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Inflection Points in Cubic Structures

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Abstract: In this paper, we introduce and study new geometric concepts in a general cubic structure. We define the concept of the inflection point in a general cubic structure and investigate relationships between inflection points and associated and corresponding points in a general cubic structure.

Keywords: cubic structure; inflection point; TSM-quasigroup; corresponding points; associated points; tangential of a point

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1. Introduction and Motivation

When studying various third- and fourth-order curves and some other geometric problems, the authors have often encountered abstract geometric structures, which seemed worth studying. In [1], we named these cubic structures. In the same paper, numerous examples of these structures are given, and the connection of these geometric structures with algebraic structures are investigated. Additionally, the connection between cubic structures and totally symmetric medial quasigroups, as well as commutative groups, was thoroughly studied. Some simple properties of cubic structures were also proven.

Let Q be a nonempty set, whose elements are called *points*, and let $[\ ] \subseteq Q^3$ be a ternary relation on Q. Such a relation and the ordered pair $(Q, [\ ])$ is called a *cubic relation* and a *cubic structure*, respectively, if the following conditions are fulfilled:

C1. For any two points $a, b \in Q$, there is a unique point $c \in Q$ such that $[a, b, c]$, i.e., $(a, b, c) \in [\ ]$.

C2. The relation $[\ ]$ is totally symmetric, i.e., $[a, b, c]$ implies $[a, c, b]$, $[b, a, c]$, $[b, c, a]$, $[c, a, b]$, and $[c, b, a]$.

C3. $[a, b, c]$, $[d, e, f]$, $[g, h, i]$, $[a, d, g]$, and $[b, e, h]$ imply $[c, f, i]$, which can be clearly written in the form shown in the following table:

$$\begin{array}{cc|c|} \cline{3-3} a & b & c \\ d & e & f \\ g & h & i \\ \cline{3-3} \end{array}$$

Throughout the paper, we use the property C2 without explicitly mentioning it.

Given a nonempty set Q and a binary operation $\cdot$ on Q, the pair $(Q, \cdot)$ is called a *quasigroup* if, for each $a, b \in Q$, unique elements x and y exist, such that $ax = b$ and $ya = b$. (From now on, whenever there is no risk of confusion, the product is simply denoted by a juxtaposition.) A quasigroup $(Q, \cdot)$ in which the identity $(ab) \cdot (cd) = (ac) \cdot (bd)$ is valid is called *medial*, and it is *totally symmetric* if it satisfies the identities $(ab) \cdot b = a$, $a \cdot (ab) = b$. A totally symmetric medial quasigroup is called TSM-quasigroup for short.

One can prove that if the equivalence

$$[a, b, c] \Leftrightarrow ab = c$$

links the ternary relation [ ] and the binary operation $\cdot$, then $(Q, [\ ])$ is a cubic structure if and only if it is a TSM-quasigroup [1] (Th. 1). The properties of TSM-quasigroups were thoroughly studied in [2].

Some geometric examples of cubic structures were considered in [1], the most important of which is perhaps [1] (Example 2.1). Let Γ be a planar cubic curve, $Q \subseteq \Gamma$, the set of all nonsingular points on Γ, and let $[a, b, c]$ mean that the points a, b, $c \in Q$ lie on the same line. Then, one can prove that $(Q, [\ ])$ is a cubic structure. In this paper, some well-known relationships that are valid on a cubic curve motivate the introduction of new concepts in a general cubic structure.

Two concepts in cubic structures are defined in [3]. The point a' is the *tangential* of point a if the statement $[a, a, a']$ holds. Each point has one and only one tangential. If point a' is the tangential of point a, then we can also say that point a is an *antecedent* of point a'. If a' is the tangential of point a and a'' is the tangential of point a', then we can say that a'' is the *second tangential* of point a. Two points are said to be *corresponding* if they have a common tangential. If the maximum number of mutually corresponding points is finite, then it is of the form 2^m for some fixed number $m \in \mathbb{N} \cup \{0\}$.

In such a case, we can say that the distinct points $a_1, \ldots, a_n$, $n \leq 2^m$ with the common tangential are *associated*. The number m is called the *rank* of the observed cubic structure $(Q, [\ ])$.

2. Inflection Points

We say that point a in a cubic structure is an *inflection point*, so the statement $[a, a, a]$ holds, i.e., if that point is self-tangential.

