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Ordered Vectorial Quasi and Almost Contractions on Ordered Vector Metric Spaces

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Abstract: In this paper, we define ordered vectorial quasi contractions. We show that ordered quasi contractions are ordered vectorial quasi contractions, but the reverse is not true. We also define ordered vectorial almost contractions and present fixed point theorems for this type of contractions. Hence, we disclose many results in the literature. With the help of examples, we illustrate the relationship between these two types of contractions and some others in the literature.

Keywords: fixed point; Riesz space; partial ordered metric spaces; vector metric space; almost contractions; ordered quasi contractions

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1. Introduction

Classical Banach's contraction principle [1] has a very important role in fixed point theory. Despite its considerable importance, it has a weakness that the function needs to be continuous if it satisfies this contraction condition. To remove continuity, many generalizations have been made, such as [2–5]. Ćirić [2] provided some fixed point results for functions satisfying the following contraction condition.

Definition 1. *Let (X, d) be a metric space. A function $\mathcal{T} : X \to X$ is called a quasi contraction if there exists a constant $h \in (0, 1)$ satisfying*

$$d(\mathcal{T}x, \mathcal{T}y) \leq h. \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}$$

for all $x, y \in X$.

Following this, Berinde presented weak contractive mappings, or (δ, L)-weak contractive mappings (later called almost contractive mappings), which are:

Definition 2 ([6]). *Let (X, d) be a metric space. A function $\mathcal{T} : X \to X$ is called (δ, L)-weak contraction if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ satisfying*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \delta d(x, y) + Ld(y, \mathcal{T}x) \tag{1}$$

for all $x, y \in X$.

He emphasized that any Kannan, Chatterjee, or Zamfirescu contraction, or any quasi contraction with $h \in (0, 1/2)$, is a weak contraction. In addition, he provided a fixed point result for weak contractions.

Theorem 1 ([6]). *Let (X, d) be a complete metric space and $\mathcal{T} : X \to X$ be a weak contraction. Then*

(1) $F(\mathcal{T}) = \{x \in X : \mathcal{T}x = x\}$ is nonempty;
(2) For any $x_0 \in X$, the Picard iteration (x_n) converges to some $x^* \in F(\mathcal{T})$;

(3) The following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1)$$
$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n)$$

hold for all $n \in \mathbb{N}^+$.

Berinde remarked that this theorem can not guarantee the uniqueness of a fixed point, and he stated that, in this theorem, if the weak contraction $\mathcal{T}$ also satisfies

$$d(\mathcal{T}x, \mathcal{T}y) \leq \theta d(x, y) + L_1 d(y, \mathcal{T}x)$$

for all $x, y \in X$ and for some $L_1 \geq 0$ where $\theta \in (0, 1)$, the uniqueness of the fixed point is attained. He gave more results and properties about weak contractive mappings in [6–9]. On the other hand, there is another tendency to extend the Banach contraction principle with partial ordering. To our knowledge, it started with Ran and Reuring [10] and was followed by many authors, notably Nieto and Lopez [11,12]. They presented some fixed point results for nondecreasing, nonincreasing and even not monotone contractions. After that, many fixed point results have been given on partially ordered metric spaces, such as [13]. While in [14,15], the notion of partial metric is combined with partial ordering, in [16], the notion of M-metric is combined with partial ordering. Moreover, in many works such as [17–20], many fixed point results have been given on cone metric spaces. On the other hand, a generalization of metric spaces was made by Cevik and Altun [21]. They presented *vector metric space* (*E-metric space*) and gave some fixed point results on this space. According to this work, any metric is a vector metric, but the converse is not true in general. In the last two decades, many extensions of the results in [21] have been completed, such as [22–25]. In [23], the authors united the concept of partial ordering with vector metric and provided some fixed point theorems on ordered vector metric space; hence, they generalized the results of [11,12,21].

In this paper, we aim to combine the results in [2,6–8] with the notion of vector metric introduced in [21] and partial ordering. Hence, we extend the results in these works as well as the ones in [11,12,23]. We define ordered vectorial quasi contraction. According to this definition, any ordered quasi contraction (extension of quasi contraction) is an ordered vectorial quasi contraction, but the converse is not true. We also define ordered vectorial almost contraction, which is an extension of (δ, L)-weak contraction and, we present some fixed point theorems for this kind of family of contractions. In addition, we provide some related examples that show the differences between our results and the ones previously mentioned.

