

Article

On a Volterra Integral Equation with Delay, via w -Distances

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Abstract: The paper deals with a Volterra integral equation with delay. In order to apply the w -weak generalized contraction theorem for the study of existence and uniqueness of solutions, we rewrite the equation as a fixed point problem. The assumptions take into account the support of w -distance and the complexity of the delay equation. Gronwall-type theorem and comparison theorem are also discussed using a weak Picard operator technique. In the end, an example is provided to support our results.

Keywords: Volterra integral equation with delay; w -distance; weakly Picard operator; abstract Gronwall lemma



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1. Introduction

The study of Volterra integral equations is an interesting area of research because of their applications in physics, biology, control theory and in other fields of sciences. In the last decades, they have been extensively and intensively studied. Numerous results on existence and uniqueness, monotonicity, stability, as well as numerical solutions have been obtained. To name a few, we refer the reader to [1–5] and the references therein. On the other hand, the results regarding the blow-up of the solutions is among the most attractive topics in qualitative theory of Volterra integral equations due to their applications, especially in biology, economics and physics (see, e.g., [5–8]). The theory of Volterra integral equations with delay have been studied by many authors (see, e.g., [1,2,9–11]). The most common approach in studying the existence of solutions for a Volterra integral equation with delay, is to rewrite (1) as a fixed point problem. Then, one can apply different fixed point principles to the above equation and establish the existence of solutions (see, e.g., [9,10,12–14]).

Very recently, many results related to mappings satisfying various contractive conditions and underlying distance spaces were obtained in [9,15–21] and the references contained therein.

In this paper, we consider a Volterra integral equation with delay of the form

$$y(x) = \varphi(x) + \int_{a_1}^x F(x, s, y(g(s))) ds \quad (1)$$

where $a_0, a_1, a_2 \in \mathbb{R}$, $a_0 < a_1 < a_2$, and $\varphi \in C([a_1, a_2], \mathbb{R})$, $F \in C([a_1, a_2] \times [a_1, a_2] \times \mathbb{R}, \mathbb{R})$, $g \in C([a_1, a_2], [a_0, a_2])$ with $g(x) \leq x$, $y(x) = \tilde{y}(x)$ for $x \in [a_0, a_1]$, are given.

In this paper, the motivation of the work has been started from the results of T. Wongyat and W. Sintunavarat [20]. Using w -weak generalized contractions theorem, we give some results in the case of Volterra integral equations with delay. In the end a Gronwall-type theorem and a comparison theorem are also obtained.

2. Preliminaries

For the convenience of the reader we recall here some definitions and preliminary results, for details, see [17,20].

Let (\mathcal{Y}, ρ) be a metric space. First we present the notion of w -distance on \mathcal{Y} and w^0 -distance on \mathcal{Y} .

Definition 1 ([17]). Let (\mathcal{Y}, ρ) be a metric space. A function $d : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ is called w -distance on \mathcal{Y} , if the following conditions are satisfied:

- (1) $d(y_1, y_2) \leq d(y_1, y_3) + d(y_3, y_2), \forall y_1, y_2, y_3 \in \mathcal{Y}$;
- (2) $d(y, \cdot) : \mathcal{Y} \rightarrow [0, \infty)$ is lower semicontinuous, $y \in \mathcal{Y}$;
- (3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(y_1, y_2) \leq \delta$ and $d(y_1, y_3) \leq \delta$ imply $\rho(y_2, y_3) \leq \varepsilon, \forall y_1, y_2, y_3 \in \mathcal{Y}$.

It is well known that each metric on a nonempty set \mathcal{Y} is a w -distance on \mathcal{Y} .

Definition 2 ([20]). A function $\psi : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ is called w^0 -distance on \mathcal{Y} , if it is a w -distance on \mathcal{Y} with $d(y, y) = 0$ for all $y \in \mathcal{Y}$.

Next, we give the definitions of an altering distance function, ceiling distance of ρ and w -generalized weak contraction mapping used in the paper [20].

Definition 3 ([20]). A function $\psi : [0, \infty) \rightarrow [0, +\infty)$ is called an altering distance function, if the following conditions are satisfied:

- (1) ψ is a continuous and nondecreasing function;
- (2) $\psi(x) = 0$ if and only if $x = 0$.

