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Analytical Investigation of the Existence of Solutions for a System of Nonlinear Hadamard-Type Integro-Differential Equations Based upon the Multivariate Mittag-Leffler Function

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Abstract: In this paper, the authors propose an investigation of the existence of solutions for a system of nonlinear Hadamard-type integro-differential equations in a Banach space. The result derived is new and based upon Babenko's approach, Leray-Schauder's nonlinear alternative, and the multivariate Mittag-Leffler function. Using an illustrative example, a demonstration of the application of the main theorem is also considered.

Keywords: Hadamard-type fractional integral; Leray-Schauder's alternative; Babenko's approach; multivariate Mittag-Leffler function

MSC: Primary 26A33; 34A08; 33E12; Secondary 34A12

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1. Introduction

Let $0 < a < b < \infty$ and $C[a,b]$ be the space given by

$$C[a,b] = \{u : [a,b] \to \mathbb{R} : \ u \text{ is continuous on [a, b] and } \|u\|_C = \max_{x\in[a,b]} |u(x)| < +\infty\}.$$

Clearly, $C[a,b]$ is a Banach space. Furthermore, the product space $C[a,b] \times C[a,b]$ (also a Banach space) is defined as

$$C[a,b] \times C[a,b] = \{(u,v) : \ u, v \in C[a,b]\},$$

with the norm given by

$$\|(u,v)\| = \|u\|_C + \|v\|_C.$$

The Hadamard-type fractional integral and derivative of order $\alpha > 0$ for a function u are defined in [1–4] (see also the recent developments on the subject of fractional calculus and its applications, which are reported in [5,6]) as follows:

$$\left(\mathcal{J}^{\alpha}_{a+,\mu} u\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{t}{x}\right)^{\mu} \left(\log\frac{x}{t}\right)^{\alpha-1} u(t)\frac{dt}{t}, \quad a < x < b$$

and

$$(\mathcal{D}^{\alpha}_{a+,\,\mu}\, u)(x) = x^{-\mu}\delta^n x^{\mu} (\mathcal{J}^{n-\alpha}_{a+,\,\mu}\, u)(x), \quad \delta = x\frac{d}{dx},$$

where $\log(\cdot) = \log_e(\cdot)$, $\mu \in \mathbb{R}$, $n = [\alpha] + 1$, and $[\alpha]$ is an integral part of α. In particular, we let

$$(\mathcal{J}^{\alpha}_{a+} u)(x) = (\mathcal{J}^{\alpha}_{a+,0} u)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1} u(t)\frac{dt}{t}.$$

There are many definitions of fractional derivatives available in the literature, such as the Riemann-Liouville derivative, which played an important role in the development of the theory of fractional analysis. However, the commonly used derivative is the Hadamard fractional derivative (with $\mu = 0$) given by Hadamard in [7]. Butzer et al. [8–10] studied various properties of the Hadamard-type derivative, which is more general than the familiar Hadamard fractional derivative.

For $u \in C[a,b]$, we have

$$\|\mathcal{J}_{a+}^{\alpha} u\|_C \leqq \frac{1}{\Gamma(\alpha+1)} \left(\log \frac{b}{a}\right)^{\alpha} \|u\|_C.$$

Indeed, we get

$$\begin{aligned} \|\mathcal{J}_{a+}^{\alpha} u\|_C &= \frac{1}{\Gamma(\alpha)} \max_{x \in [a,b]} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} u(t) \frac{dt}{t} \\ &\leqq \frac{1}{\Gamma(\alpha)} \int_a^b \left(\log \frac{b}{t}\right)^{\alpha-1} \frac{dt}{t} \|u\|_C \leqq \frac{1}{\Gamma(\alpha+1)} \left(\log \frac{b}{a}\right)^{\alpha} \|u\|_C. \end{aligned}$$

Let $X_\mu(a,b)$ be the space of those Lebesgue measurable functions u on $[a,b]$ for which $x^{\mu-1}u(x)$ is absolutely integrable [2]:

$$X_\mu(a,b) = \left\{ u : [a,b] \to C : \quad \|u\|_{X_\mu} = \int_a^b x^{\mu-1} |u(x)| dx < \infty \right\}.$$

Obviously, $C[a,b] \subset X_\mu(a,b)$. Then, it follows from Lemma 2.2 in [2] that the following semigroup property holds true:

$$\mathcal{J}_{a+}^{\alpha} \mathcal{J}_{a+}^{\beta} u = \mathcal{J}_{a+}^{\alpha+\beta} u,$$

for all α, $\beta > 0$, and $u \in C[a,b]$.