A partially ordered set, whose any two elements contain both supremum and infimum, is called lattice. An ordered vector space E is a real vector space with an order relation "$\leqslant$", which is compatible with the algebraic structure of E. In other words, $a \leqslant b$ implies $a + c \leqslant b + c$ and $a\lambda \leqslant b\lambda$ for all $c \in E$, $\lambda \in [0, \infty)$. An ordered vector space is called Riesz space whenever it is a lattice. A Riesz space is labelled Archimedean if $\frac{1}{n}a \downarrow 0$ for all $a \in E^+$. By the notation $a_n \downarrow a$, we mean the sequence (a_n) is order-reversing and the infimum of the set $\{a_n : n \in \mathbb{N}\}$ is a. For other facts and concepts related to Riesz space, we refer [26,27]. Now, let us recall some concepts from [21], especially the definition of vector metric space. A map $d : X^2 \to E$ is named *vector metric* (*E-metric*) if it satisfies the conditions

(1) $d(x, y) = 0 \Leftrightarrow x = y$, for all $x, y \in X$

(2) $d(x, y) \leqslant d(x, z) + d(y, z)$ for all $x, y, z \in X$

where X is a nonempty set and $(E, \leqslant)$ is a Riesz space. Hence, (X, d, E) (briefly X) is called *vector metric space* (*E-metric space*). A sequence (x_n) of X *vectorially converges* (*E-converges*) to $x \in X$ if there is a sequence (b_n) in E such that $b_n \downarrow 0$ and $d(x_n, x) \leqslant b_n$ for all n.

The sequence (x_n) is labelled *E-Cauchy if there* is a sequence (b_n) in E such that $b_n \downarrow 0$ and $d(x_{n+p}, x_n) \leqslant b_n$ for all n and p. Additionally, if every E-Cauchy sequence in X is E-convergent, then X is said to be *E-complete*.

Note that a function $f : X \to Y$ is called *vectorial continuous if* $f(x_n) \overset{\rho,E_2}{\to} f(x)$ whenever $x_n \overset{d,E_1}{\to} x$ where (X, d, E_1) and (Y, ρ, E_2) are two vector metric spaces [28].

2. Ordered Vectorial Quasi and Almost Contractions

Initially, we present the definitions of ordered vectorial quasi contractions and ordered vectorial almost contractions in this part. Unless otherwise stated, we assume that (X, ρ, E) (shortly X) is a vector metric space, and it is equipped with the ordering "$\preceq$". We also use the notations M_x^y and S_x^y, respectively, for the maximum (if exists) and the supremum of the set $\{\rho(x,y), \rho(x, \mathcal{T}x), \rho(y, \mathcal{T}y), \rho(x, \mathcal{T}y), \rho(y, \mathcal{T}x)\}$.

Definition 3. *A function $\mathcal{T}$ on an ordered vector metric space X to itself is named ordered vectorial quasi contraction if there is a number $h \in (0,1)$ satisfying*

$$\rho(\mathcal{T}x, \mathcal{T}y) \leqslant h.S_x^y \tag{2}$$

for all $x \preceq y$ where $x, y \in X$.

In general, according to a given partial ordering, the concepts of supremum and maximum do not coincide, even on finite sets. In case any two elements of a finite subset of a Riesz space are not comparable, this set has a supremum but may not have a maximum. On the other hand, a finite set whose elements are taken from a Riesz space has a maximum when all the elements are comparable. For instance, if we assume $E = \mathbb{R}$ endowed with usual ordering, it can be clearly understood that M_x^y and S_x^y coincide. Hence, every ordered quasi contraction is an ordered vectorial quasi contraction, but the reverse is not true (see Example 1). Since any ordered Kannan, Chattarjee and Zamfirescu contraction is an ordered quasi contraction, similar deductions can be made for this type of contractions.

