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Inertial Krasnoselski–Mann Iterative Method for Solving Hierarchical Fixed Point and Split Monotone Variational Inclusion Problems with Its Applications

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Abstract: In this article, we discuss the hierarchical fixed point and split monotone variational inclusion problems and propose a new iterative method with the inertial terms involving a step size to avoid the difficulty of calculating the operator norm in real Hilbert spaces. A strong convergence theorem of the proposed method is established under some suitable control conditions. Furthermore, the proposed method is modified and used to derive a scheme for solving the split problems. Finally, we compare and demonstrate the efficiency and applicability of our schemes for numerical experiments as well as an example in the field of image restoration.

Keywords: variational inclusion problem; inertial technique; strong convergence; image restoration; hierarchical fixed point problem

MSC: 47H05; 47J20; 47J25; 65K05

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1. Introduction

The variational inequality problem (VIP for short) is a significant branch of mathematics. Over the decades, this problem has been extensively studied for solving many real-world problems in various applied research areas such as physics, economics, finance, optimization, network analysis, medical images, water resources and structural analysis. Moreover, it contains fixed point problems, optimization problems arising in machine learning, signal processing and linear inverse problems, (see [1–3]). The set of solutions of the variational inequality problem is denoted by

$$VI(C, A) = \{u \in C : \langle v - u, Au\rangle \geq 0\}, \ \forall v \in C,$$

where C is a nonempty closed convex subset of the Hilbert space H and $A : C \to H$ is a mapping.

Moudafi and Mainge [4] established the hierarchical fixed point problem for a nonexpansive mapping T with respect to another nonexpansive mapping S on H by utilizing the concept of the variational inequality problem: Find $x^* \in Fix(T)$ such that

$$\langle x^* - Sx^*, x^* - x\rangle \leq 0, \ \forall x \in Fix(T), \tag{1}$$

where $S : H \to H$ is a nonexpansive mapping and $Fix(T) = \{x \in C : x = Tx\}$. It is easy to see that (1) is equivalent to the following fixed point problem: Find $x^* \in H$ such that

$$x^* \in P_{Fix(T)} \cdot Sx^*, \tag{2}$$

where $P_{Fix(T)}$ stands for the metric projection on the closed convex set $Fix(T)$. The solution set of HFPP (1) is denoted by $\Phi = \{x^* \in H : \langle x^* - Sx^*, x^* - x\rangle \leq 0, \ \forall x \in Fix(T)\}$. It is obvious that $\Phi = VI(Fix(T), I - S)$. It is worth noting (1) covers monotone variational inequality on fixed point sets as well as minimization problem, etc. For several fixed point theory and solving the hierarchical fixed point problem in (1), many iterative methods have been studied and developed, (see [4–8]).

On the other hand, The split monotone variational inclusion problem (in short, SpMVIP) is proposed and studied by Moudafi [9]. It had been applied to the intensity-modulated radiation therapy treatment planning as a model, (see [10]). This concept has appeared in many inverse problems which emerge in phase retrieval and other real-world problems such as sensor networks, data compression as well as comprised tomography, (see [11,12]): Find $x^* \in H_1$ such that

$$0 \in f(x^*) + F(x^*) \tag{3}$$

and such that

$$y^* = Ax^* \text{ solves } 0 \in g(y^*) + G(y^*), \tag{4}$$

where $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are multi-valued monotone operators, $f : H_1 \to H_1$, $g : H_2 \to H_2$ are two single-valued operators and $A : H_1 \to H_2$ is a bounded linear operator. The solution set of SpMVIP is denoted by $\Omega = \{x^* \in H_1 : x^* \in \text{Sol(MVIP (3))}\}$ and $Ax^* \in \text{Sol(MVIP (4))}\}$.

They introduced the following iterative method and studied the weak convergence theorem for SpMVIP: For a given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = U(x_n + rA^*(V - I)Ax_n), \text{ for } r > 0,$$

where $U = J_\lambda^F(I - \lambda f)$, $V = J_\lambda^G(I - \lambda g)$ such that J_λ^F and J_λ^G are the resolvent mappings of F and G, respectively (see the definition in Section 2) and I is the identity mapping. The operator A is a bounded linear operator with A^* is adjoint of A and $r \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. They proved that the sequence $\{x_n\}$ converges weakly to a solution of hierarchical fixed point and split monotone variational inclusion problems.

In 2017, Kazmi et al. [13] developed a Krasnoselski–Mann type iterative method to approximate a common solution set of a hierarchical fixed point problem for nonexpansive mappings S, T and a split monotone variational inclusion problem which was defined as follows:

$$\begin{aligned} x_0 &\in C; \\ u_n &= (1 - \alpha_n)x_n + \alpha_n(\beta_n S x_n + (1 - \beta_n)T x_n); \\ x_{n+1} &= U(u_n + \lambda A^*(V - I)Au_n), \forall n \geq 0, \end{aligned} \tag{5}$$

where $U = J_\lambda^F(I - \lambda_n f)$, $V = J_\lambda^G(I - \lambda_n g)$ and the step size $\lambda \in (0, \frac{1}{L})$ where L is the spectral radius of the operator A^*A. Under some suitable conditions on $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$:

$$\text{(C1) } \sum_{n=0}^{\infty} \beta_n < \infty; \quad \text{(C2) } \lim_{n\to\infty} \frac{\|x_n - u_n\|}{\alpha_n \beta_n} = 0; \quad \text{(C3) } \liminf_{n\to\infty} \lambda_n > 0.$$

They proved that the sequence $\{x_n\}$ converges weakly to a solution of hierarchical fixed point and split monotone variational inclusion problems. Normally, an interesting question is how to construct the strongly convergence results which approximate the solution of the split monotone variational inclusion and hierarchical fixed point problems.

In 2021, Dao-Jun Wen [14] modified Krasnoselski–Mann type iteration (5) which replaced operator T with $(I - \mu_n D)$ where D is a strongly monotone operator and L-Lipschitzian. Define a sequence $\{x_n\}$ in the following manner:

$$\begin{aligned} u_n &= (1 - \alpha_n)x_n + \alpha_n T(\beta_n S x_n + (1 - \beta_n)(I - \mu_n D)x_n), \\ x_{n+1} &= U(u_n + \lambda A^*(V - I)Au), \end{aligned}$$

where $U = J_\lambda^F(I - \lambda f)$, $V = J_\lambda^G(I - \lambda g)$, T, S are two nonexpansive mappings and the step size $\lambda \in (0, \frac{1}{L})$ where L is the spectral radius of the operator A^*A. They established the strong convergence result and constructed the new condition of coefficients which replaced of condition (C2).

If $f = 0$ and $g = 0$, then the split monotone variational inclusion problem reduced to the following split variational inclusion problem (SVIP): Find $x^* \in H_1$ such that

$$0 \in F(x^*) \text{ and such that } y^* = Ax^* \text{ solves } 0 \in G(y^*). \tag{6}$$

Byrne et al. [15] have attempted to solve a special case of (6) and defined $\{x_n\}$ in the following: $x_{n+1} = J_\lambda^F(x_n + rA^*(J_\lambda^G - I)Ax_n)$. Moreover, they obtained the weak and strong convergence results of problem (6) with resolvent operator technique and the stepsize $r \in (0, \frac{2}{\|A^*A\|})$. In order to speed up the convergence speed, Alvarez and Attouch [16] considered the following iterative scheme: $x_{n+1} = J_\lambda^F(x_n + \theta_n(x_n - x_{n-1}))$, where F is a maximal monotone operator, $\lambda > 0$ and $\theta_n \in [0, 1]$, such the iterative scheme is called the inertial proximal method, and $\theta_n(x_n - x_{n-1})$ is referred to as the inertial extrapolation term which is a procedure of speeding up the convergence properties under the condition that $\sum_{n=1}^{\infty} \|x_n - x_{n-1}\|^2 < \infty$ (see [17]). At the same time, the idea of the inertial technique plays an important role in the optimization community as a technology to build an accelerating iterative method, (see [18,19]).

On the other hand, the iterative methods mentioned above share a common manner, that is, their step size requires a knowlegde of the prior information of the operator (matrix) norm $\|A\|$. It may be difficult to calculate $\|A\|$ and the fixed step size of iterative methods have an impact on the implementation. To conquer, the construction of self-adaptive step size has aroused interest among researchers. Recently, there are many iterative methods that do not require the prior information of the operator (matrix) norm, (see [20–22]).

Motivated and inspired by the work mentioned above, we further investigate the self-adaptive inertial Krasnoselski-mann iterative method for solving hierarchical fixed point and split monotone variational inclusion problems. This manuscript aims to suggest modifications of the results that appeared in [14] by applying the inertial scheme that is effective for speeding up the iteration process and adding the step size that the prior information of the operator (matrix) norm is not required. A strong convergence theorem of the proposed iterative method is established under some suitable conditions. Furthermore, we also present some numerical experiments to demonstrate the advantages of the stated iterative method over other existing methods [9,13,14] and apply our main results to solving the image restoration problem.

2. Materials and Methods

In this section, we recall some basic definitions and properties which will be frequently used in our later investigation. Some useful results proved already in the literature are also summarized.

Definition 1. *A mapping $f : H \to H$ is said to be:*

(i) *monotone, if*

$$\langle fx - fy, x - y \rangle \geq 0, \ \ \forall x, y \in H;$$

(ii) *α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle fx - fy, x - y \rangle \geq \alpha \|fx - fy\|^2, \ \ \forall x, y \in H;$$

(iii) *β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|fx - fy\| \leq \beta \|x - y\| \ \ \forall x, y \in H.$$

Let $F : H \to 2^H$ be a multivalued operator on H. Then the graph $G(F)$ of F is defined by

$$G(F) = \{(x, y) \in H \times H : y \in F(x)\},$$

and

(i) the operator F is called a monotone operator, if

$$\langle u - v, x - y \rangle \geq 0, \text{ whenever } u \in F(x), v \in F(y);$$

(ii) the operator F is called a maximal monotone operator, if F is monotone and the graph of F is not properly contained in the graph of other monotone mappings.