The goal of this paper is to study the existence of solutions for the following nonlinear integro-differential system involving the fractional Hadamard-type operators by using Leray-Schauder's alternative and the multivariate Mittag-Leffler function in the product space $C[a,b] \times C[a,b]$:

$$\begin{cases} u(x) + a_n (\mathcal{J}_{a+}^{\alpha_n} u)(x) + \cdots + a_1 (\mathcal{J}_{a+}^{\alpha_1} u)(x) = f_1(x, u(x), v(x)), \\ v(x) + b_n (\mathcal{J}_{a+}^{\beta_n} v)(x) + \cdots + b_1 (\mathcal{J}_{a+}^{\beta_1} v)(x) = f_2(x, u(x), v(x)), \end{cases} \tag{1}$$

where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 1$, $\beta_n > \beta_{n-1} > \cdots > \beta_1 > 1$, and the functions f_1 and f_2 are mappings from $[a,b] \times \mathbb{R}^2$ to $\mathbb{R}$ satisfying certain conditions. To the best of the authors' knowledge, this is a new development, and such an existence problem has presumably not been investigated before.

Babenko's approach [11] provides a powerful tool in solving differential and integral equations by treating bounded integral operators like variables. The method itself is similar to the Laplace transform method for the equations with constant coefficients, but it can be used to deal with integral or fractional differential equations with variable coefficients or generalized functions whose Laplace transforms do not exist in the classical sense [6,12,13]. In order to illustrate Babenko's approach in detail, we shall solve the following fractional integro-differential equation for $\alpha > 0$ and $f \in X_\mu(a,b)$ (see also [14]):

$$u(x) + \mathcal{J}_{a+}^{\alpha} u(x) = f(x).$$

Clearly, the above equation proves to be of the form:

$$(1 + \mathcal{J}_{a+}^{\alpha}) u(x) = f(x),$$

which is informally arrived at through Babenko's method,

$$u(x) = (1 + \mathcal{J}^{\alpha}_{a+})^{-1} f(x) = \sum_{k=0}^{\infty} (-1)^k \mathcal{J}^{\alpha k}_{a+} f(x),$$

where

$$(\mathcal{J}^{\alpha}_{a+})^k = \mathcal{J}^{\alpha k}_{a+},$$

by the semigroup property. It follows from Lemma 2.1 in [2] that

$$\begin{aligned} \|u\|_{X_\mu} &\leqq \sum_{k=0}^{\infty} \left\| \mathcal{J}^{\alpha k}_{a+} f(x) \right\|_{X_\mu} \leqq \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \left(\log \frac{b}{a} \right)^{\alpha k} \|f\|_{X_\mu} \\ &= E_{\alpha,1}\left(\log^{\alpha} \frac{b}{a} \right) \|f\|_{X_\mu} < \infty, \end{aligned}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

is the two-parameter Mittag-Leffler function (see, for details, [6]; see also a recent expository article [15]). Therefore, u is the solution of the integral equation and is well defined in the space $X_\mu(a,b)$.

Theorem 1 (Leray-Schauder's alternative [16])**.** *Consider the continuous and compact mapping T of a Banach space S into itself. The boundedness of*

$$\{x \in S : x = \lambda T x \textit{ for some } 0 \leqq \lambda \leqq 1\}$$

implies that T has a fixed point.

Leray-Schauder's alternative is a useful theorem for showing the existence of solutions to nonlinear fractional differential equations [17–24]. In the year 2004, Bai and Fang and Gao [25] considered the existence of a positive solution to the following singular coupled system using Leray-Schauder's alternative and Krasnoselskii's fixed point theorem in a cone:

$$\begin{cases} D^s u(t) = f(t, v(t)), & 0 < t < 1, \\ D^p v(t) = g(t, u(t)), & 0 < t < 1, \end{cases}$$

where $0 < s < 1$, $0 < p < 1, D^s$, D^p are two standard Riemann-Liouville fractional derivatives, f, g $: (0,1] \times [0, +\infty) \to [0, +\infty)$ are two given functions, and