Example 1. *Let $X = \{(0,0), (0,1), (1,0), (2,2)\}$ and*

$$\begin{aligned} \beta = \quad & \{((0,0),(0,0)),\ ((0,0),(2,2)),\ ((0,1),(0,1)),\ ((0,1),(2,2)), \\ & ((1,0),(1,0)),\ ((1,0),(2,2)),\ ((2,2),(2,2))\}. \end{aligned}$$

It is clear that the relation $\preceq$ defined as

$$x \preceq y \Leftrightarrow (x,y) \in \beta$$

is a partial ordering on X. Let X be equipped with this relation and $E = \mathbb{R}^2$ be equipped with coordinatewise ordering. Then E is a Riesz space and X is a vector metric space with the map $\rho : X^2 \to E$ defined as $\rho((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|)$. Suppose the function $\mathcal{T} : X \to X$is defined as

$$\begin{aligned} \mathcal{T}((0,0)) &= \mathcal{T}((1,0)) = (0,1) \text{ and} \\ \mathcal{T}((0,1)) &= \mathcal{T}((2,2)) = (1,0). \end{aligned} \tag{3}$$

It is clear that, for all $x, y \in X$ with $x \preceq y$, the function $\mathcal{T}$ assures

$$\rho(\mathcal{T}x, \mathcal{T}y) \leqslant h.S_x^y$$

for $h = 1/2$. That is $\mathcal{T}$ is an ordered vectorial quasi contraction. However, $\mathcal{T}$ is neither a quasi contraction nor an ordered quasi contraction; hence, the results for quasi contractions and ordered quasi contractions can not be applicable to this example. In particular, if we assume $x = (0,1)$ and $y = (2,2)$, we see that

$$\{\rho(x,y),\rho(x,\mathcal{T}x),\rho(y,\mathcal{T}y),\rho(x,\mathcal{T}y),\rho(y,\mathcal{T}x)\} = \{(2,1),(1,1),(1,2)\}. \quad (4)$$

As a result, M_x^y does not exist while $S_x^y = (2,2)$.

As already mentioned, an ordered quasi contraction is an extension of a quasi contraction. Likewise, an ordered vectorial quasi contraction is an extension of an ordered quasi contraction. However, we do not give fixed point results for this type of contraction because the contraction we focused on, which is ordered vectorial almost contraction, is a generalization of ordered vectorial quasi contraction.

Definition 4. *A function $\mathcal{T}$ on an ordered vector metric space X to itself is named ordered vectorial almost contraction if there is a real number $\delta \in (0,1)$ and some $L \geq 0$ satisfying*

$$\rho(\mathcal{T}(x),\mathcal{T}(y)) \leqslant \delta.\rho(x,y) + L\rho(y,\mathcal{T}(x)) \quad (5)$$

for all $x \preceq y$ where $x,y \in X$.

The following proposition gives us a relationship between ordered vectorial quasi contractions and ordered vectorial almost contractions.

Proposition 1. *Let S_x^y be an element of the set*

$$\{\rho(x,y),\rho(x,\mathcal{T}x),\rho(y,\mathcal{T}y),\rho(x,\mathcal{T}y),\rho(y,\mathcal{T}x)\}.$$

Then any ordered vectorial quasi contraction with $h \in (0,1/2)$ is an ordered vectorial almost contraction.

Since the proof can be made in a similar way as in Proposition 3 in [6], it is omitted. This proposition is not a necessary condition for an ordered vectorial quasi contraction to be an ordered vectorial almost contraction. In Example 1 for $x = (0,1)$ and $y = (2,2)$, we have $S_x^y = (2,2)$, which is not a member of the set $\{\rho(x,y),\rho(x,\mathcal{T}x),\rho(y,\mathcal{T}y),\rho(x,\mathcal{T}y),\rho(y,\mathcal{T}x)\}$ and $h \notin (0,1/2)$ since $h = 1/2$. However, if we take $\delta = 1/2$ and $L \geq 1/2$, then we see that $\mathcal{T}$ is an ordered vectorial almost contraction.

Now, we give a fixed point theorem for ordered vectorial almost contractions. Throughout the rest of the work, unless otherwise stated, we assume that X is E-complete vector metric space and it is endowed with an order relation "$\preceq$", and $(E,\leqslant)$ is an Archimedean Riesz space.

Theorem 2. *Let $\mathcal{T} : X \to X$ is an ordered vectorial almost contraction where $\delta \in (0,1)$ and $L \geq 0$. Suppose that $\mathcal{T}$ is order-preserving, and one of the following is satisfied*

(i) $\mathcal{T}$ is vectorial continuous;

(ii) For any order-preserving sequence (x_n) in X, if $x_n \overset{\rho,E}{\to} x$ then $x_n \preceq x$ for all $n \in \mathbb{N}^+$.