Definition 4 ([20]). A w -distance q on a metric space (\mathcal{Y}, ρ) is called a ceiling distance of ρ if and only if $d(y_1, y_2) \geq \rho(y_1, y_2), \forall y_1, y_2 \in \mathcal{Y}$.

Definition 5 ([20]). Let d be a w -distance on a metric space (\mathcal{Y}, ρ) . An operator $V : \mathcal{Y} \rightarrow \mathcal{Y}$ is called a w -generalized weak contraction mapping if

$$\psi(d(V(y_1), V(y_2))) \leq \psi(m(y_1, y_2)) - \phi(d(y_1, y_2)), \forall y_1, y_2 \in \mathcal{Y}, \tag{2}$$

where

$$m(y_1, y_2) := \max \left\{ d(y_1, y_2), \frac{1}{2} [d(y_1, V(y_2)) + d(V(y_1), y_2)] \right\}, \tag{3}$$

$\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function, and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\phi(x) = 0$ if and only if $x = 0$. If $d = \rho$, then the mapping V is called generalized weak contraction mapping.

Let (\mathcal{Y}, ρ) be a complete metric space. We present below some results of fixed point of the operatorial equation $V(y) = y, y \in \mathcal{Y}$ via w -distances.

Theorem 1 ([20]). Let $d : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ be a w^0 -distance on \mathcal{Y} and a ceiling distance of ρ . Suppose that $V : \mathcal{Y} \rightarrow \mathcal{Y}$ is a continuous w -generalized weak contraction. Then, the operator V has a unique fixed point in \mathcal{Y} . Moreover, for each $y_0 \in \mathcal{Y}$, the successive approximation sequence $\{y_n\}_{n \in \mathbb{N}}$, defined by $y_n = V^n(y_0)$, for all $n \in \mathbb{N}$ converges to the unique fixed point of the operator V .

Theorem 2 ([20]). Let $d : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ be a continuous w^0 -distance on \mathcal{Y} and a ceiling distance of ρ . Suppose that $V : \mathcal{Y} \rightarrow \mathcal{Y}$ is a w -generalized weak contraction. Then, the operator V has a unique fixed point in \mathcal{Y} . Moreover, for each $y_0 \in \mathcal{Y}$, the successive approximation sequence

$\{y_n\}_{n \in \mathbb{N}}$, defined by $y_n = V^n(y_0)$, for all $n \in \mathbb{N}$ converges to the unique fixed point of the operator V .

Theorem 3 ([20]). Let $d : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ be a continuous w -distance on \mathcal{Y} and a ceiling distance of ρ . Suppose that $V : \mathcal{Y} \rightarrow \mathcal{Y}$ is a continuous operator such that, for all $y_1, y_2 \in \mathcal{Y}$

$$\psi(d(V(y_1), V(y_2))) \leq \psi(m(y_1, y_2)) - \phi(d(y_1, y_2)), \tag{4}$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function, and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\phi(x) = 0$ if and only if $x = 0$. Then, the operator V has a unique fixed point in \mathcal{Y} . Moreover, for each $y_0 \in \mathcal{Y}$, the successive approximation sequence $\{y_n\}_{n \in \mathbb{N}}$, defined by $y_n = V^n(y_0)$, for all $n \in \mathbb{N}$ converges to the unique fixed point of the operator V .

We now give some definitions and lemmas (see [22–24]), which are needed in advance.

Let (\mathcal{Y}, ρ) be a metric space. Let us consider a given operator $V : \mathcal{Y} \rightarrow \mathcal{Y}$. In this setting, V is called weakly Picard operator (briefly WPO) if, for all $y \in \mathcal{Y}$, the sequence of Picard iterations, $\{V^n(y)\}_{n \in \mathbb{N}}$, converges in (\mathcal{Y}, ρ) and its limit (which generally depend on y) is a fixed point of V . We denote by F_V the fixed point set of V , i.e., $F_V = \{y \in \mathcal{Y} : V(y) = y\}$. If an operator V is WPO with $F_V = \{y^*\}$, then, by definition, V is called a Picard operator (briefly PO).