$$\lim_{t \to 0^+} f(t, \cdot) = \lim_{t \to 0^+} g(t, \cdot) = +\infty.$$

In 2014, Ahmad and Ntouyas [26] studied the existence of solutions for a couple system of Hadamard-type fractional differential equations (also with $\mu = 0$) and integral boundary conditions based on Leray-Schauder's alternative. In addition, Toumi and EI Abidine [27] investigated the following nonlinear fractional differential problem on $\mathbb{R}^+ = (0, +\infty)$

$$\begin{cases} D^{\alpha} u(t) + f(t, u(t), D^p u(t)) = 0, & t \in \mathbb{R}^+, \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, \end{cases}$$

where $2 \leqq m \in \mathbb{N}, m-1 < \alpha \leqq m, 0 < p \leqq \alpha - 1$, and f ia a Borel measurable function in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ satisfying certain conditions. They showed the existence of multiple unbounded positive solutions by Schauder's fixed point theorem, which is a special case of Leray-Schauder's alternative.

Recently, Ding et al. [28] applied the fixed-point index and non-negative matrices to study the existence of positive solutions for a system of Hadamard-type fractional differential equations with semipositone nonlinearities.

We assume that the functions $f_1(x, z_1, z_2)$ and $f_2(x, z_1, z_2)$ satisfy the Lipschitz conditions in the second and third variables. Then, the uniqueness of a system for the nonlinear Hadamard-type integro-differential equations, with all $\mu \in \mathbb{R}$ and positive orders, in the Banach space $X_\mu(a, b) \times X_\mu(a, b)$, was studied very recently by Li in [29] by using Banach's fixed point theorem.

The multivariate Mittag-Leffler function was initially given by Hadid and Luchko [30] for solving linear fractional differential equations with constant coefficients:

$$E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1+\cdots+\alpha_m k_m+\beta)},$$

where $\alpha_i, \beta > 0$ for $i = 1, 2, \cdots, m$.

2. Main Results

In this section, we shall present our main theorem dealing with the existence of solutions to the nonlinear system (1) by Babenko's approach, Leray-Schauder's alternative, and the multivariate Mittag-Leffler function.

Theorem 2. *Assume that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 1$, $\beta_n > \beta_{n-1} > \cdots > \beta_1 > 1$, and the functions $f_1(x, z_1, z_2)$ and $f_2(x, z_1, z_2)$ are continuous mappings from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$ satisfying the following conditions for non-negative constants C_0, C_1 and C_2 :*

$$|f_1(x, y_1, y_2)| \leqq C_0 + C_1|y_1| + C_2|y_2|$$

and

$$|f_2(x, y_1, y_2)| \leqq C_0 + C_1|y_1| + C_2|y_2|.$$

In addition, suppose that f'_{1x} and f'_{2x} are bounded and

$$\max\{C_1, C_2\}(E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) + E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right)) < 1.$$

Then, there exists a solution to the system (1) in the space $C[a, b] \times C[a, b]$.

Proof. Let $f \in C[a, b]$ with $0 < a < b < \infty$. Then, the following equation

$$u(x) + a_n(\mathcal{J}_{a+}^{\alpha_n}u)(x) + \cdots + a_1(\mathcal{J}_{a+}^{\alpha_1}u)(x) = f(x),$$

has a unique and global solution in the space $C[a, b]$ by Babenko's approach and the semigroup property

$$u(x) = \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \cdots, k_n} a_n^{k_n}\cdots a_1^{k_1}(\mathcal{J}_{a+}^{k_n\alpha_n+\cdots+k_1\alpha_1}f)(x),$$

where we define

$$\mathcal{J}_{a+}^{0}f(x) = f(x).$$

Indeed,

$$\begin{aligned}
\|u\|_C &\leqq \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \cdots, k_n} |a_n|^{k_n} \cdots |a_1|^{k_1} \left\| \mathcal{J}_{a+}^{k_n\alpha_n+\cdots+k_1\alpha_1} f \right\|_C \\
&\leqq \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \cdots, k_n} |a_n|^{k_n} \cdots |a_1|^{k_1} \\
&\quad \cdot \frac{1}{\Gamma(k_n\alpha_n+\cdots+k_1\alpha_1+1)} \left(\log \frac{b}{a}\right)^{k_n\alpha_n+\cdots+k_1\alpha_1} \|f\|_C \\
&= E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1}, \cdots, |a_n|\left(\log\frac{b}{a}\right)^{\alpha_n} \right) \|f\|_C < +\infty. \qquad (2)
\end{aligned}$$

This claims that the series is uniformly convergent on $[a,b]$, and hence $u(x)$ is continuous.

