathtt{f}}\|\mathtt{u}\|}{\daleth'_{\mathtt{f}}+\|\mathtt{u}\|}+|\zeta(0)|+\max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))|+\max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(\mathtt{t},0)|\\ <&2+\Delta_0,\end{aligned}$$

which implies that $\|\mathtt{Su}\| < 2+\Delta_0$, where $\Delta_0 = |\zeta(0)| + \max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(0,\zeta(\mathtt{u}(\mathtt{t})))| + \max_{\mathtt{t}\in[0,\mathtt{T}_i]}|\mathtt{f}(\mathtt{t},0)|$ and hence $\mathtt{S}(\mathtt{X})$ is bounded. Therefore by Theorem 7, the operator $\mathtt{G}(\mathtt{u},\mathtt{v}) = \mathtt{Tu} + \mathtt{Sv}$ has a coupled fixed point in $\widetilde{\Omega}$. Accordingly, the system (3) has a solution in $\widetilde{\Omega}$. □

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