If $V : \mathcal{Y} \rightarrow \mathcal{Y}$ is a WPO, we can define the operator $V^\infty : \mathcal{Y} \rightarrow F_V$, by $V^\infty(y) := \lim_{n \rightarrow +\infty} V^n(y)$.

If \mathcal{Y} is a nonempty set, then the triple $(\mathcal{Y}, \rho, \leq)$ is an ordered metric space, where \leq is a partial order relation on \mathcal{Y} .

In the setting of ordered metric spaces, we have some properties related to WPOs and POs.

Theorem 4 (Rus [22,23] Characterization theorem). Let (\mathcal{Y}, ρ) be a metric space and $V : \mathcal{Y} \rightarrow \mathcal{Y}$ an operator. Then V is WPO if and only if there exists a partition of \mathcal{Y} , $\mathcal{Y} = \bigcup_{\lambda \in \Lambda} \mathcal{Y}_\lambda$, such that

- (i) $\mathcal{Y}_\lambda \in I(V)$, for all $\lambda \in \Lambda$;
- (ii) $V|_{\mathcal{Y}_\lambda} : \mathcal{Y}_\lambda \rightarrow \mathcal{Y}_\lambda$ is PO, for all $\lambda \in \Lambda$.

Theorem 5 ([23] Abstract Gronwall Lemma). Let $(\mathcal{Y}, \rho, \leq)$ be an ordered metric space and $V : \mathcal{Y} \rightarrow \mathcal{Y}$ be an increasing WPO. Then we have the following:

- (j) for $y \in \mathcal{Y}, y \leq V(y) \Rightarrow y \leq V^\infty(y)$;
- (jj) for $y \in \mathcal{Y}, y \geq V(y) \Rightarrow y \geq V^\infty(y)$.

Theorem 6 ([23] Abstract Comparison Lemma). Let $(\mathcal{Y}, \rho, \leq)$ be an ordered metric space and $V_1, V_2, V_3 : \mathcal{Y} \rightarrow \mathcal{Y}$ be such that:

- (h) $V_1 \leq V_2 \leq V_3$;
- (hh) the operators V_1, V_2, V_3 are WPO;
- (hhh) the operator V_2 is increasing.

Then, for $y_1, y_2, y_3 \in \mathcal{Y}, y_1 \leq y_2 \leq y_3 \Rightarrow V_1^\infty(y_1) \leq V_2^\infty(y_2) \leq V_3^\infty(y_3)$.

For the theory of weakly Picard operators, its generalization and applications, see [9,11,12,14,22–28].

3. Main Result

Throughout this paper it will be assumed that:

- (C₁) $y(x) = \tilde{y}(x), x \in [a_0, a_1]$;
- (C₂) $\tilde{y}(a_1) = \varphi(a_1)$;

$$(C_3) \quad \sup_{x \in [a_0, a_2]} y(x) = \sup_{x \in [a_1, a_2]} y(x).$$

With respect to the Equation (1) we consider the equation (in $\alpha \in \mathbb{R}$)

$$\alpha = \varphi(a_1). \tag{5}$$

Let S_φ be the solution set of the Equation (5).

Now we consider the operator $V : C([a_0, a_2], \mathbb{R}) \rightarrow C([a_0, a_2], \mathbb{R})$ defined by

$$V(y)(x) := \varphi(x) + \int_{a_1}^x F(x, s, y(g(s))) ds \tag{6}$$

for all $y \in C([a_0, a_2], \mathbb{R})$ and $x \in [a_1, a_2]$.

Let $\mathcal{Y} := C([a_0, a_2], \mathbb{R})$ and $\mathcal{Y}_{\tilde{y}} := \{y \in \mathcal{Y} \mid y|_{[a_0, a_1]} = \tilde{y}\}$. Then

$$\mathcal{Y} = \bigcup_{\tilde{y} \in C([a_0, a_1], \mathbb{R})} \mathcal{Y}_{\tilde{y}}$$

is a partition of \mathcal{Y} .

Lemma 1. We suppose that the conditions (C_1) , (C_2) and (C_3) are satisfied. Then it is obvious that $V(\mathcal{Y}) \subset \mathcal{Y}_{\tilde{y}}$ and $V(\mathcal{Y}_{\tilde{y}}) \subset \mathcal{Y}_{\tilde{y}}$.