Let $F : H \to 2^H$ be a set-valued maximal monotone mapping, which Graph (F) is not property contained in the graph of any other monotone mapping. Then the resolvent operator $J_\lambda^F : H \to H$ is defined by

$$J_\lambda^F = (I + \lambda F)^{-1}(x), \ \forall x \in H,$$

where I stands for the identity operator on H. It is well known that resolvent operator J_λ^F is single-valued, nonexpansive and firmly nonexpansive for $\lambda > 0$.

Lemma 1. *Let f be a κ- inverse strongly monotone mapping and F be a maximal monotone mapping, then $J_\lambda^F(I - \lambda F)$ is averaged for all $\lambda \in (0, 2\kappa)$.*

Lemma 2 ([9])**.** *Let f be a mapping and F be a maximal monotone mapping, then $0 \in f(x^*) + F(x^*)$ if and only if $x^* = J_\lambda^F(I - \lambda f)x^*$, i.e., $x^* \in Fix(J_\lambda^F(I - \lambda f))$ for $\lambda > 0$.*

Lemma 3. *For any $x, y \in H$, the following results hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \ \forall t \in [0, 1]$.

Lemma 4 ([23])**.** *Let $D : H \to H$ be η-strongly monotone and L-Lipschitz continuous. Then $I - \mu_n D$ is a $(1 - \mu_n \rho)$-contraction, i.e.,*

$$\|(I - \mu_n D)x - (I - \mu_n D)y\| \leq (1 - \mu_n \rho)\|x - y\|, \ \forall x, y \in H,$$

where $\{\mu_n\}$ is a sequence such that $\mu_n \in (0, \mu]$ and $\rho = \frac{2\eta - \mu L^2}{2}$ with $\mu < \frac{2\mu}{L^2}$.

Lemma 5 ([24])**.** *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \tau_n)a_n + b_n + c_n, \ n \geq 0,$$

where $\{\tau_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$. Assume that $\Sigma_{n=0}^{\infty} c_n < \infty$. If $b_n \leq \tau_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.

Lemma 6 ([25])**.** *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\tau_n\}$ be a sequence of real numbers in $(0, 1)$ with $\Sigma_{n=1}^{\infty} \tau_n = \infty$ and $\{b_n\}$ be a sequence of real numbers such that*

$$a_{n+1} \leq (1 - \tau_n)a_n + \tau_n b_n, \ \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

3. Results

In this section, we propose a self-adaptive method with inertial extrapolation term for solving hierarchical fixed point and split variational inclusion problems. To begin with, the control conditions need to be satisfied by Condition 1:

(A1) $\{\theta_n\} \subset (0, \theta)$ for some $\theta > 0$ such that $\theta_n = O(\tau_n)$, i.e., $\lim_{n\to\infty} \frac{\theta_n}{\tau_n} = 0$;

(A2) $\{\tau_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=0}^{\infty} \tau_n = \infty$;

(A3) $\{\alpha_n\}, \{\beta_n\} \subset (a,b) \subset (0,1)$ such that $\liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=0}^{\infty} \beta_n < \infty$;

(A4) $\{\mu_n\}$ is a positive sequence such that $\lim_{n\to\infty} \mu_n = 0$ and $\sum_{n=0}^{\infty} \mu_n = \infty$;

(A5) $\{\delta_n\} \subset (a,b) \subset (0, 1-\tau_n)$, $\{\alpha_n\beta_n\} \subset (a,b) \subset (0,\tau_n)$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n\mu_n < \infty$.

Remark 1. *From* $\bar{\theta_n} = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$ *and Algorithm 1, we have*

$$\theta_n \|x_n - x_{n-1}\| \leq \bar{\theta_n}\|x_n - x_{n-1}\| \leq \epsilon_n.$$

Therefore $\Sigma_{n=1}^{\infty}\theta_n(x_n - x_{n-1}) < \infty$.

Algorithm 1: Inertial Krasnoselski-Mann Iterative Method

Initialization : Let $x_0, x_1 \in H_1$, $\theta > 0$, $\{\sigma_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,\infty)$ and $\sum_{n=0}^{\infty} \epsilon_n < \infty$.
Iterative steps : Calculate x_{n+1} follows:
Step 1. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose $0 \leq \theta_n \leq \bar{\theta_n}$ where
$\bar{\theta_n} = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise.} \end{cases}$
Step 2. Compute
$w_n = x_n + \theta_n(x_n - x_{n-1})$;
$u_n = (1-\alpha_n)w_n + \alpha_n T((1-\beta_n)Sw_n + \beta_n(1-\mu_n D)w_n)$.
Step 3. Compute $U = J_\lambda^F(I - \lambda f)$, $V = J_\lambda^G(I - \lambda g)$ and
$y_n = U(u_n + r_n(A^*(V-I)Au_n))$, where
$r_n = \begin{cases} \dfrac{\sigma_n\|(V-I)Au_n\|^2}{\|A^*(V-I)Au_n\|^2}, & \text{if } \|A^*(V-I)Au_n\|^2 \neq 0; \\ 0, & \text{otherwise.} \end{cases}$
Step 4. Compute
$x_{n+1} = (1 - \delta_n - \tau_n)x_n + \delta_n y_n$.
Set $n = n+1$, and return to **Step 1**.

Lemma 7. *Assume that H_1 and H_2 are real Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*. Let $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be κ_1, κ_2- inverse strongly monotone mappings with $\kappa = \min\{\kappa_1, \kappa_2\}$. Then, for any $x, y \in H$, $\lambda \in (0, 2\kappa)$ and $L = \|A\|^2$, the following statements hold:*

(i) $U = J_\lambda^F(I - \lambda f)$ *and* $V = J_\lambda^G(I - \lambda g)$ *are nonexpansive;*
(ii) $\|Wx - Wy\|^2 \leq \|x-y\|^2 - r_n(1 - r_n L)\|(V-I)Ax - (V-I)Ay\|^2$,
where $W = I + r_n A * (V - I)A$.

Proof. (i) Since f and g are κ_1, κ_2- inverse strongly monotone mappings, respectively. From Lemma 1, we obtain that $U = J_\lambda^F(I - \lambda f)$ is averaged. So U is nonexpansive for $\lambda \in (0, 2\kappa)$.

Similarly, we can prove that $V = J_\lambda^G(I - \lambda g)$ is nonexpansive.
(ii) From $V = J_\lambda^G(I - \lambda g)$ is nonexpansive, we have

$$\begin{aligned}
\|(I-V)Ax-(I-V)Ay\|^2 &= \langle (I-V)(Ax-Ay),(I-V)(Ax-Ay)\rangle \\
&= \|(Ax-Ay)-(V(Ax)-V(Ay))\|^2 \\
&= \|Ax-Ay\|^2 - 2\langle Ax-Ay, V(Ax)-V(Ay)\rangle \\
&\quad + \|V(Ax-Ay)\|^2 \\
&\leq 2\|Ax-Ay\|^2 - 2\langle Ax-Ay, V(Ax)-V(Ay)\rangle \\
&= 2\langle Ax-Ay,(Ax-Ay)-(V(Ax-Ay))\rangle \\
&= 2\langle Ax-Ay,(I-V)Ax-(I-V)Ay\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|Wx-Wy\|^2 &= \|(I+r_nA^*(V-I)A)x-(I+r_nA^*(V-I)A)y\|^2 \\
&= \|x-y\|^2 - 2r_n\langle x-y, A^*(I-V)Ax - A^*(I-V)Ay\rangle \\
&\quad + r_n^2\|A^*(I-V)Ax - A^*(I-V)Ay\|^2 \\
&\leq \|x-y\|^2 - r_n\|(I-V)Ax-(I-V)Ay\|^2 \\
&\quad + r_n^2\|A\|^2\|(I-V)Ax-(I-V)Ay\|^2 \\
&= \|x-y\|^2 - r_n(1-r_nL)\|(V-I)Ax-(V-I)Ay\|^2.
\end{aligned}$$

□

Lemma 8. *The sequence $\{r_n\}$ formed by Algorithm 1., then $\{r_n\}$ is bounded.*

Proof. If $\|A^*(J_\lambda^G(I-\lambda g)-I)Au_n\|^2 \neq 0$, then

$$\inf\left\{\frac{1\|(J_\lambda^G(I-\lambda g)-I)Au_n\|^2}{\|A^*(J_\lambda^G(I-\lambda g)-I)Au_n\|^2} - r_n\right\} > 0.$$

From A is bounded and linear, we get

$$\begin{aligned}
\frac{\sigma_n\|(J_\lambda^G(I-\lambda g)-I)Au_n\|^2}{\|A^*(J_\lambda^G(I-\lambda g)-I)Au_n\|^2} &\geq \frac{\sigma_n\|(J_\lambda^G(I-\lambda g)-I)Au_n\|^2}{\|A\|^2\|(J_\lambda^G(I-\lambda g)-I)Au_n\|^2} \\
&= \frac{\sigma_n}{\|A\|^2}.
\end{aligned} \tag{7}$$

Hence $\sup r_n > \infty$, thus $\{r_n\}$ is bounded. □

Lemma 9. Assume that $\{y_n\}$ and $\{u_n\}$ are formed by Algorithm 1. If $\{u_{n_k}\}$ converges weakly to z and $\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\| = \lim_{k\to\infty}\|(V-I)Au_{n_k}\| = 0$, then $z \in \Omega = \{x^* \in H_1 : x^* \in$ Sol(MVIP (3))$\}$ and $Ax^* \in$ Sol(MVIP (4))$\}$.