If there exists $x_0 \in X$ with $x_0 \preceq \mathcal{T}(x_0)$, then $\mathcal{T}$ has a fixed point.

Proof. Let us define a sequence (x_n) as $x_n = \mathcal{T}(x_{n-1})$ for all $n \geq 1$. Since $\mathcal{T}$ is order-preserving, we obtain $x_0 \preceq \mathcal{T}(x_0)$. Following this process leads us to the result

$$x_0 \preceq x_1 = \mathcal{T}(x_0) \preceq \ldots \preceq x_{n+1} = \mathcal{T}(x_n) = \mathcal{T}^n(x_0)$$

for all $n \in \mathbb{N}^+$. It is clear that (x_n) is an order-preserving sequence. The function $\mathcal{T}$ is an ordered vectorial almost contraction; hence, for a real number $\delta \in (0,1)$ and some $L \geq 0$

$$\rho(x_n,x_{n+1}) = \rho(\mathcal{T}(x_{n-1}),\mathcal{T}(x_n)) \leqslant \delta.\rho(x_{n-1},x_n) + L\rho(x_n,\mathcal{T}(x_{n-1})).$$

Since $x_n = \mathcal{T}(x_{n-1})$, we have

$$\rho(x_n, x_{n+1}) = \rho(\mathcal{T}(x_{n-1}), \mathcal{T}(x_n)) \leqslant \delta.\rho(x_{n-1}, x_n)$$

and we obtain

$$\rho(x_n, x_{n+1}) \leqslant \delta^n.\rho(x_1, x_0)$$

for all $n \in \mathbb{N}^+$. Hence, for any $n, p \in X$, we deduce that

$$\begin{aligned} \rho\big(x_n, x_{n+p}\big) &\leqslant \rho(x_n, x_{n+1}) + \cdots + \rho\big(x_{n+p-1}, x_{n+p}\big) \\ &\leqslant \Big(\delta^n + \cdots + \delta^{n+p-1}\Big)\rho(x_1, x_0) \\ &\leqslant \frac{\delta^n}{1-\delta}\rho(x_1, x_0). \end{aligned}$$

Due to the fact that $(\delta^n/(1-\delta))\rho(x_1, x_0) \downarrow 0$ and E is an Archimedean Riesz space, it is clear that (x_n) is E-Cauchy sequence. By E-completeness of X, we deduce that there exists an element x in X such that $x_n \overset{\rho,E}{\to} x$. That is, there exists a sequence (a_n) such that $\rho(x_n, x) \leqslant a_n$ for all n and $a_n \downarrow 0$. For the rest, we investigate two cases separately.

(i) In case $\mathcal{T}$ is vectorial continuous, there exists a sequence (b_n) such that $b_n \downarrow 0$ and $\rho(\mathcal{T}(x_n), \mathcal{T}(x)) \leqslant b_n$ for all n where (x_n) is a sequence in X and $x_n \overset{\rho,E}{\to} x$. So

$$\begin{aligned} \rho(x, \mathcal{T}(x)) &\leqslant \rho(x, \mathcal{T}(x_n)) + \rho(\mathcal{T}(x_n), \mathcal{T}(x)) \\ &\leqslant \rho(x, x_{n+1}) + \rho(\mathcal{T}(x_n), \mathcal{T}(x)) \\ &\leqslant a_{n+1} + b_n \\ &\leqslant a_n + b_n \end{aligned}$$

for all $n \in \mathbb{N}^+$. Since $a_n + b_n \downarrow 0$, we have $\mathcal{T}(x) = x$.