Let $(u,\ v) \in C[a,] \times C[a,b]$. Define a mapping T on the space $C[a,b] \times C[a,b]$ as

$$T(u,v) = (T_1(u,v),\ T_2(u,v)),$$

where

$$\begin{aligned}
T_1(u,v) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \cdots, k_n} a_n^{k_n} \cdots a_1^{k_1} \\
\cdot (\mathcal{J}_{a+}^{k_n\alpha_n+\cdots+k_1\alpha_1} f_1)(x, u(x), v(x)),
\end{aligned}$$

and

$$\begin{aligned}
T_2(u,v) = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \cdots, k_n} b_n^{k_n} \cdots b_1^{k_1} \\
\cdot (\mathcal{J}_{a+}^{k_n\beta_n+\cdots+k_1\beta_1} f_2)(x, u(x), v(x)),
\end{aligned}$$

It follows from the inequality (2) that

$$\begin{aligned}
\|T_1(u,v)\|_C &\leqq E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1}, \cdots, |a_n|\left(\log\frac{b}{a}\right)^{\alpha_n} \right) \|f_1(x,u,v)\|_C \\
&\leqq E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1}, \cdots, |a_n|\left(\log\frac{b}{a}\right)^{\alpha_n} \right) \\
&\quad \cdot (C_0 + C_1\|u\|_C + C_2\|v\|_C), \\
\|T_2(u,v)\|_C &\leqq E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1}, \cdots, |b_n|\left(\log\frac{b}{a}\right)^{\beta_n} \right) \|f_2(x,u,v)\|_C \\
&\leqq E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1}, \cdots, |b_n|\left(\log\frac{b}{a}\right)^{\beta_n} \right) \\
&\quad \cdot (C_0 + C_1\|u\|_C + C_2\|v\|_C).
\end{aligned}$$

Therefore, T is a continuous mapping from the space $C[a,b] \times C[a,b]$ to itself, since f_1 and f_2 are continuous.

Suppose that B is a proper bounded subset of $C[a,b] \times C[a,b]$; then, we can find constants $W_1,\ W_2 > 0$ such that

$$\|f_1(x,u,v)\|_C \leqq W_1, \quad \|f_2(x,u,v)\|_C \leqq W_2,$$

for all $(u, v) \in B$, which deduces that

$$\|T_1(u,v)\|_C \leqq W_1 E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right), \text{ and}$$
$$\|T_2(u,v)\|_Y \leqq W_2 E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right).$$

Thus, TB is uniformly bounded in the space $C[a,b] \times C[a,b]$. We need to show that T is equicontinuous on $C[a,b] \times C[a,b]$. Letting $\tau_1, \tau_2 \in [a,b]$ with $\tau_1 < \tau_2$, we come to

$$\begin{aligned}|T_1(u,v)(\tau_2) - T_1(u,v)(\tau_1)| &\leqq |f_1(\tau_2,u(\tau_2),v(\tau_2)) - f_1(\tau_1,u(\tau_1),v(\tau_1))| \\ &+ \sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|a_1|^{k_1}\cdots|a_n|^{k_n}\frac{1}{\Gamma(\lambda_1)} \\ &\cdot\left|\int_a^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t} - \int_a^{\tau_1}\left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t}\right|,\end{aligned}$$

where

$$\lambda_1 = k_1\alpha_1 + \cdots + k_n\alpha_n \geqq \alpha_1 > 1,$$

for $k = k_1 + \cdots + k_n \geqq 1$.

Since f'_{1x} is bounded, there is a constant $M_1 > 0$ such that

$$|f_1(\tau_2,u(\tau_2),v(\tau_2)) - f_1(\tau_1,u(\tau_1),v(\tau_1))| \leqq M_1(\tau_2 - \tau_1),$$

by the mean value theorem.