Proof. Since U is nonexpansive, we get

$$\begin{aligned}
\|y_{n_k}-Uu_{n_k}\| &= \|U(u_{n_k}+r_{n_k}A^*(V-I)Au_{n_k})-Uu_{n_k}\| \\
&\leq \|r_{n_k}(A^*(V-I)Au_{n_k})\| \\
&= r_{n_k}\|A\|\|(V-I)Au_{n_k}\|,
\end{aligned}$$

which together $\lim_{k\to\infty}\|(V-I)Au_{n_k}\| = 0$ gives that

$$\lim_{k\to\infty}\|y_{n_k}-Uu_{n_k}\| = 0.$$

Since

$$\|u_{n_k} - Uu_{n_k}\| = \|u_{n_k} - y_{n_k}\| + \|y_{n_k} - Uu_{n_k}\|,$$

we get

$$\lim_{k\to\infty} \|u_{n_k} - Uu_{n_k}\| = 0.$$

This combines with Lemma 2 and $u_{n_k} \rightharpoonup z$ yields $z \in Fix(J_\lambda^F(I - \lambda f))$. In view of the fact A is a linear operator, we get $Au_{n_k} \rightharpoonup Az$. From $\lim_{k\to\infty} \|(V - I)Au_{n_k}\| = 0$, we obtain $Az \in Fix(J_\lambda^G(I - \lambda g))$. Therefore $z \in \Omega$. □

Theorem 1. *Let H_1 and H_2 be are two real Hilbert spaces and C be a nonempty closed convex subset of H_1 and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Assume that $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are set-valued maximal monotone operators. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be are κ_1, κ_2- inverse strongly monotone mappings with $\kappa = \min\{\kappa_1, \kappa_2\}$. Let $D : C \to C$ be η-strongly monotone and L-Lipschitzian, $T : C \to C$ be a nonexpansive mapping, $S : C \to C$ be a continuous quasi-nonexpansive mapping such that $I - S$ is monotone and $\Psi = \Phi \cap \Omega \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by Algorithm 1 and the Condition 1 holds. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Psi$ in the norm, where $\|x^*\| = \min\{\|z\| : z \in \Omega\}$.*

Proof. This proof is divided the remaining proof into several steps.
Step 1. We will show that $\{x_n\}$ is bounded. Let us assume that $x^* \in \Psi$. From the definition of w_n, we get

$$\begin{aligned} \|w_n - x^*\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - x^*\| + \tau_n\frac{\theta_n}{\tau_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{8}$$

According to Condition 1, one has $\frac{\theta_n}{\tau_n}\|x_n - x_{n-1}\| \to 0$. Therefore, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_n}{\tau_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1. \tag{9}$$

Combining (8) and (9), we obtain

$$\|w_n - x^*\| \leq \|x_n - x^*\| + \tau_n M_1, \quad \forall n \geq 1. \tag{10}$$

Set $v_n = (1 - \beta_n)Sw_n + \beta_n B_n w_n$ and $B_n = I - \mu_n D$, we get

$$\begin{aligned} \|u_n - x^*\| &= \|(1 - \alpha_n)w_n + \alpha_n(Tv_n) - x^*\| \\ &= \|(1 - \alpha_n)(w_n - x^*) + \alpha_n(Tv_n - x^*)\| \\ &\leq (1 - \alpha_n)\|w_n - x^*\| + \alpha_n\|Tv_n - x^*\| \\ &\leq (1 - \alpha_n)\|w_n - x^*\| + \alpha_n\|v_n - x^*\|. \end{aligned} \tag{11}$$

From Lemma 4, we have

$$\begin{aligned} \|v_n - x^*\| &= \|(1 - \beta_n)Sw_n + \beta_n B_n w_n - x^*\| \\ &\leq (1 - \beta_n)\|Sw_n - x^*\| + \beta_n\|B_n w_n - x^*\| \\ &\leq (1 - \beta_n)\|w_n - x^*\| + \beta_n\Big(\|B_n w_n - B_n x^*\| + \|B_n x^* - x^*\|\Big) \\ &\leq (1 - \beta_n)\|w_n - x^*\| + \beta_n\Big((1 - \mu_n\rho)\|w_n - x^*\| + \mu_n\|Dx^*\|\Big) \\ &\leq \|w_n - x^*\| + \beta_n\mu_n\|Dx^*\|. \end{aligned} \tag{12}$$

From (11) and (12), we get

$$\begin{aligned} \|u_n - x^*\| &\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\|v_n - x^*\| \\ &\leq (1-\alpha_n)\|w_n - x^*\| + \alpha_n\Big(\|w_n - x^*\| + \beta_n\mu_n\|Dx^*\|\Big) \\ &\leq \|w_n - x^*\| + \alpha_n\beta_n\mu_n\|Dx^*\|. \end{aligned} \tag{13}$$

Indeed, it follows Lemma 7, $Ax^* = V(Ax^*)$ and (7), we have

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - r_n(1 - r_n\|A\|^2)\|(V-I)Au_n\|^2 \\ &\leq \|u_n - x^*\|^2 - r_n(1-\sigma_n)\|(V-I)Au_n\|^2. \end{aligned} \tag{14}$$

It follows from $\{\sigma_n\} \subset (0,1)$, we obtain

$$\|y_n - x^*\| \leq \|u_n - x^*\|. \tag{15}$$

Combining (10), (13) and (15), we get

$$\|y_n - x^*\| \leq \|w_n - x^*\| + \alpha_n\beta_n\mu_n\|Dx^*\| \leq \|x_n - x^*\| + \tau_n M_1 + \alpha_n\beta_n\mu_n\|Dx^*\|.$$

Set $M_2 = \tau_n M_1 + \alpha_n\beta_n\mu_n\|Dx^*\| \geq 0$, it follows that

$$\|y_n - x^*\| \leq \|x_n - x^*\| + M_2. \tag{16}$$

By the definition of x_{n+1}, we also have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\delta_n-\tau_n)x_n + \delta_n y_n - x^*\| \\ &= \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*) + \tau_n x^*\| \\ &\leq \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\| + \tau_n\|x^*\|. \end{aligned} \tag{17}$$

Further, we have

$$\begin{aligned} \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\|^2 &= (1-\delta_n-\tau_n)^2\|x_n - x^*\|^2 + \delta_n^2\|y_n - x^*\|^2 \\ &\quad +2(1-\delta_n-\tau_n)\delta_n\langle x_n - x^*, y_n - x^*\rangle \\ &\leq (1-\delta_n-\tau_n)^2\|x_n - x^*\|^2 + \delta_n^2\|y_n - x^*\|^2 \\ &\quad +(1-\delta_n-\tau_n)\delta_n\|x_n - x^*\|^2 \\ &\quad +(1-\delta_n-\tau_n)\delta_n\|y_n - x^*\|^2 \\ &= (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 \\ &\quad +(1-\tau_n)\delta_n\|y_n - x^*\|^2. \end{aligned} \tag{18}$$

By (16) and Condition 1, we get

$$\begin{aligned} & (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\|y_n - x^*\|^2 \\ \leq\ & (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\|x_n - x^*\| + M_2\Big)^2 \\ =\ & (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\|x_n - x^*\|^2 + 2M_2\|x_n - x^*\| + M_2^2\Big) \\ =\ & (1-\tau_n)^2\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(2M_2\|x_n - x^*\| + M_2^2\Big) \\ =\ & (1-\tau_n)^2\|x_n - x^*\|^2 + 2(1-\tau_n)\delta_n M_2\|x_n - x^*\| + (1-\tau_n)\delta_n M_2^2 \\ \leq\ & \Big((1-\tau_n)\|x_n - x^*\| + M_2\Big)^2. \end{aligned} \tag{19}$$

Substituting (19) into (17), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \|(1-\delta_n-\tau_n)(x_n-x^*)+\delta_n(y_n-x^*)\|+\tau_n\|x^*\| \\ &\leq (1-\tau_n)\|x_n-x^*\|+M_2+\tau_n\|x^*\| \\ &= (1-\tau_n)\|x_n-x^*\|+\tau_n M_1+\alpha_n\beta_n\mu_n\|Dx^*\|+\tau_n\|x^*\| \\ &= (1-\tau_n)\|x_n-x^*\|+\tau_n(M_1+\|x^*\|)+\alpha_n\beta_n\mu_n\|Dx^*\|.\end{aligned}$$

From Lemma 5, this implies that $\{x_n\}$ is bounded. Together with (10), (13) and (16), we get $\{w_n\}$, $\{u_n\}$ and $\{y_n\}$ are also bounded.

Step 2. We will show that

$$\delta_n(1-\tau_n)\Big(\alpha_n(1-\alpha_n)\|Tv_n-w_n\|^2+\delta_n r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big) \leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\tau_n M_5.$$

By the definition of w_n, we get

$$\begin{aligned}\|w_n-x^*\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-x^*\|^2 \\ &= \|(x_n-x^*)+\theta_n(x_n-x_{n-1})\|^2 \\ &= \|x_n-x^*\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\langle x_n-x^*,x_n-x_{n-1}\rangle \\ &\leq \|x_n-x^*\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\|x_n-x^*\|\|x_n-x_{n-1}\| \\ &= \|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|\Big(\theta_n\|x_n-x_{n-1}\|+2\|x_n-x^*\|\Big) \\ &\leq \|x_n-x^*\|^2+\tau_n M_3, \qquad (20)\end{aligned}$$

for some $M_3>0$. From Lemma 3, we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \|(1-\alpha_n)w_n+\alpha_n(Tv_n)-x^*\|^2 \\ &= \|(1-\alpha_n)(w_n-x^*)+\alpha_n(Tv_n-x^*)\|^2 \\ &\leq (1-\alpha_n)\|w_n-x^*\|^2+\alpha_n\|Tv_n-x^*\|^2-\alpha_n(1-\alpha_n)\|Tv_n-w_n\|^2 \\ &\leq (1-\alpha_n)\|w_n-x^*\|^2+\alpha_n\|v_n-x^*\|^2-\alpha_n(1-\alpha_n)\|Tv_n-w_n\|^2. \qquad (21)\end{aligned}$$