(ii) For any order-preserving sequence (x_n) in X, let $x_n \preceq x$ for all n whenever $x_n \overset{d,E}{\to} x$. Then, we have

$$\begin{aligned} \rho(x, \mathcal{T}(x)) &\leqslant \rho(x, x_{n+1}) + \rho(x_{n+1}, \mathcal{T}(x)) \\ &= \rho(x, x_{n+1}) + \rho(\mathcal{T}(x_n), \mathcal{T}(x)) \\ &\leqslant \rho(x, x_{n+1}) + \delta.\rho(x_n, x) + L\rho(x, \mathcal{T}(x_n)) \\ &= \rho(x, x_{n+1}) + \delta.\rho(x_n, x) + L\rho(x, x_{n+1}) \\ &\leqslant (1+L).\rho(x, x_{n+1}) + \delta.\rho(x_n, x) \\ &= (1+L).a_{n+1} + \delta.a_n \\ &\leqslant (1+L+\delta).a_n \end{aligned}$$

for all $n \in \mathbb{N}^+$. Since $a_n \downarrow 0$, so $(1+L+\delta).a_n$ does. Thus $\mathcal{T}(x) = x$, that is, x is a fixed point of $\mathcal{T}$. □

Although this theorem presents the existence of a fixed point, it is not sufficient to say that the fixed point is unique.

Example 2. *Let $X = C[0,1]$, and it is equipped with an ordering $\preceq$ defined for any $f, g \in X$ as*

$$f \preceq g \Leftrightarrow f(x) \leq g(x)$$

for all $x \in [0,1]$ where "$\leq$" is usual ordering on $[0,1]$. Let also $E = \mathbb{R}^2$ be equipped with coordinatewise ordering, and the map $\rho : X^2 \to E$ is defined as

$$\rho(f, g) = \left(\sup_{x\in[0,1]} |f(x) - g(x)|, 2 \sup_{x\in[0,1]} |f(x) - g(x)| \right)$$

for all $f, g \in X$. Suppose $\mathcal{T} : X \to X$ be the identity map. That is, $\mathcal{T}f = f$ for all $f \in X$. It is clear that $\mathcal{T}$ is an ordered vectorial almost contraction with a positive $\delta \in (0,1)$ and some $L \geq 1 - \delta$. All hypotheses of Theorem 2 are satisfied, but $\mathcal{T}$ has infinitely many fixed points.

Another important role of this example is showing us that any ordered vectorial almost contraction need not to be an ordered vectorial quasi contraction. Indeed, since $\mathcal{T}$ is the identity map, $\rho(\mathcal{T}f, \mathcal{T}g) = \rho(f, g)$ and for any $f, g \in X$ with $f \preceq g$, we have

$$\begin{aligned} S_f^g &= \sup\{\rho(f,g), \rho(f,\mathcal{T}f), \rho(g,\mathcal{T}g), \rho(f,\mathcal{T}g), \rho(g,\mathcal{T}f)\} \\ &= \sup\{\rho(f,g), \rho(f,f), \rho(g,g), \rho(f,g), \rho(g,f)\} \\ &= \rho(f,g). \end{aligned}$$

Hence, there is not at least one $h \in (0,1)$ satisfying

$$\rho(\mathcal{T}f, \mathcal{T}g) \leq h.S_f^g. \tag{6}$$

That is, $\mathcal{T}$ is not an ordered vectorial quasi contraction.

By applying an additional condition to Theorem 2, we can obtain the uniqueness of the fixed point.

Theorem 3. *Let all hypotheses of Theorem 2 be satisfied. Let the function $\mathcal{T}$ satisfy the property*

$$\rho(\mathcal{T}(x), \mathcal{T}(y)) \leqslant \delta_1.\rho(x,y) + L_1\rho(x, \mathcal{T}(x)) \tag{7}$$

for all $x \preceq y$ where $x, y \in X$, a $\delta_1 \in (0,1)$ and some $L_1 \geq 0$. If there exists a comparable element for any two elements in X, then T has a unique fixed point.

Proof. Let x and y be two fixed points of $\mathcal{T}$. We have to investigate two cases separately again.

(i) Let x and y be comparable elements. Since

$$\rho(x,y) = \rho(\mathcal{T}(x), \mathcal{T}(y)) \leqslant \delta_1.\rho(x,y) + L_1\rho(x, \mathcal{T}(x))$$

we obtain $(1-\delta_1).\rho(x,y) \leqslant 0$. That is $x = y$.