The main purpose of this section is to prove a new result of the existence, uniqueness and approximation of the solution for nonlinear Volterra integral equation with delay by using Theorem 3.

Theorem 7. We consider the integral Equation (1) where $a_0, a_1, a_2 \in \mathbb{R}$, $a_0 < a_1 < a_2$, and $\varphi \in C([a_1, a_2], \mathbb{R})$, $F \in C([a_1, a_2] \times [a_1, a_2] \times \mathbb{R}, \mathbb{R})$, $g \in C([a_1, a_2], [a_0, a_2])$ with $g(x) \leq x$, are given functions. We suppose the following:

- (i) the mapping $V : C([a_0, a_2], \mathbb{R}) \rightarrow C([a_0, a_2], \mathbb{R})$ defined by (6) is continuous;
- (ii) the altering distance function $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfies $\psi(x) < x$, for all $x > 0$, and the continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfies $\phi(x) = 0$, if and only if $x = 0$;
- (iii)

$$\begin{aligned} & |F(x, s, y_1(g(s)))| + |F(x, s, y_2(g(s)))| \leq \\ & \leq \frac{1}{a_2 - a_1} \left[\psi(|y_1(g(s))| + |y_2(g(s))|) - \phi\left(\sup_{t \in [a_0, a_2]} |y_1(g(t))| + \sup_{t \in [a_0, a_2]} |y_2(g(t))|\right) - 2|\varphi(x)| \right], \end{aligned}$$

for all $y_1, y_2 \in C([a_0, a_2], \mathbb{R})$, $x, s \in [a_1, a_2]$.

Then the integral Equation (1) has a unique solution;

Moreover, for each $y_0 \in C([a_0, a_2], \mathbb{R})$, the sequence of Picard iterations $\{y_n\}_{n \in \mathbb{N}}$, defined by $y_n = V^n(y_0)$, for all $n \in \mathbb{N}$, converges to the unique solution of the integral Equation (1).

Proof. Let $\mathcal{Y} = C([a_0, a_2], \mathbb{R})$ and we consider the metric $\rho : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, \infty)$ given by $\rho(y_1, y_2) := \sup_{x \in [a_1, a_2]} |y_1(x) - y_2(x)|$, for all $y_1, y_2 \in C([a_0, a_2], \mathbb{R})$. It is clear that (\mathcal{Y}, ρ) is a complete metric space. Now, we define the function $d : C([a_0, a_2], \mathbb{R}) \times C([a_0, a_2], \mathbb{R}) \rightarrow [0, \infty)$ by the relation:

$$d(y_1, y_2) := \sup_{x \in [a_0, a_2]} |y_1(x)| + \sup_{x \in [a_0, a_2]} |y_2(x)|, \text{ for all } y_1, y_2 \in C([a_0, a_2], \mathbb{R}) \tag{7}$$

and it is easy to see that d is a w -distance on $C([a_0, a_2], \mathbb{R})$ and a ceiling distance of ρ .

We intend to show that the operator V satisfies the condition (4). We have

$$\begin{aligned}
 & |V(y_1)(x)| + |V(y_2)(x)| \\
 &= \left| \varphi(x) + \int_{a_1}^x F(x, s, y_1(g(s))) ds \right| + \left| \varphi(x) + \int_{a_1}^x F(x, s, y_2(g(s))) ds \right| \\
 &\leq |\varphi(x)| + \left| \int_{a_1}^x F(x, s, y_1(g(s))) ds \right| + |\varphi(x)| + \\
 &\quad + \left| \int_{a_1}^x F(x, s, y_2(g(s))) ds \right| \\
 &\leq 2|\varphi(x)| + \int_{a_1}^x (|F(x, s, y_1(g(s)))| + |F(x, s, y_2(g(s)))|) ds \\
 &\leq 2|\varphi(x)| + \frac{1}{a_2 - a_1} \int_{a_1}^x [\psi(|y_1(g(s))| + |y_2(g(s))|) \\
 &\quad - \phi\left(\sup_{t \in [a_0, a_2]} |y_1(g(t))| + \sup_{t \in [a_0, a_2]} |y_2(g(t))|\right) - 2|\varphi(x)|] ds \\
 &\leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)).
 \end{aligned}$$