From Lemma 3, Lemma 4 and (10), we obtain

$$\begin{aligned}\|v_n-x^*\|^2 &= \|(1-\beta_n)Sw_n+\beta_n B_n w_n-x^*\|^2 \\ &= \|(1-\beta_n)(Sw_n-x^*)+\beta_n(B_n w_n-x^*)\|^2 \\ &= \|(1-\beta_n)(Sw_n-x^*)+\beta_n((1-\mu_n D)w_n-x^*)\|^2 \\ &= \|(1-\beta_n)(Sw_n-x^*)+\beta_n\Big((1-\mu_n D)(w_n-x^*)-\mu_n Dx^*\Big)\|^2 \\ &\leq (1-\beta_n)\|Sw_n-x^*\|^2+\beta_n\|(1-\mu_n D)(w_n-x^*)-\mu_n Dx^*\|^2 \\ &\quad -(1-\beta_n)\beta_n\|Sw_n-B_n w_n\|^2 \\ &\leq (1-\beta_n)\|Sw_n-x^*\|^2+\beta_n\Big((1-\mu_n D)^2\|w_n-x^*\|^2 \\ &\quad -2\langle\mu_n Dx^*,(1-\mu_n D)(w_n-x^*)-\mu_n Dx^*\rangle\Big) \\ &\leq (1-\beta_n)\|w_n-x^*\|^2+\beta_n\Big((1-\mu_n\rho)\|w_n-x^*\|^2 \\ &\quad -2\langle\mu_n Dx^*,(1-\mu_n D)(w_n-x^*)-\mu_n Dx^*\rangle\Big) \\ &\leq \|w_n-x^*\|^2+2\beta_n\mu_n\|Dx^*\|\Big(\|x^*-w_n\|+\mu_n\|Dw_n\|\Big) \\ &\leq \|w_n-x^*\|^2+2\beta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\|+\tau_n M_1+\mu_n\|Dw_n\|\Big). \qquad (22)\end{aligned}$$

Substituting (20) and (22) into (21), we get

$$\begin{aligned}\|u_n - x^*\|^2 &\le (1-\alpha_n)\|w_n - x^*\|^2 + \alpha_n\Big(\|w_n - x^*\|^2 + 2\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| \\ &\quad + \tau_n M_1 + \mu_n\|Dw_n\|\Big)\Big) - \alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 \\ &\le \|w_n - x^*\|^2 + 2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 + \mu_n\|Dw_n\|\Big) \\ &\quad -\alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2. \end{aligned} \tag{23}$$

From (14) and (23), it follows that

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|u_n - x^*\|^2 - r_n(1-\sigma_n)\|(V-I)Au_n\|^2 \\ &\le \|w_n - x^*\|^2 + 2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 + \mu_n\|Dw_n\|\Big) \\ &\quad -\alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 - r_n(1-\sigma_n)\|(V-I)Au_n\|^2. \end{aligned} \tag{24}$$

By the definition of x_{n+1} , we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\delta_n-\tau_n)x_n + \delta_n y_n - x^*\|^2 \\ &= \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*) - \tau_n x^*\|^2 \\ &= \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\|^2 + \tau_n^2\|x^*\|^2 \\ &\quad -\tau_n\langle(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*), x^*\rangle \\ &\le \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\|^2 + \tau_n M_4, \end{aligned} \tag{25}$$

for some $M_4 > 0$. Substituting (18), (24) and using Condition 1 into (25), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\|^2 + \tau_n M_4 \\ &\le (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\|y_n - x^*\|^2 + \tau_n M_4 \\ &\le (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\|x_n - x^*\|^2 + \tau_n M_3 \\ &\quad +2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 + \mu_n\|Dw_n\|\Big) \\ &\quad -\alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 - r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big) + \tau_n M_4 \\ &\le (1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\tau_n M_3 + 2\tau_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| \\ &\quad +\tau_n M_1 + \mu_n\|Dw_n\|\Big) - \alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 \\ &\quad -r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big) + \tau_n M_4 \\ &\le \|x_n - x^*\|^2 + \tau_n M_5 - \delta_n(1-\tau_n)\Big(\alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 \\ &\quad -r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big), \end{aligned} \tag{26}$$

for some $M_5 > 0$. Thus

$$\begin{aligned}&\delta_n(1-\tau_n)\Big(\alpha_n(1-\alpha_n)\|Tv_n - w_n\|^2 + \delta_n r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big) \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \tau_n M_5.\end{aligned}$$

Step 3. We will show that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1-\tau_n)\|x_n - x^*\|^2 + \tau_n\Big(\delta_n M_3 + 2\delta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| \\ &\quad +\tau_n M_1 + \mu_n\|Dw_n\|\Big) + 2\delta_n\|y_n - x_n\|\|x^* - x_{n+1}\| + 2\langle x^*, x^* - x_{n+1}\rangle\Big).\end{aligned}$$

From the definition of x_{n+1}, we get

$$\begin{aligned} x_{n+1} &= (1-\delta_n-\tau_n)x_n+\delta_n y_n \\ &= (1-\delta_n)x_n+\delta_n y_n-\tau_n x_n. \end{aligned}$$

Let $t_n=(1-\delta_n)x_n+\delta_n y_n$. Then we have $x_n-t_n=\delta_n(y_n-x_n)$ and

$$\begin{aligned} \|t_n-x^*\|^2 &= \|(1-\delta_n)x_n+\delta_n y_n-x^*\|^2 \\ &= \|(1-\delta_n)(x_n-x^*)+\delta_n(y_n-x^*\|^2 \\ &\le (1-\delta_n)\|x_n-x^*\|^2+\delta_n\|y_n-x^*\|^2-\delta_n(1-\delta_n)\|x_n-y_n\|^2 \\ &\le (1-\delta_n)\|x_n-x^*\|^2+\delta_n\|y_n-x^*\|^2. \end{aligned} \tag{27}$$

From the definition of t_n, we obtain

$$\begin{aligned} x_{n+1} &= t_n-\tau_n x_n \\ &= (1-\tau_n)t_n-\tau_n(x_n-t_n) \\ &= (1-\tau_n)t_n-\tau_n\delta_n(y_n-x_n). \end{aligned} \tag{28}$$

This implies that

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \|(1-\tau_n)t_n-\tau_n\delta_n(y_n-x_n)-x_*\|^2 \\ &= \|(1-\tau_n)(t_n-x^*)-\tau_n(\delta_n(y_n-x_n)-x^*)\|^2 \\ &\le (1-\tau_n)\|t_n-x^*\|^2+2\langle\tau_n\delta_n(y_n-x_n)+\tau_n x^*,x^*-x_{n+1}\rangle \\ &\le (1-\tau_n)\|t_n-x^*\|^2+2\tau_n\delta_n\langle y_n-x_n,x^*-x_{n+1}\rangle+2\tau_n\langle x^*,x^*-x_{n+1}\rangle \\ &\le (1-\tau_n)\|t_n-x^*\|^2+2\tau_n\delta_n\|y_n-x_n\|\|x^*-x_{n+1}\|+2\tau_n\langle x^*,x^*-x_{n+1}\rangle. \end{aligned} \tag{29}$$

From (20), (24), (27) and (29), we have

$$\begin{aligned} \|x_{n+1}-x^*\|^2 \le\ & (1-\tau_n)\Big((1-\delta_n)\|x_n-x^*\|^2+\delta_n\|y_n-x^*\|^2\Big) \\ & +2\tau_n\delta_n\|y_n-x_n\|\|x^*-x_{n+1}\|+2\tau_n\langle x^*,x^*-x_{n+1}\rangle \\ \le\ & (1-\tau_n)\Big((1-\delta_n)\|x_n-x^*\|^2+\delta_n\Big(\|x_n-x^*\|^2+\tau_n M_3 \\ & +2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\|+\tau_n M_1+\mu_n\|Dw_n\|\Big) \\ & -\alpha_n(1-\alpha_n)\|Tv_n-w_n\|^2-r_n(1-\sigma_n)\|(V-I)Au_n\|^2\Big)\Big) \\ & +2\tau_n\delta_n\|y_n-x_n\|\|x^*-x_{n+1}\|+2\tau_n\langle x^*,x^*-x_{n+1}\rangle \\ \le\ & (1-\tau_n)\|x_n-x^*\|^2+\tau_n\Big(\delta_n M_3+2\delta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\|+\tau_n M_1 \\ & +\mu_n\|Dw_n\|\Big)\Big)+2\delta_n\|y_n-x_n\|\|x^*-x_{n+1}\|+2\langle x^*,x^*-x_{n+1}\rangle. \end{aligned} \tag{30}$$

Step 4. We will show $x^*\in\Psi$. Assume that $\{\|x_{n_k}-x^*\|\}$ is a subsequence of $\{\|x_n-x^*\|\}$ such that

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|\Big)\ge 0.$$

Then,

$$\begin{aligned} & \liminf_{k\to\infty}\Big(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\Big) \\ & =\liminf_{k\to\infty}\Big(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|\Big)\Big(\|x_{n_{k+1}}-x^*\|+\|x_{n_k}-x^*\|\Big)\ge 0. \end{aligned}$$

From Step 2, we see that

$$\begin{aligned}
&\delta_{n_k}\alpha_{n_k}(1-\alpha_{n_k})\|Tv_{n_k}-w_{n_k}\|^2+\delta_{n_k}r_{n_k}(1-\sigma_{n_k})\|(V-I)Au_{n_k}\|^2\\
&\leq \limsup_{k\to\infty}\Big(\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2+\tau_{n_k}M_5\Big)\\
&\leq \limsup_{k\to\infty}\Big(\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\Big)+\limsup_{k\to\infty}\tau_{n_k}M_5\\
&\leq 0,
\end{aligned}$$

which indicates that

$$\lim_{k\to\infty}\|Tv_{n_k}-w_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|(V-I)Au_{n_k}\|=0. \tag{31}$$

From the definition of u_n, we get

$$\|u_{n_k}-w_{n_k}\|=\alpha_{n_k}\|Tv_{n_k}-w_{n_k}\|.$$

Thus

$$\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0. \tag{32}$$

From $w_{n_k}-x_{n_k}=\theta_{n_k}(x_{n_k}-x_{n_k-1})$, we have

$$\begin{aligned}
\|u_{n_k}-x_n\| &\leq \|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\\
&= \|u_{n_k}-w_{n_k}\|+\theta_{n_k}\|x_{n_k}-x_{n_{k-1}}\|\\
&= \|u_{n_k}-w_{n_k}\|+\tau_{n_k}\frac{\theta_{n_k}}{\tau_{n_k}}\|x_{n_k}-x_{n_{k-1}}\|.
\end{aligned}$$