(ii) Let x and y be incomparable elements. Then there exists an element w comparable with x and y. Since $\mathcal{T}$ is order-preserving, $\mathcal{T}^n(w)$ is comparable with w and as a result with x and y for any n. Hence, we have

$$\begin{aligned} \rho(x,y) &\leqslant \rho(x, \mathcal{T}^n(w)) + \rho(\mathcal{T}^n(w), y) \\ &= \rho(\mathcal{T}(x), \mathcal{T}^n(w)) + \rho(\mathcal{T}^n(w), \mathcal{T}(y)) \\ &\leqslant \delta_1.\rho\left(x, \mathcal{T}^{n-1}(w)\right) + L_1\rho(x, \mathcal{T}(x)) + \delta_1.\rho\left(y, \mathcal{T}^{n-1}(w)\right) + L_1\rho(y, \mathcal{T}(y)) \\ &\leqslant \delta_1.\left[\rho\left(x, \mathcal{T}^{n-1}(w)\right) + \rho\left(y, \mathcal{T}^{n-1}(w)\right)\right] \\ &\quad \vdots \\ &\leqslant \delta_1^n.[\rho(x,w) + \rho(y,w)] \end{aligned}$$

for all n. Since $\delta_1 \in (0,1)$ and E is an Archimedean Riesz space, $\delta_1^n.[\rho(x,w) + \rho(y,w)] \downarrow 0$. Thus, we have $x = y$. By (i) and (ii), we say that the fixed point of $\mathcal{T}$ is unique. □

Example 3. *Let $X = \{0,1,2\}$ and $\beta = \{(0,0),(1,1),(2,2),(0,1)\}$. It is clear that the relation $\preceq$ defined as*

$$x \preceq y \Leftrightarrow (x,y) \in \beta$$

is a partial ordering on X. Let X be equipped with this relation and $E = \mathbb{R}^2$ be equipped with coordinatewise ordering. E is an Archimedean Riesz space, and X is an E-complete vector metric space with the map $\rho : X^2 \to E$ defined as $\rho(x,y) = (|x-y|, |x-y|)$. Suppose that the function $\mathcal{T} : X \to X$ is defined as

$$\mathcal{T}(0) = \mathcal{T}(1) = 0 \text{ and } \mathcal{T}(2) = 1.$$

It is clear that $\mathcal{T}$ is order-preserving, and for all $x, y \in X$ with $x \preceq y$, $\mathcal{T}$ satisfies

$$\rho(\mathcal{T}(x), \mathcal{T}(y)) \leqslant \delta.\rho(x,y) + L\rho(x, \mathcal{T}(y))$$

for $\delta = 1/3$ and any $L \geq 0$. In addition to that, $\mathcal{T}$ satisfies

$$\rho(x,y) = \rho(\mathcal{T}(x), \mathcal{T}(y)) \leqslant \delta_1.\rho(x,y) + L_1\rho(x, \mathcal{T}(x))$$

for $\delta_1 = 1/2$ and any $L_1 \geq 0$. Additionally, if a sequence (x_n) in X is order-preserving, then $x_n \preceq x$ for all $n \in \mathbb{N}^+$. Hence, we see that all hypotheses of Theorem 3 are satisfied, and 0 is the only fixed point of $\mathcal{T}$. On the other hand, if we take $x = 2$ and $y = 1$, we see that $\rho(\mathcal{T}(x), \mathcal{T}(y)) = (1,1)$, while $\rho(x,y) = (1,1)$ and $\rho(y, \mathcal{T}(x)) = (0,0)$. As a result, we could not find at least one $L \geq 0$ satisfying (1). That is, we can not apply Theorem 1 to this example. Another important aspect of this example is the function $\mathcal{T}$ is an ordered vectorial quasi contraction for any $h \in (0,1)$. However, again if we assume $x = 2$ and $y = 1$, we see that there is no such a $h \in (0,1)$ satisfying (2). So, the results for quasi contraction on metric spaces can not be applicable to this example.

3. Conclusions

In this paper, by combining the notion vector metric and partial ordering with the notion quasi contraction and almost contraction, the ordered vectorial quasi and almost contractions were defined. Additionally, it was showed that every ordered quasi contraction is an ordered vectorial quasi contraction, but the reverse may not be true in general. Moreover, some fixed point results for ordered vectorial almost contractions were presented. Furthermore, it was emphasized that, since in some cases ordered vectorial quasi contractions are ordered vectorial almost contractions, our results are extensions of not only the ones made for ordered almost contractions but also the ones made for ordered quasi contractions.

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