Lemma 1. *If points a and b are inflection points and if the statement $[a, b, c]$ holds, then point c is also an inflection point.*

Proof. The proof follows by applying the table

$$\begin{array}{cc|c|} a & b & c \\ a & b & c \\ a & b & c \end{array}. \quad \square$$

Example 1. *For a more visual representation of Lemma 1, consider the TSM-quasigroup given by the Cayley table*

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Lemma 2. *If inflection point a is the tangential point of point b, then a and b are corresponding points.*

Proof. Point a is the common tangential of points a and b. $\square$

Example 2. *For a more visual representation of Lemma 2, consider the TSM-quasigroup given by the Cayley table*

	a	b	c	d
a	a	b	d	c
b	b	a	c	d
c	d	c	b	a
d	c	d	a	b

Proposition 1. *If a' and b' are the tangentials of points a and b, respectively, and if c is an inflection point, then $[a, b, c]$ implies $[a', b', c]$.*

Proof. According to [3] (Th. 2.1), $[a,b,c]$ implies $[a',b',c']$, where c' is the tangential of c. However, in our case $c'=c$. $\square$

Lemma 3. *If a' and b' are the tangentials of points a and b respectively, and if $[a,b,c]$ and $[a',b',c]$, then c is an inflection point.*

Proof. The statement is followed by applying the table

$$\begin{array}{cc|c|} \cline{3-3} a & b & c \\ a & b & c \\ a' & b' & c \\ \cline{3-3} \end{array}. \quad \square$$

Example 3. *For a more visual representation of Proposition 1 and Lemma 3, consider the TSM-quasigroup given by the Cayley table*

	a	b	c	d	e
a	d	c	b	a	e
b	c	e	a	d	b
c	b	a	c	e	d
d	a	d	e	b	c
e	e	b	d	c	a

Lemma 4. *If a' and b' are the tangentials of points a and b, respectively, and if c is an inflection point, then $[a,b,d]$ and $[a',b',c]$ imply that c and d are corresponding points.*

Proof. From the table

$$\begin{array}{cc|c|} \cline{3-3} a & b & d \\ a & b & d \\ a' & b' & c \\ \cline{3-3} \end{array}$$

it follows that point d has the tangential c, which itself is self-tangential. $\square$

Example 4. *For a more visual representation of Lemma 4, consider the TSM-quasigroup given by the Cayley table*

	a	b	c	d	e	f	g	h
a	e	d	g	b	a	h	c	f
b	d	f	h	a	g	b	e	c
c	g	h	c	d	f	e	a	b
d	b	a	d	c	e	f	h	g
e	a	g	f	e	d	c	b	h
f	h	b	e	f	c	d	g	a
g	c	e	a	h	b	g	f	d
h	f	c	b	g	h	a	d	e

Lemma 5. *If the corresponding points a_1, a_2, and their common second tangential a'' satisfy $[a_1,a_2,a'']$, then a'' is an inflection point.*

Proof. The statement follows on from the table

$$\begin{array}{cc|c|} \cline{3-3} a_1 & a_2 & a'' \\ a_1 & a_2 & a'' \\ a' & a' & a'' \\ \cline{3-3} \end{array}$$

where a' is the common tangential of points a_1 and a_2. $\square$

Example 5. *For a more visual representation of Lemma 5, consider the TSM-quasigroup given by the Cayley table*

	a_1	a_2	a_3	a_4
a_1	a_3	a_4	a_1	a_2
a_2	a_4	a_3	a_2	a_1
a_3	a_1	a_2	a_4	a_3
a_4	a_2	a_1	a_3	a_4

Lemma 6. *Let a_1, a_2, and a_3 be pairwise corresponding points with the common tangential a', such that $[a_1, a_2, a_3]$. Then, a' is an inflection point.*

Proof. The proof follows from the table

$$\begin{array}{cc|c|} \cline{3-3} a_1 & a_1 & a' \\ a_2 & a_2 & a' \\ a_3 & a_3 & a' \\ \cline{3-3} \end{array}\,. \qquad \square$$

Example 6. *For a more visual representation of Lemma 6, consider the TSM-quasigroup given by the Cayley table*

	a_1	a_2	a_3	a_4
a_1	a_4	a_3	a_2	a_1
a_2	a_3	a_4	a_1	a_2
a_3	a_2	a_1	a_4	a_3
a_4	a_1	a_2	a_3	a_4

Corollary 1. *Let a_1, a_2, and a_3 be pairwise corresponding points with the common tangential a', which is not an inflection point. Then, $[a_1, a_2, a_3]$ does not hold.*

Lemma 7. *Let $[b, c, d]$, $[a, b, e]$, $[a, c, f]$, and $[a, d, g]$. Point a is an inflection point if and only if $[e, f, g]$.*

Proof. Each of the *if* and *only if* statements follow on from one of the respective tables:

$$\begin{array}{cc|c|} \cline{3-3} b & e & a \\ c & f & a \\ d & g & a \\ \cline{3-3} \end{array} \qquad \begin{array}{cc|c|} \cline{3-3} a & b & e \\ a & c & f \\ a & d & g \\ \cline{3-3} \end{array}\,. \qquad \square$$