So

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0. \tag{33}$$

Since T is nonexpansive, S is a continuous quasi-nonexpansive mapping, we have

$$\begin{aligned}
\|Tv_{n_k}-x^*\|^2 &\leq \|v_{n_k}-x^*\|^2\\
&= \|((1-\beta_{n_k})Sw_{n_k}+\beta_{n_k}B_{n_k}w_{n_k})-x^*\|^2\\
&= \|(1-\beta_{n_k})(Sw_{n_k}-x^*)+\beta_{n_k}(w_{n_k}-x^*)-\beta_{n_k}\mu_{n_k}Dw_{n_k}\|^2\\
&\leq \|(1-\beta_{n_k})(Sw_{n_k}-x^*)+\beta_{n_k}(w_{n_k}-x^*)\|^2-2\beta_{n_k}\mu_{n_k}\langle Dw_{n_k},v_{n_k}-x^*\rangle\\
&\leq (1-\beta_{n_k})\|Sw_{n_k}-x^*\|^2+\beta_{n_k}\|w_{n_k}-x^*\|^2-\beta_{n_k}(1-\beta_{n_k})\|Sw_{n_k}-w_{n_k}\|^2\\
&\quad -2\beta_{n_k}\mu_{n_k}\langle Dw_{n_k},v_{n_k}-x^*\rangle\\
&\leq \|w_{n_k}-x^*\|^2-\beta_{n_k}(1-\beta_{n_k})\|Sw_{n_k}-w_{n_k}\|^2-2\beta_{n_k}\mu_{n_k}\langle Dw_{n_k},v_{n_k}-x\rangle.
\end{aligned} \tag{34}$$

So

$$\begin{aligned}
\beta_{n_k}(1-\beta_{n_k})\|Sw_{n_k}-w_{n_k}\|^2 &\leq \|w_{n_k}-x^*\|^2-\|Tv_{n_k}-x^*\|^2-2\beta_{n_k}\mu_{n_k}\langle Dw_{n_k},v_{n_k}-x^*\rangle\\
&\leq \|w_{n_k}-Tv_{n_k}\|(\|w_{n_k}-x^*\|-\|Tv_{n_k}-x^*\|)\\
&\quad -2\beta_{n_k}\mu_{n_k}\langle Dw_{n_k},v_{n_k}-x^*\rangle.
\end{aligned}$$

Thus

$$\lim_{k\to\infty}\|Sw_{n_k}-w_{n_k}\|=0. \tag{35}$$

Note that

$$\begin{aligned} \|w_{n_k} - v_{n_k}\| &= \|w_{n_k} - \Big((1-\beta_{n_k})Sw_{n_k} + \beta_{n_k}(1-\mu_{n_k}D)w_{n_k}\Big)\| \\ &= \|(1-\beta_{n_k})(Sw_{n_k} - w_{n_k}) + \beta_{n_k}\mu_{n_k}Dw_{n_k}\| \\ &\leq (1-\beta_{n_k})\|Sw_{n_k} - w_{n_k}\| + \beta_{n_k}\mu_{n_k}\|Dw_{n_k}\|. \end{aligned} \tag{36}$$

From (35) and Condition 1, we get

$$\lim_{k\to\infty} \|w_{n_k} - v_{n_k}\| = 0. \tag{37}$$

By the definition of u_n, we have

$$\begin{aligned} \|v_{n_k} - Tv_{n_k}\| &= \|v_{n_k} - w_{n_k}\| + \|w_{n_k} - u_{n_k}\| + \|u_{n_k} - Tv_{n_k}\| \\ &= \|v_{n_k} - w_{n_k}\| + \|w_{n_k} - u_{n_k}\| + (1-\alpha_{n_k})\|w_{n_k} - Tv_{n_k}\| \\ &= \|v_{n_k} - w_{n_k}\| + \alpha_{n_k}\|w_{n_k} - Tv_{n_k}\| + (1-\alpha_{n_k})\|w_{n_k} - Tv_{n_k}\| \\ &= \|v_{n_k} - w_{n_k}\| + \|w_{n_k} - Tv_{n_k}\|. \end{aligned}$$

From (31) and (37), we obtain

$$\lim_{k\to\infty} \|v_{n_k} - Tv_{n_k}\| = 0. \tag{38}$$

Since $w_n = x_n + \theta_n(x_n - x_{n-1})$, we have $w_n - x_n = \theta_n(x_n - x_{n-1})$. It follows that

$$\|w_n - x_n\| = \tau_n \frac{\theta_n}{\tau_n}\|x_n - x_{n-1}\|.$$

So

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{39}$$

Set $q_n = u_n + r_n A^*(V-I)Au_n$. Now we estimate

$$\begin{aligned} \|q_{n_k} - x_{n_k}\| &= \|u_{n_k} + r_{n_k}A^*(V-I)Au_{n_k} - x_{n_k}\| \\ &\leq \|u_{n_k} - x_{n_k}\| + r_{n_k}\|A\|\|(V-I)Au_{n_k}\|. \end{aligned}$$

From (31) and (33), we get

$$\lim_{k\to\infty} \|q_{n_k} - x_{n_k}\| = 0. \tag{40}$$

Since

$$\|u_{n_k} - q_{n_k}\| \leq \|u_{n_k} - x_{n_k}\| + \|x_{n_k} - q_{n_k}\|,$$

by (33) and (40), we have

$$\lim_{k\to\infty} \|u_{n_k} - q_{n_k}\| = 0. \tag{41}$$

By the definition of y_n and Lemma 7, we get

$$\begin{aligned} \|y_{n_k} - x^*\|^2 &= \|J_\lambda^F(I-\lambda f)q_{n_k} - J_\lambda^F(I-\lambda f)x^*\|^2 \\ &\leq \|(q_{n_k} - x^*) - \lambda(fq_n - fx^*)\|^2 \\ &= \|q_{n_k} - x^*\|^2 + \lambda^2\|fq_n - fx^*\|^2 - 2\lambda\langle q_n - x^*, fq_n - fx^*\rangle \\ &\leq \|q_{n_k} - x^*\|^2 + \lambda^2\|fq_n - fx^*\|^2 - 2\kappa_1\lambda\|fq_n - fx^*\| \\ &= \|q_{n_k} - x^*\|^2 - \lambda(2\kappa_1 - \lambda)\|fq_n - fx^*\|. \end{aligned} \tag{42}$$

By the definition of x_{n+1} and (21), (22), (25), (42), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \|(1-\delta_n-\tau_n)(x_n - x^*) + \delta_n(y_n - x^*)\|^2 + \tau_n M_4 \\
&\leq (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\|y_n - x^*\|^2 + \tau_n M_4 \\
&\leq (1-\delta_n-\tau_n)(1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\|x_n - x^*\|^2 + \tau_n M_3 \\
&+\alpha_n\Big(2\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 + \mu_n\|Dw_n\|\Big)\Big) - r_n(1-\sigma_n)\|(V-I)Au_n\|^2 \\
&-\lambda(2\kappa_1 - \lambda)\|fq_n - fx^*\|\Big) + \tau_n M_4 \\
&\leq (1-\tau_n)\|x_n - x^*\|^2 + (1-\tau_n)\delta_n\Big(\tau_n M_3 + \alpha_n\Big(2\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 \\
&+\mu_n\|Dw_n\|\Big)\Big) - r_n(1-\sigma_n)\|(V-I)Au_n\|^2 - \lambda(2\kappa_1-\lambda)\|fq_n - fx^*\|\Big) + \tau_n M_4. \qquad (43)
\end{aligned}$$

Thus

$$\begin{aligned}
(1-\tau_n)\delta_n\lambda(2\kappa_1-\lambda)\|fq_n - fx^*\| \quad &\leq \quad \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1-\tau_n)\delta_n\Big(\tau_n M_3 \\
&+\alpha_n\Big(2\beta_n\mu_n\|Dx^*\|\Big(\|x_n - x^*\| + \tau_n M_1 + \mu_n\|Dw_n\|\Big) \\
&-r_n(1-r_n\sigma_n)\|(V-I)Au_n\|^2\Big) + \tau_n M_4. \qquad (44)
\end{aligned}$$

Substituting n into n_k, from the boundness of $\{x_n\}$, Condition 1, (31), and taking $k \to \infty$, we get

$$\lim_{k\to\infty} \|fq_{n_k} - fx^*\| = 0. \qquad (45)$$

Since J_λ^F is firmly nonexpansive, we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 \quad &= \quad \|J_\lambda^F(I-\lambda f)q_n - J_\lambda^F(I-\lambda f)x^*\| \\
&\leq \quad \langle (I-\lambda f)q_n - (I-\lambda f)x^*, y_n - x^*\rangle \\
&= \quad \frac{1}{2}\Big(\|(I-\lambda f)q_n - (I-\lambda f)x^*\|^2 + \|y_n - x^*\|^2 - \|q_n - y_n - \lambda(fq_n - fx^*)\|^2\Big) \\
&= \quad \frac{1}{2}\Big(\|q_n - x^*\|^2 + \lambda^2\|fq_n - fx^*\|^2 - 2\lambda\langle q_n - x^*, fq_n - fx^*\rangle + \|y_n - x^*\|^2 \\
&\quad -\|q_n - y_n - \lambda(fq_n - fx^*)\|^2\Big) \\
&= \quad \frac{1}{2}\Big(\|q_n - x^*\|^2 + \lambda^2\|fq_n - fx^*\|^2 - 2\lambda\langle q_n - x^*, fq_n - fx^*\rangle + \|y_n - x^*\|^2 \\
&\quad -\Big(\|q_n - y_n\|^2 + \lambda\|fq_n - fx^*\|^2 - 2\lambda\langle q_n - y_n, fq_n - fx^*\rangle\Big)\Big) \\
&\leq \quad \frac{1}{2}\Big(\|q_n - x^*\|^2 + 2\lambda\|x^* - y_n\|\|fq_n - fx^*\| + \|y_n - x^*\|^2 - \|q_n - y_n\|^2\Big).
\end{aligned}$$