This implies that

$$\sup_{x \in [a_0, a_2]} |Vy_1(x)| + \sup_{x \in [a_0, a_2]} |Vy_2(x)| \leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)).$$

This leads to

$$d(Vy_1, Vy_2) \leq \psi(m(y_1, y_2)) - \phi(d(y_1, y_2)), \text{ for all } y_1, y_2 \in \mathcal{Y}.$$

Furthermore, we have

$$\psi(d(Vy_1, Vy_2)) \leq d(Vy_1, Vy_2) \leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)), \text{ for all } y_1, y_2 \in \mathcal{Y}.$$

We obtain that V satisfies the condition (4) and thus V is a Picard operator. This implies that there exists a unique solution of the integral Equation (1). \square

Since the operator V defined in (6) is a PO, we can establish the following Gronwall-type lemma for the Equation (1).

Theorem 8. We consider the integral Equation (1) where $a_1, a_2 \in \mathbb{R}$, $a_1 < a_2$, and the functions $\varphi \in C([a_1, a_2], \mathbb{R})$, $F \in C([a_1, a_2] \times [a_1, a_2] \times \mathbb{R}, \mathbb{R})$, $g \in C([a_1, a_2], [a_0, a_2])$ with $g(x) \leq x$, are given. We assume that the conditions (i)–(iii) from Theorem 7 hold. Furthermore, we suppose that

(iv) $F(x, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function with respect to the last argument, for all $x, s \in [a_1, a_2]$.

Let $y^* \in C([a_0, a_2], \mathbb{R})$ be the unique solution of the system. Then, the following implications hold:

(1) for all $y \in C([a_0, a_2], \mathbb{R})$ with

$$y(x) \leq \varphi(x) + \int_{a_1}^x F(x, s, y(g(s))) ds,$$

for all $y \in [a_1, a_2]$, we have $y \leq y^*$;

(2) for all $y \in C([a_0, a_2], \mathbb{R})$ with

$$y(x) \geq \varphi(x) + \int_{a_1}^x F(x, s, y(g(s))) ds,$$

for all $x \in [a_1, a_2]$, we have $y \geq y^*$.

Proof. From (iv), we have that the operator V defined in (6) is increasing with respect to the partial order.

By the proof of Theorem 7, it follows that V is a Picard operator. The conclusion of the theorem follows from Theorem 5. \square

In a similar way, a comparison theorem for Equation (1) can be obtained, using the abstract comparison theorem given in Section 2 of this paper.

Theorem 9. We consider the integral Equation (1) where $a_1, a_2 \in \mathbb{R}$, $a_1 < a_2$, and the functions $\varphi_i \in C([a_1, a_2], \mathbb{R})$, $F_i \in C([a_1, a_2] \times [a_1, a_2] \times \mathbb{R}, \mathbb{R})$ and $g_i \in C([a_1, a_2], [a_0, a_2])$, $i = 1, 2, 3$ are given. We assume that the conditions (i)–(iii) from Theorem 7 hold. Furthermore, we suppose that

- (i) $\varphi_1 \leq \varphi_2 \leq \varphi_3, F_1 \leq F_2 \leq F_3, g_1 \leq g_2 \leq g_3$;
- (ii) φ_2, F_2, g_2 are increasing;
- (iii) $S_{\varphi_1} = S_{\varphi_2} = S_{\varphi_3}$.

Let $y_i \in C([a_1, b], \mathbb{R})$ be a solution of the equation

$$y_i(x) = \varphi_i(x) + \int_{a_1}^x F_i(x, s, y(g_i(s))) ds, \quad x \in [a, b], \quad i = 1, 2, 3.$$

If $y_1(x) \leq y_2(x) \leq y_3(x)$, $x \in [a_1, a]$ then $y_1(x) \leq y_2(x) \leq y_3(x)$, $x \in [a_1, a_2]$.

Proof. The proof follows from the Theorem 6. \square

Next we study the existence and uniqueness of solutions of the following integral equation using Theorem 7.