Furthermore,

$$\begin{aligned}\int_a^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t} &= \int_a^{\tau_1}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t} \\ &+ \int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|\int_a^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t} - \int_a^{\tau_1}\left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1} f_1(t,u(t),v(t))\frac{dt}{t}\right| \\ &\leqq \left|\int_a^{\tau_1}\left(\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1} - \left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1}\right) f_1(t,u(t),v(t))\frac{dt}{t}\right| \\ &\quad + \int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1}|f_1(t,u(t),v(t))|\frac{dt}{t}.\end{aligned}$$

Obviously,

$$\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1}|f_1(t,u(t),v(t))|\frac{dt}{t} \leqq W_1\frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{\lambda_1}}{\lambda_1},$$

and

$$\begin{aligned}
&\left|\int_a^{\tau_1}\left(\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1}-\left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1}\right)f_1(t,u(t),v(t))\frac{dt}{t}\right|\\
&\leqq W_1\int_a^{\tau_1}\left(\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1}-\left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1}\right)\frac{dt}{t}\\
&\leqq W_1\left(\frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{\lambda_1}}{\lambda_1}+\frac{\left(\log\frac{\tau_2}{a}\right)^{\lambda_1}}{\lambda_1}-\frac{\left(\log\frac{\tau_1}{a}\right)^{\lambda_1}}{\lambda_1}\right).
\end{aligned}$$

Again, by the mean value theorem, we deduce that

$$0<\frac{\left(\log\frac{\tau_2}{a}\right)^{\lambda_1}}{\lambda_1}-\frac{\left(\log\frac{\tau_1}{a}\right)^{\lambda_1}}{\lambda_1}\leqq(\tau_2-\tau_1)\left(\log\frac{b}{a}\right)^{\lambda_1-1}.$$

Hence, we have

$$\begin{aligned}
&\left|\int_a^{\tau_1}\left(\left(\log\frac{\tau_2}{t}\right)^{\lambda_1-1}-\left(\log\frac{\tau_1}{t}\right)^{\lambda_1-1}\right)f_1(t,u(t),v(t))\frac{dt}{t}\right|\\
&\leqq W_1\frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{\lambda_1}}{\lambda_1}+W_1(\tau_2-\tau_1)\left(\log\frac{b}{a}\right)^{\lambda_1-1}.
\end{aligned}$$

In summary, therefore, we find that

$$\begin{aligned}
&|T_1(u,v)(\tau_2)-T_1(u,v)(\tau_1)|\leqq M_1(\tau_2-\tau_1)\\
&+\ 2W_1\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|a_1|^{k_1}\cdots|a_n|^{k_n}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)}\\
&+W_1\ (\tau_2-\tau_1)\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|a_1|^{k_1}\cdots|a_n|^{k_n}\\
&\cdot\frac{\left(\log\frac{b}{a}\right)^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n)}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&2W_1\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|a_1|^{k_1}\cdots|a_n|^{k_n}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)}\\
&\leqq 2W_1\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|a_1|^{k_1}\cdots|a_n|^{k_n}\frac{\left(\log\frac{b}{a}\right)^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)},
\end{aligned}$$

which implies that the series of the left-hand side is uniformly convergent on $[a,b]$, and every term in the series has the factor $\left(\log\frac{\tau_2}{\tau_1}\right)^{\alpha_1}$. Therefore, T_1 is equicontinuous on $C[a,b]$.

Regarding T_2, we let M_2 be a constant, such that

$$|f'_{2x}|\leqq M_2.$$

Then, it follows from a similar step that

$$
\begin{aligned}
&|T_2(u,v)(\tau_2) - T_2(u,v)(\tau_1)| \leqq M_2(\tau_2-\tau_1) \\
&+ 2W_2 \sum_{k=1}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} |b_1|^{k_1}\cdots|b_n|^{k_n} \frac{\left(\log\frac{\tau_2}{\tau_1}\right)^{k_1\beta_1+\cdots+k_n\beta_n}}{\Gamma(k_1\beta_1+\cdots+k_n\beta_n+1)} \\
&+ W_2\,(\tau_2-\tau_1) \sum_{k=1}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} |b_1|^{k_1}\cdots|b_n|^{k_n} \\
&\cdot \frac{\left(\log\frac{b}{a}\right)^{k_1\beta_1+\cdots+k_n\beta_n}}{\Gamma(k_1\beta_1+\cdots+k_n\beta_n)}.
\end{aligned}
$$

So, clearly, T_2 is also equicontinuous on $C[a,b]$. This further infers that T is a compact mapping by the Arzela-Ascoli theorem. It remains to be proven that the set

$$W = \{(u,v) \in C[a,b] \times C[a,b] : (u,v) = \lambda T(u,v) \text{ for some } 0 \leqq \lambda \leqq 1\}$$

is bounded.