So

$$\|y_n - x^*\|^2 \leq \|q_n - x^*\|^2 + 2\lambda\|x^* - y_n\|\|fq_n - fx^*\| - \|q_n - y_n\|^2. \qquad (46)$$

From (23), (25) and (46), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \|(1-\delta_n-\tau_n)(x_n-x^*)+\delta_n(y_n-x^*)\|^2+\tau_n M_4 \\
&\leq (1-\delta_n-\tau_n)(1-\tau_n)\|x_n-x^*\|^2+(1-\tau_n)\delta_n\|y_n-x^*\|^2+\tau_n M_4 \\
&\leq (1-\delta_n-\tau_n)(1-\tau_n)\|x_n-x^*\|^2+(1-\tau_n)\delta_n\Big(\|x_n-x^*\|^2+\tau_n M_3 \\
&\quad +2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\|+\tau_n M_1+\mu_n\|Dw_n\|\Big) \\
&\quad +2\lambda\|x^*-y_n\|\|fq_n-fx^*\|-\|q_n-y_n\|^2\Big)+\tau_n M_4 \\
&\leq (1-\tau_n)\|x_n-x^*\|^2+(1-\tau_n)\delta_n\Big(\tau_n M_3+2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\| \\
&\quad +\tau_n M_1+\mu_n\|Dw_n\|\Big)+2\lambda\|x^*-y_n\|\|fq_n-fx^*\|-\|q_n-y_n\|^2\Big)+\tau_n M_4. \quad (47)
\end{aligned}$$

Therefore

$$\begin{aligned}
&(1-\tau_n)\delta_n\|q_n-y_n\|^2 \\
&\leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(1-\tau_n)\delta_n\Big(\tau_n M_3+2\alpha_n\beta_n\mu_n\|Dx^*\|\Big(\|x_n-x^*\|+\tau_n M_1 \\
&+\mu_n\|Dw_n\|\Big)+2\lambda\|x^*-y_n\|\|fq_n-fx^*\|-\|q_n-y_n\|^2\Big)+\tau_n M_4.
\end{aligned}$$

Substituting n into n_k, from the boundness of $\{x_n\}$, Condition 1, (45) and taking limit $k\to\infty$, we get

$$\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0. \quad (48)$$

Since

$$\|y_{n_k}-u_{n_k}\|\leq\|y_{n_k}-q_{n_k}\|+\|q_{n_k}-u_{n_k}\|,$$

it follows from (41) and (48), we obtain

$$\lim_{k\to\infty}\|y_{n_k}-u_{n_k}\|=0. \quad (49)$$

Since

$$\|x_{n_k}-y_{n_k}\|=\|x_{n_k}-q_{n_k}\|+\|q_{n_k}-y_{n_k}\|,$$

we have

$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0. \quad (50)$$

By the definition of x_{n+1}, we get

$$\|x_{n_{k+1}}-x_{n_k}\|=\delta_{n_k}\|y_{n_k}-x_{n_k}\|-\tau_{n_k}x_{n_k}. \quad (51)$$

Using (50) and Condition 1, we have

$$\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0. \quad (52)$$

Note that

$$\begin{aligned}
\|x_{n_k}-Tx_{n_k}\| &= \|x_{n_k}-v_{n_k}\|+\|v_{n_k}-Tv_{n_k}\|+\|Tv_{n_k}-Tx_{n_k}\| \\
&\leq 2\|x_{n_k}-v_{n_k}\|+\|v_{n_k}-Tv_{n_k}\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to x^*. Assume that $x^* \neq Tx^*$, by Opial property

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - x^*\| &< \liminf_{k\to\infty} \|x_{n_k} - Tx^*\| \\
&\leq \liminf_{k\to\infty} \left(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\| \right) \\
&\leq \liminf_{k\to\infty} \left(2\|x_{n_k} - v_{n_k}\| + \|v_{n_k} - Tv_{n_k}\| + \|x_{n_k} - x^*\| \right) \\
&= \liminf_{k\to\infty} \|x_{n_k} - x^*\|.
\end{aligned}$$

This is a contradiction. Hence $x^* = Tx^*$ and $x^* \in Fix(T)$. On the other hand, we note that $u_n = (1-\alpha_n)w_n + \alpha_n Tv_n$ and

$$\begin{aligned}
w_n - Tv_n &= (w_n - v_n) + (v_n - Tv_n) \\
&= ((1-\beta_n)(w_n - Sw_n) - \beta_n\mu_n Dw_n) + (v_n - Tv_n).
\end{aligned}$$

Setting

$$\varphi_n = \frac{w_n - u_n}{\alpha_n(1-\beta_n)} = (w_n - Sw_n) - \frac{\beta_n}{(1-\beta_n)}\mu_n Dw_n + \frac{1}{(1-\beta_n)}(v_n - Tv_n).$$

In particular, for each $z \in Fix(T)$, we have

$$\begin{aligned}
\langle \varphi_n, w_n - z\rangle &= \langle (I-S)w_n - \frac{\beta_n}{(1-\beta_n)}\mu_n Dw_n + \frac{1}{(1-\beta_n)}(I-T)v_n, w_n - z\rangle \\
&= \langle (I-S)w_n - (I-S)z, w_n - z\rangle + \langle (I-S)z, w_n - z\rangle \\
&\quad - \frac{\beta_n}{(1-\beta_n)}\mu_n\langle Dw_n, w_n - z\rangle + \frac{1}{(1-\beta_n)}\langle (I-T)v_n, w_n - z\rangle \\
&= \langle (I-S)w_n - (I-S)z, w_n - z\rangle + \langle (I-S)z, w_n - z\rangle \\
&\quad - \frac{\beta_n}{(1-\beta_n)}\mu_n\langle Dw_n, w_n - z\rangle + \frac{1}{(1-\beta_n)}\langle (I-T)v_n, w_n - v_n\rangle \\
&\quad + \frac{1}{(1-\beta_n)}\langle (I-T)v_n, v_n - z\rangle,
\end{aligned}$$

which comes from the monotonicity of $I-S$ and $I-T$,

$$\langle \varphi_n, w_n - z\rangle \geq \langle (I-S)z, w_n - z\rangle + \frac{1}{(1-\beta_n)}\langle (I-T)v_n, w_n - v_n\rangle - \frac{\beta_n}{(1-\beta_n)}\mu_n\langle Dw_n, w_n - z\rangle.$$

Replacing n with n_k, we have

$$\begin{aligned}
&\langle \varphi_{n_k}, w_{n_k} - z\rangle \\
&\geq \langle (I-S)z, w_{n_k} - z\rangle + \frac{1}{(1-\beta_{n_k})}\langle (I-T)v_{n_k}, w_{n_k} - v_{n_k}\rangle - \frac{\beta_{n_k}}{(1-\beta_{n_k})}\mu_{n_k}\langle Dw_{n_k}, w_{n_k} - z\rangle.
\end{aligned} \tag{53}$$

Moreover, by (32) and Condition 1, we get

$$\lim_{k\to\infty} \varphi_{n_k} = \lim_{k\to\infty} \frac{w_{n_k} - u_{n_k}}{\alpha_{n_k}(1-\beta_{n_k})} = 0.$$

Since $\|x_{n_k} - w_{n_k}\| \to 0$, one has $w_{n_k} \rightharpoonup x^*$. Taking limit on both sides of (53) for $k \to \infty$, it follows from (37) and (38) that

$$\langle (I-S)z, x^* - z\rangle \leq 0 \ , \forall z \in Fix(T).$$

If z is substituted by $tz + (1-t)x^*$ for $t \in (0,1)$, we get

$$\langle (I-S)(tz+(1-t)x^*), x^* - z\rangle \leq 0 \ , \forall z \in Fix(T).$$

Since $(I-S)$ is continuous, on taking limit $t \to 0^+$, we have

$$\langle (I-S)x^*, x^* - z\rangle \leq 0 \ , \forall z \in Fix(T).$$

That is $x^* \in \Phi$. Next we will show that $x^* \in \Omega$.

Note that $y_{n_k} = U(u_{n_k} + r_{n_k}(A^*(V-I)Au_{n_k}))$ can be rewritten as

$$\frac{y_{n_k} - u_{n_k} + r_{n_k}(A^*(V-I)Au_{n_k}}{\lambda} \in f(Wu_{n_k}) - F(y_{n_k}), \tag{54}$$

where $W = I + r_n A^*(I-V)A$ is nonexpansive for Lemma 7. Taking $k \to \infty$ in (54) by (31), (49) and the graph of maximal monotone mapping is weakly strongly closed, we obtain $0 \in f(x^*) + F(x^*)$. Moreover, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, we have $Au_{n_k} \to Ax^*$. Since the resolvent operator $V = J_\lambda^G(I - \lambda g)$ is average and hence nonexpansive, we can obtain

$$0 \in g(Ax^*) + G(Ax^*).$$

This shows that $x^* \in \Omega$ and so $x^* \in \Psi$.