Example

We consider the integral equation

$$y(x) = \frac{x}{4} + \int_0^x xs^2y(\lambda s)ds, \tag{8}$$

where $x \in [0, 1]$, $\lambda \in [0, 1]$, and the following condition

$$\left| xs^2y_1(\lambda x) \right| + \left| xs^2y_2(\lambda x) \right| \leq \frac{1}{2} (|y_1(\lambda x)| + |y_2(\lambda x)|) - \frac{1}{2}x.$$

Now let $\mathcal{Y} = C[0, 1]$ with the metric $\rho : X \times X \rightarrow [0, \infty)$ given by

$$\rho(y_1, y_2) = \sup_{x \in [0, 1]} |y_1(x) - y_2(x)|,$$

for all $y_1, y_2 \in \mathcal{Y}$.

It is clear that (\mathcal{Y}, ρ) is a complete metric space. For all $y_1, y_2 \in \mathcal{Y}$, the function

$$d(y_1, y_2) = \sup_{x \in [0, 1]} |y_1(x)| + \sup_{x \in [0, 1]} |y_2(x)|$$

is a w -distance on \mathcal{Y} and a ceiling distance of ρ . Next, we define the operator $V : \mathcal{Y} \rightarrow \mathcal{Y}$, defined by

$$V(y)(x) = \frac{x}{4} + \int_0^x xs^2y(\lambda s)ds, \quad \text{for all } y \in \mathcal{Y}.$$

The functions $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ defined by $\psi(x) = \frac{x}{2}$ and $\phi(x) = \frac{x}{4}$ verify that $\psi(x) < x$ for all $x > 0$ and $\phi(x) < \psi(x)$, for all $x > 0$.

Thus

$$\begin{aligned}
|V(y_1)(x)| + |V(y_2)(x)| &= \left| \frac{x}{4} + \int_0^x xs^2 y_1(\lambda s) ds \right| + \left| \frac{x}{4} + \int_0^x xs^2 y_2(\lambda s) ds \right| \\
&\leq \frac{x}{2} + \int_0^x xs^2 (|y_1(\lambda s)| + |y_2(\lambda s)|) ds \\
&\leq \frac{x}{2} + \frac{1}{2} (|y_1(\lambda x)| + |y_2(\lambda x)|) - \frac{x}{2} \\
&\leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)).
\end{aligned}$$

From this we get

$$\sup_{x \in [0,1]} |V(y_1)(x)| + \sup_{x \in [0,1]} |V(y_2)(x)| \leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)).$$

We obtain that

$$d(V(y_1), V(y_2)) \leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)), \text{ for all } y_1, y_2 \in \mathcal{Y}.$$

Finally,

$$\psi(d(V(y_1), V(y_2))) \leq d(V(y_1), V(y_2)) \leq \psi(d(y_1, y_2)) - \phi(d(y_1, y_2)), \text{ for all } y_1, y_2 \in \mathcal{Y}.$$

Hence, by Theorem 7, V has a unique fixed point and we conclude that Equation (8) has a unique solution.

4. Conclusions

In this paper, we have investigated a Volterra integral equation with delay. Using w -weak generalized contractions theorem and the assumptions (C_1) – (C_3) , we obtain an existence and uniqueness result, a Gronwall-type theorem and a comparison theorem for Equation (1). We employed the Picard operator method, fixed point theorems and abstract Gronwall lemma, to obtain our results. In the end, an example is presented. The theorems obtained in this paper are also applicable to systems of integral equations with delay. As for a future study, several numerical examples can be taken and a comparative study with previously published results or theory can be done.

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References

- Burton, T.A. *Volterra Integral and Differential Equations*; Academic Press: New York, NY, USA, 1983.
- Corduneanu, C. *Integral Equations and Stability of Feedback Systems*; Academic Press: New York, NY, USA, 1973.
- Gripenberg, G.; Londen, S.O.; Staffans, O. *Volterra Integral and Functional Equations*; Encyclopedia of Mathematics and its Applications 34; Cambridge University Press: Cambridge, UK, 1990.
- Guo, D.; Lakshmikantham, V.; Liu X. *Nonlinear Integral Equations in Abstract Spaces*; Kuwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
- Sidorov, D. *Integral Dynamical Models: Singularities, Signals and Control*; World Scientific Publishing Company: Singapore, 2014.