Example 7. *For a more visual representation of Lemma 7, consider the TSM-quasigroup given by the Cayley table*

	a	b	c	d	e	f	g
a	a	e	f	g	b	c	d
b	e	f	d	c	a	b	g
c	f	d	g	b	e	a	c
d	g	c	b	e	d	f	a
e	b	a	e	d	c	g	f
f	c	b	a	f	g	d	e
g	d	g	c	a	f	e	b

3. Inflection Points in Cubic Structures of Rank 2

Let $(Q, [\ ])$ be a cubic structure of rank 2, i.e., associated points form quadruples.

Lemma 8. *Let inflection point a' be the common tangential of distinct points a_1 and a_2, and let a_3 be a point such that $[a_1, a_2, a_3]$. Then, a' is also the tangential of point a_3, i.e., a', a_1, a_2, and a_3 are associated points.*

Proof. The proof follows by applying the table

$$\begin{array}{cc|c|} a' & a' & a' \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{array}\,. \quad \square$$

Proposition 2. *Let a' be the common tangential of points a_1, a_2, and a_3, and let these four points be distinct. If a' is an inflection point, then $[a_1, a_2, a_3]$.*

Proof. Let b be a point such that $[a_1, a_2, b]$. By Lemma 8, points a', a_1, a_2, and b are associated, and $b = a_3$. $\square$

Theorem 1. *Let a_1, a_2, a_3, and a_4 be associated points, and let $[a_1, a_2, a_3]$. Then, a_4 is an inflection point and it is also the common tangential of points a_1, a_2, and a_3.*

Proof. Let a' be the common tangential of points a_1, a_2, a_3, and a_4. By Lemma 6, a' is an inflection point, i.e., the common tangential of points a_1, a_2, a_3, a_4, and a'. Therefore, point a' is actually one of points a_1, a_2, a_3, or a_4. If $a' = a_1$, then a_1 would be an inflection point and the common tangential of points a_2, a_3, and a_4, and by Proposition 2, it follows that $[a_2, a_3, a_4]$, which is, by C1, impossible because $[a_1, a_2, a_3]$ holds. In the same way, we get contradictions by assuming $a' = a_2$ or $a' = a_3$. Therefore, $a' = a_4$. $\square$

For a more visual representation of Lemma 8, Proposition 2, and Theorem 1 consider the TSM-quasigroup in Example 6.

In [3] (Th. 4.3), we proved the following: If a_1, a_2, a_3, and a_4 are associated points with the common tangential a', then points p, q, and r exist such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, $[a_1, a_3, q]$, $[a_2, a_4, q]$, $[a_1, a_4, r]$ and $[a_2, a_3, r]$, and points a', p, q, and r are associated.

Theorem 2. *Let a_1, a_2, a_3, and a_4 be associated points with the first and second tangentials a' and a'', where $a' \neq a''$. If a'' is an inflection point, then it is one of points p, q, or r, such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, $[a_1, a_3, q]$, $[a_2, a_4, q]$, $[a_1, a_4, r]$, and $[a_2, a_3, r]$. If, e.g., $a'' = r$, then $[a', p, q]$.*

Proof. The points a', p, q, and r are associated, and their common tangential is the tangential a'' of point a'. Point a'' is self-tangential. Because of the rank 2, there are only four different associated points, and since $a'' \neq a'$, point a'' must be equal to one of points p, q, or r. Let, e.g., $a'' = r$. Since a'' is an inflection point and also the tangential of points a', p, and q, it follows from Proposition 2 that $[a', p, q]$. $\square$

Example 8. *For a more visual representation of Theorem 2, consider the TSM-quasigroup given by the Cayley table*

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_5	a_6	a_7	a_8	a_1	a_2	a_3	a_4
a_2	a_6	a_5	a_8	a_7	a_2	a_1	a_4	a_3
a_3	a_7	a_8	a_5	a_6	a_3	a_4	a_1	a_2
a_4	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1
a_5	a_1	a_2	a_3	a_4	a_8	a_7	a_6	a_5
a_6	a_2	a_1	a_4	a_3	a_7	a_8	a_5	a_6
a_7	a_3	a_4	a_1	a_2	a_6	a_5	a_8	a_7
a_8	a_4	a_3	a_2	a_1	a_5	a_6	a_7	a_8

4. Conclusions

Various concepts, which appear in any cubic structure, and relations between them, are introduced and studied in [3] and in this paper. In the future, the authors intend to use cubic structures to study the properties of some types of configurations (see [4–7]) among which are, for example, Steiner's triplets.

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