Step 5. We will show that $x_n \to x^*$ where $\|x^*\| = \min\{\|z\| : z \in \Omega\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $x_{n_j} \rightharpoonup z$. Moreover,

$$\begin{aligned} \limsup_{k\to\infty} \langle x^*, x^* - x_{n_k}\rangle &= \limsup_{j\to\infty} \langle x^*, x^* - x_{n_j}\rangle \\ &= \langle x^*, x^* - z\rangle. \end{aligned} \tag{55}$$

Since $\|x_{n_k} - u_{n_k}\| \to 0$, one has $u_{n_j} \rightharpoonup z$, which together with $\|y_{n_k} - u_{n_k}\| \to 0$ and Lemma 9, we get $z \in \Omega$. From the definition of x^* and (55), we obtain

$$\limsup_{k\to\infty} \langle x^*, x^* - x_{n_k}\rangle = \langle x^*, x^* - z\rangle \leq 0. \tag{56}$$

Combining (56) and (52), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \langle x^*, x^* - x_{n_{k+1}}\rangle &\leq \limsup_{k\to\infty} \langle x^*, x^* - x_{n_k}\rangle \\ &\leq \langle x^*, x^* - z\rangle \leq 0. \end{aligned} \tag{57}$$

This together with (50), Step 3. and Lemma 6, we can conclude that $\{x_n\}$ strongly converges to $x^* \in \Psi$. Hence we obtain the desired conclusion. □

Remark 2. *We note that our results directly improve and extend some known results in the literature as follows:*

(i) Our proposed iterative method has a strong convergence in real Hilbert spaces which is more preferable than the weak convergence results of Kazmi et al. [7].
(ii) We improve and extend Theorem 3.1 of Dao-Jun Wen [14]. Especially, we use the quasi-nonexpansive mappings instead of the nonexpansive mappings.
(iii) The selection of the step size in the iterative method provided by Dao-Jun Wen [14] and Kazmi et al. [7] requires the prior information of the operator (matrix) norm while our iterative method can update the step size of each iteration.
(iv) Our iterative method improve and extend iterative method of Dao-Jun Wen [14] and Kazmi et al. [7].

4. Theoretical Applications

In this section, we derive a scheme for solving hierarchical fixed point and the split problems from Algorithm 1 and also extend and generalize the known results.

4.1. Split Variational Inclusion Problem

The split variational inclusion problem is one of the important special case of the split monotone variational inclusion problem. This is a fundamental problem in optimization theory, which is applied in a wide range of disciplines. In other words, if $f = 0$ and $g = 0$, then the split monotone variational inclusion problem is reduced to split variational inclusion problem. Let us denote Sol(SVIP) $= \{x^* \in H_1 : 0 \in F(x^*)$ and $0 \in G(Ax^*)\}$ by the solution set of the split variational inclusion problem.

Theorem 2. *Let H_1 and H_2 be are two real Hilbert spaces and C be a nonempty closed convex subset of H_1 and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Assume that $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are set-valued maximal monotone operators. Let $D : C \to C$ be η-strongly monotone and L-Lipschitzian, $T : C \to C$ be a nonexpansive mapping, $S : C \to C$ be a continuous quasi-nonexpansive mapping such that $I - S$ is monotone and $\Gamma =$ Sol(SVIP) $\cap \Phi \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ be the a sequence which satisfies the Condition 1 generated by the following scheme:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}); \\ u_n = (1-\alpha_n)w_n + \alpha_n T((1-\beta_n)Sw_n + \beta_n(1-\mu_n D)w_n); \\ y_n = J_\lambda^F(u_n + r_n(A^*(J_\lambda^G - I)Au_n)); \\ x_{n+1} = (1-\delta_n-\tau_n)x_n + \delta_n y_n, \end{cases}$$

where $\{r_n\}$ and $\{\theta_n\}$ satisfy the Algorithm 1. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$ in the norm, where $\|x^*\| = \min\{\|z\| : z \in \Gamma\}$.

4.2. Split Variational Inequality Problem

Let C be a nonempty closed convex subset of a Hilbert space H_1. Define the normal cone $N_C(x)$ of C at a point $x \in C$ by

$$N_C(x) = \{z \in H_1 : \langle z, y - x\rangle \leq 0, \forall y \in C\}.$$

It is known that for each $\lambda > 0$, we get

$$\begin{aligned} y &= (I + \lambda N_C)^{-1}x \Leftrightarrow x \in y + \lambda N_C(y) \Leftrightarrow x - y \in \lambda N_C(y) \\ &\Leftrightarrow \langle x - y, z - y\rangle \leq 0,\ \forall z \in C \\ &\Leftrightarrow y \in P_C x. \end{aligned}$$

This implies that $(I + \lambda N_C)^{-1}x = P_C x$. Let C and Q be nonempty closed convex subsets of Hilbert spaces of H_1 and H_2, respectively. If $F = N_C$ and $G = N_Q$ in the split monotone variational inclusion problem, the following split variational inequality problem is obtained: Find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0 \ \ \forall x \in C \ \text{ and } \langle g(Ax^*), y - Ax^*\rangle \geq 0 \ \ \forall y \in Q,$$

where $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are κ_1, κ_2- inverse strongly monotone mappings with $\kappa = \min\{\kappa_1, \kappa_2\}$. In particular, if f is a κ_1- inverse strongly monotone mapping with $\lambda \in (0, 2\kappa_1)$, then $P_C(I - \lambda f)$ is average. Then, the following results can be obtained from our Theorem 1.

Theorem 3. *Let H_1, H_2, C, Q, f, g be the same as above. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. Select arbitrary initial points $x_0, x_1 \in H_1$, $\{x_n\}$ is generated by the following scheme:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}); \\ u_n = (1-\alpha_n)w_n + \alpha_n T((1-\beta_n)Sw_n + \beta_n(1-\mu_n D)w_n); \\ y_n = P_C(I-\lambda f)(u_n + r_n(A^*(P_Q(I-\lambda g) - I)Au_n)); \\ x_{n+1} = (1-\delta_n-\tau_n)x_n + \delta_n y_n, \end{cases}$$

where $\{r_n\}$ and $\{\theta_n\}$ satisfy the Algorithm 1. Suppose that Condition 1 are satisfied and $\Upsilon = Sol(SVIP) \cap \Phi \cap Fix(S) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Upsilon$ in the norm, where $\|x^*\| = \min\{\|z\| : z \in \Upsilon\}$.

5. Numerical Experiments

Some numerical results will be presented in this section to demonstate the effeciveness of our proposed method. The MATLAB codes were run in MATLAB version 9.5 (R2018b) on MacBook Pro 13-inch, 2019 with 2.4 GHz Quad-Core Intel Core i5 processor. RAM 8.00 GB.

Example 1. *We consider an example in infinite dimensional Hilbert spaces. Assume $H_1 = H_2 = L^2([0,1])$ with inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$ and induced norm $\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}$ for all $x, y \in L^2([0,1])$. Let $F, G : L_2(0,1) \to L_2(0,1)$ be defined by $Fx(t) = 3x(t)$ and $Gx(t) = 5x(t)$ where $x(t) \in L^2([0,1]), t \in [0,1]$. Let $f, g : L_2(0,1) \to L_2(0,1)$ be defined by $fx(t) = 2x(t)$ and $gx(t) = 4x(t)$ where $x(t) \in L^2([0,1]), t \in [0,1]$. Then f is 2-inverse strongly monotone, g is 4-inverse strongly monotone and F, G are maximal monotone. Further, we have $\lambda > 0$ by a direct calculation that*

$$\begin{aligned} J_\lambda^F(x-\lambda fx) &= (I+\lambda F)^{-1}(x-\lambda fx) \\ &= \Big(\frac{1-2\lambda}{1+3\lambda}\Big)x(t) \end{aligned}$$

$$\begin{aligned} J_\lambda^G(x-\lambda gx) &= (I+\lambda G)^{-1}(x-\lambda gx) \\ &= \Big(\frac{1-4\lambda}{1+5\lambda}\Big)x(t), \end{aligned}$$

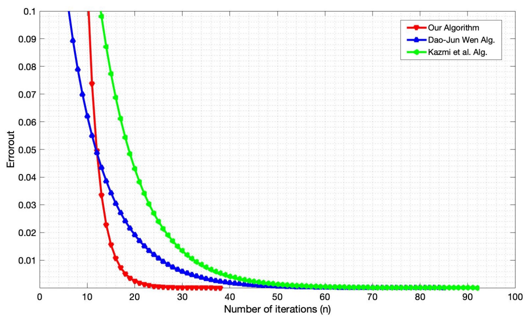
where $x(t) \in L^2[0,1]$, $t \in [0,1]$. Choose $\theta = 0.6$ and $\varepsilon_n = \frac{1}{(n+1)^3}$, we have

$$\bar{\theta_n} = \begin{cases} \min\{0.6, \frac{1}{(n+1)^3\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}; \\ 0.6, & \text{otherwise.} \end{cases}$$

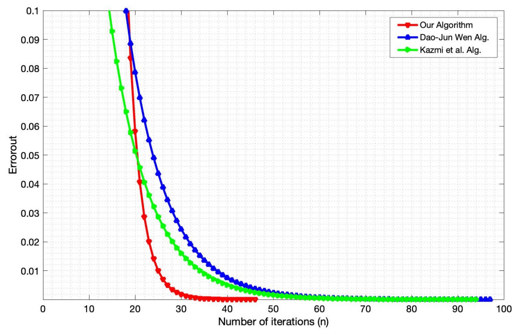
The mapping $A : H_1 \to H_2$ is defined by $Ax(t) = -\frac{9}{4}x(t)$ and $\|A\| = \|A^*\| = \frac{9}{4}$. Let $Tx(t) = x(t)$, $Sx(t) = x(t)\cos(x(t))$ and $Dx(t) = 10x(t)$ where $x(t) \in L^2([0,1])$, $t \in [0,1]$. Then T is a nonexpansive mapping with $Fix(T) = (-\infty, \infty)$ and S is a continuous qusi-nonexpansive mapping with $Fix(S) = \{0\}$ and $(I-S)$ is monotone but S is not a nonexpansive mapping. Hence $\Phi = \mathrm{Sol(HFPP)} = \{0\}$. Furthermore, it easy to prove that $\Omega = \{0\}$. Therefore $\Psi = \Psi \cap \Omega = \{0\} \neq \emptyset$. We choose the iterative coefficients

$$\alpha_n = \frac{1}{2}, \beta_n = \frac{1}{n^2}, \mu_n = \frac{1}{2n}, \delta_n = 0.98 - \tau_n, \tau_n = \frac{1}{n+2}.$$

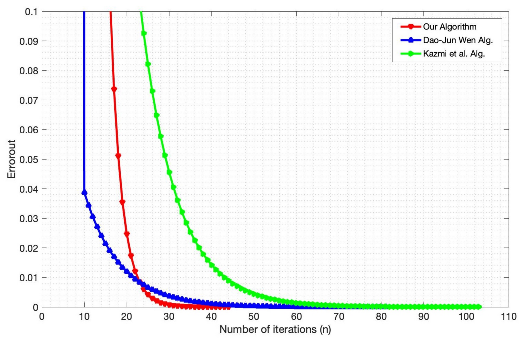
The further comparison of convergence behavior between our proposed method and Dao-Jun Wen. [14] and Kazmi et al. [7] is displayed in Figure 1 and Table 1 with the stopping criteria $\|x_n - x^*\| \leq 10^{-5}$.