6. Olmstead, W.E.; Roberts, C.A.; Deng, K. Coupled Volterra equations with blow-up solutions. *J. Integral Equ. Appl.* **1997**, *7*, 499–516. [[CrossRef](#)]
7. Panin, A.A. On local solvability and blow-up of solutions of an abstract nonlinear Volterra integral equation. *Math. Notes* **2015**, *97*, 892–908. [[CrossRef](#)]
8. Sidorov, D. Existence and blow-up of Kantorovich principal continuous solutions of nonlinear integral equations. *Diff. Equ.* **2014**, *50*, 1217–1224. [[CrossRef](#)]
9. Ilea, V.A.; Otrocol, D. An application of the Picard operator technique to functional integral equations. *J. Nonlinear Convex Anal.* **2017**, *18*, 405–413.
10. Ilea, V.A.; Otrocol, D. On the Burton method of progressive contractions for Volterra integral equations. *Fixed Point Theory* **2020**, *21*, 585–594. [[CrossRef](#)]
11. Şerban, M.A. Data dependence for some functional-integral equations. *J. Appl. Math.* **2008**, *1*, 219–234.
12. Dobriţoiu, M. An integral equation with modified argument. *Studia Univ. Babeş-Bolyai Math.* **2004**, *49*, 27–33.
13. Ilea, V.A.; Otrocol, D. Existence and uniqueness of the solution for an integral equation with supremum, via w -distances. *Symmetry* **2020**, *12*, 1554 [[CrossRef](#)]
14. Marian, D.; Ciplea, S.; Lungu, N. Ulam-Hyers stability of Darboux-Ionescu problem. *Carpathian J. Math.* **2021**, *37*, 211–216. [[CrossRef](#)]
15. Aguirre Salazar, L.; Reich, S. A remark on weakly contractive mappings. *J. Nonlinear Conv. Anal.* **2015**, *16*, 767–773.
16. Dobriţoiu, M. An application of the w -weak generalized contractions theorem. *J. Fixed Point Theory Appl.* **2019**, *21*, 93. [[CrossRef](#)]
17. Kada, O.; Suzuki, T.; Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Math. Jpn.* **1996**, *44*, 381–391.
18. Suzuki, T.; Takahashi, W. Fixed points theorems and characterizations of metric completeness. *Topol. Methods Nonlinear Anal. J. Juliusz Schauder Cent.* **1996**, *8*, 371–382. [[CrossRef](#)]
19. Takahashi, W.; Wong, N.C.; Yao, J.C. Fixed point theorems for general contractive mappings with w -distances in metric spaces. *J. Nonlinear Conv. Anal.* **2013**, *14*, 637–648.
20. Wongyat, T.; Sintunavarat, W. The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fraction differential equations via w -distances. *Adv. Diff. Equ.* **2017**, *2017*, 211. [[CrossRef](#)]
21. Wongyat, T.; Sintunavarat, W. On new existence and uniqueness results concerning solutions to nonlinear Fredholm integral equations via w -distances and weak altering distance functions. *J. Fixed Point Theory Appl.* **2019**, *21*, 7. [[CrossRef](#)]
22. Rus, I.A. *Generalized Contractions and Applications*; Cluj University Press: Cluj-Napoca, Romania, 2001.
23. Rus, I.A. Picard operators and applications. *Sci. Math. Jpn.* **2003**, *58*, 191–219.
24. Rus, I.A. Fixed points, upper and lower fixed points: Abstract Gronwall lemmas. *Carpathian J. Math.* **2004**, *20*, 125–134.
25. Marian, D.; Ciplea, S.; Lungu, N. On a Functional Integral Equation. *Symmetry* **2021**, *13*, 13–21. [[CrossRef](#)]
26. Marian, D.; Ciplea, S.A.; Lungu, N. Optimal and Nonoptimal Gronwall Lemmas. *Symmetry* **2020**, *12*, 17–28. [[CrossRef](#)]
27. Reich, S.; Zaslavski, A.J. Almost all nonexpansive mappings are contractive. *C. R. Math. Rep. Acad. Sci. Can.* **2000**, *22*, 118–124.
28. Reich, S.; Zaslavski, A.J. The set of noncontractive mappings is sigma-porous in the space of all nonexpansive mappings. *C. R. Acad. Sci. Paris Ser. I Math.* **2001**, *333*, 539–544. [[CrossRef](#)]