For any $x \in [a,b]$,

$$u(x) = \lambda T_1(u,v)(x), \quad v(x) = \lambda T_2(u,v)(x).$$

From Inequality (2), we have

$$
\begin{aligned}
\|u\|_C \leqq \|T_1(u,v)\|_C \leqq E_{(\alpha_1,\cdots,\alpha_n,\,1)}&\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) \\
&\cdot (C_0 + C_1\|u\|_C + C_2\|v\|_C), \quad \text{and} \\
\|v\|_C \leqq \|T_2(u,v)\|_C \leqq E_{(\beta_1,\cdots,\beta_n,\,1)}&\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right) \\
&\cdot (C_0 + C_1\|u\|_C + C_2\|v\|_C).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&A_1\|u\|_C + A_2\|v\|_C \leqq \\
&\quad \leqq C_0 E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) \\
&\qquad + C_0 E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 = 1 - C_1(&E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) \\
&+ E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right)) > 0, \quad \text{and} \\
A_2 = 1 - C_2(&E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1},\cdots,|a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) \\
&+ E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1},\cdots,|b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right)) > 0,
\end{aligned}
$$

by our hypothesis.

Let

$$A_0 = \min\{A_1, A_2\} > 0.$$

Then, we have

$$A_0(\|u\|_C + \|v\|_C) \leqq A_1\|u\|_C + A_2\|v\|_C$$

and

$$\begin{aligned} &\|u\|_C + \|v\|_C \\ &\quad \leqq \frac{C_0}{A_0} E_{(\alpha_1,\cdots,\alpha_n,\,1)}\left(|a_1|\left(\log\frac{b}{a}\right)^{\alpha_1}, \cdots, |a_n|\left(\log\frac{b}{a}\right)^{\alpha_n}\right) \\ &\qquad + \frac{C_0}{A_0} E_{(\beta_1,\cdots,\beta_n,\,1)}\left(|b_1|\left(\log\frac{b}{a}\right)^{\beta_1}, \cdots, |b_n|\left(\log\frac{b}{a}\right)^{\beta_n}\right). \end{aligned}$$

Hence, W is bounded for all $\lambda \in [0, 1]$. Using Leray-Schauder's alternative, we imply that system (1) has a solution in the space $C[a, b] \times C[a, b]$. □

Remark 1. *From Theorem 2, we can derive that, if f_1, f'_{1x}, f_2 and f'_{2x} are continuous and bounded (that is, $C_1 = C_2 = 0$), then the system (1) has a solution in the space $C[a, b] \times C[a, b]$.*

Example 1. *As an illustrative example, the following nonlinear Hadamard-type integro-differential system with all integral orders bigger than 1 and arbitrary coefficients a_1, a_2, b_1, and b_2*

$$\begin{cases} u(x) + a_2\ \mathcal{J}^{2.7}_{a+}u(x) + a_1\ \mathcal{J}^{2.1}_{a+}u(x) = \cos(x + u(x) + v(x)), \\ v(x) + b_2\ \mathcal{J}^{2.1}_{a+}v(x) + b_1\ \mathcal{J}^{1.1}_{a+}v(x) = \sin^2(x + u(x)), \end{cases} \tag{3}$$

has a solution in the space $C[a, b] \times C[a, b]$ ($0 < a < b < +\infty$), since

$$\begin{aligned} &f_1(x, y_1, y_2) = \cos(z + y_1 + y_2) \ \ \textit{and} \\ &f_2(x, y_1, y_2) = \sin^2(x + y_1), \end{aligned}$$

are continuous and bounded with their partial derivatives with respect to x, by noting that

$$|f'_{1x}(x, y_1, y_2)| \leqq 1, \quad |f'_{2x}(x, y_1, y_2)| \leqq 2.$$

Thus, $C_0 = 2$, and $C_1 = C_2 = 0$ in Theorem 2. By Remark 1, the system (3) has a solution in the space $C[a, b] \times C[a, b]$.

3. Conclusions

Using Babenko's approach, Leray-Schauder's alternative, and the multivariate Mittag-Leffler function, we have studied the existence of solutions to the nonlinear Hadamard-type integro-differential system (1), which is new. The results obtained are fresh and interesting. We have also included an example showing the application of the main theorem.

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