(**a**) $x_0 = x_1 = 100t^2$



(**b**) $x_0 = x_1 = 500(t^3 + 2t)$



(**c**) $x_0 = x_1 = 500\sin(t)$

Figure 1. Numerical behavior of all algorithms with different initial values in Example 1.

Table 1. The result of all algorithms with different inertial points in Example 1.

Algorithms	$x_1 = 100t^2$		$x_1 = 500(t^3 + 2t)$		$x_1 = 500\sin(t)$	
	Iter.	**Time (s)**	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**
Our Algorithm	38	0.13	47	0.13	45	0.12
Dao-Jun Wen Alg.	86	0.15	97	0.12	81	0.16
Kazmi et al. Alg.	93	0.18	95	0.24	103	0.21

Remark 3. *Table 1 shows that the different inertial points $x_0 = x_1$ have almost on effect on the number of iterations and shows that our proposed method has a better number of iterations and CPU time than Dao-Jun Wen. [14] and Kazmi et al. [7].*

Example 2. *In this example, we apply our main result to solve the image restoration problem by using Algorithm 1. We consider the convex minimization problem as follows:*

$$\min_x \{h(x) + g(x)\}, \tag{58}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex loss function and differentiable with L-Lipschitz continuous gradient ∇h where $L > 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function. It is well known that the convex minimization problem (58) is equivalent to the following problem:

$$\textit{find a point } x^* \in C \textit{ such that } 0 \in \nabla h(x^*) + \partial g(x^*). \tag{59}$$

We consider the degradation model that represents image restoration problems as the following mathematical model:

$$b = Ax + \varepsilon, \tag{60}$$

where $A \in \mathbb{R}^{m\times n}$ is a blurring matrix and $\varepsilon \in \mathbb{R}^{m\times 1}$is a noise term. The goal is to recover the original image $x \in \mathbb{R}^{n\times 1}$ by minimizing a noise term. We consider a model which produces the restored image given by the following minimization problem:

$$\min_{x}\{\frac{1}{2}\|Ax-b\|_2^2+\omega\|x\|_1\}, \tag{61}$$

for some regularization parameter $\omega > 0$. In this situation, we choose $h(x) = \frac{1}{2}\|Ax-b\|_2^2$ and $g(x) = \mu\|x\|_1$ and set operators $f = \nabla h$, $F = \partial g$, $A = -\frac{9x}{4}$, $D = 10x$, $G = 0$, $T = \frac{3nx}{3n+1}$, $S = prox_{rg}(I - r\nabla h)$ and θ_n is defined as Example 1. We set parameters as follows:

$$\alpha_n = \frac{60n-9}{100n}, \beta_n = \frac{1}{150n^2}, \tau_n = \frac{1}{2000(2n+1)}, \delta_n = 0.4 - \tau_n, \mu_n = 0.5.$$

For this example, we choose the regularization parameter $\omega = 10^{-3}$ and the original gray and RGB images (see in Figure 2a,f). We use an average and motion blur create the blurred and noise images (see in Figure 2b,g). The performance for image restoring process is quantitatively measured by signal-to-noise ratio (SNR), which is defined as

$$SNR = 20\log_{10}\left(\frac{\|x\|_2^2}{\|x-x_n\|_2^2}\right),$$

where x and x_n denote the original image and the restored image at iteration n, respectively. A higher SNR implies that the recovered image is of higher quality. Our numerical results are explained in Table 2 and Figure 3.

Remark 4. *It can be observed from Figure 2 that the restoration quality of the the gray and RGB images restored by our algorithm is better than the quality of the image restored by Dao-Jun Wen. Alg. [14] and Kazmi et al. Alg. [7], and Tabel 2 and Figure 3 are verified by the higher SNR values of our algorithm.*

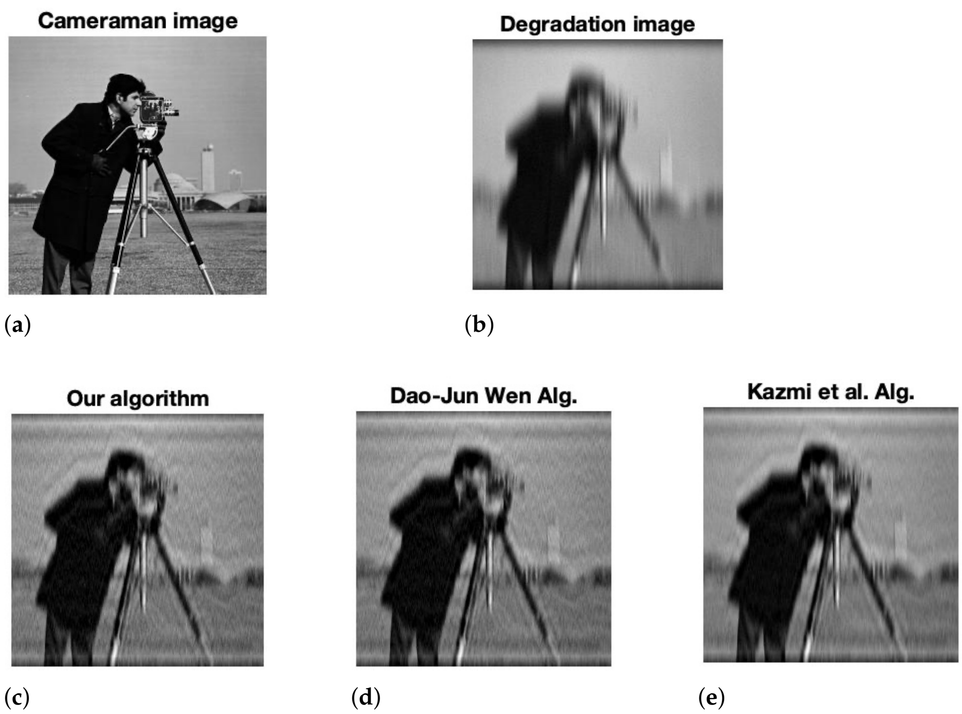
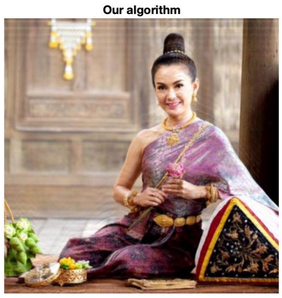


Figure 2. *Cont.*

(**f**)

(**g**)

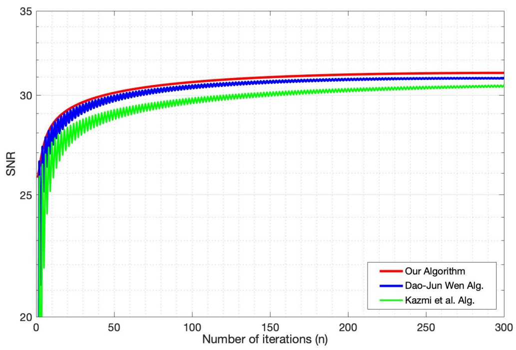


(**h**)

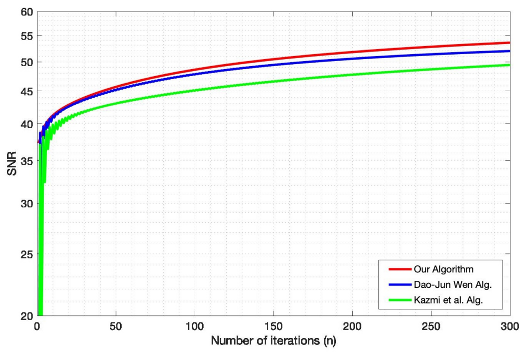
(**i**)

(**j**)

Figure 2. The original images are shown as (**a**) "Cameraman image" and (**f**) "Anchalee image", Degradation images by an average and motion blur are shown as (**b**,**g**), repectively. (**c**,**h**) show the reconstruction by our algorithm, (**d**,**i**) show the reconstruction by Dao-Jun Wen algorithm and (**e**,**j**) show the reconstruction by Kazmi et al. algorithm.



(**a**) SNR of Camaraman image



(**b**) SNR of Anchalee image

Figure 3. Numerical behavior of all algorithms with SNR in Example 2.

Table 2. The performance of signal-to-noise ratio (SNR) in the gray and RGB images.

	SNR					
	Cameraman Image			**Anchalee Image**		
n	**Our Alg.**	**Dao-Jun Wen**	**Kazmi et al.**	**Our Alg.**	**Dao-Jun Wen**	**Kazmi et al.**
1	26.2564	8.1116	4.0332	37.2865	9.5366	4.7484
50	30.1687	30.0091	29.2302	45.6664	45.1171	42.9992
100	30.7660	30.5809	29.8699	48.5931	47.7755	45.0396
200	31.1938	30.9517	30.3947	51.7846	50.5634	47.7038
300	31.2939	31.0011	30.5988	53.5650	51.9977	49.4282
400	31.2432	30.9072	30.6739	54.7291	52.8411	50.6422

6. Conclusions

In this article, the main contribution is to introduce a novel self-adaptive inertial Krasnoselski-mann iterative method for solving hierarchical fixed point and split monotone variational inclusion problems in Hilbert speaces. The main advantage of this scheme involves both the use of an inertial technique and the self-adaptive step size criterion which does not require prior knowledge of the Lipschitz constant of the cost operator. Under standard assumptions, the strong convergence of the proposed method is established. A modified scheme derived from the proposed method is given for solving hierarchical fixed points and the split problems. The application of the proposed method in image recovery and comparison with Dao-Jun Wen. Alg. [14] and Kazmi et al. Alg. [7] are presented.

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