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Convergence Criteria of Three Step Schemes for Solving Equations

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Abstract: We develop a unified convergence analysis of three-step iterative schemes for solving nonlinear Banach space valued equations. The local convergence order has been shown before to be five on the finite dimensional Euclidean space assuming Taylor expansions and the existence of the sixth derivative not on these schemes. So, the usage of them is restricted six or higher differentiable mappings. But in our paper only the first Frèchet derivative is utilized to show convergence. Consequently, the scheme is expanded. Numerical applications are also given to test convergence.

Keywords: iterative schemes; convergence criterion; Banach space



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1. Introduction

We study the problem of finding a simple solution x_* of

$$F(x) = 0, \quad (1)$$

provided $F : D \subset E \rightarrow E_1$ is an operator acting between Banach spaces E and E_1 with $D \neq \emptyset$. Since a closed form solution is not possible in general, iterative schemes are used for solving (1). Many iterative schemes are studied for approximating x_* . In this paper, we consider the iterative schemes, developed for each $n = 0, 1, 2, \dots$, as

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - A_n F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - B_n F'(x_n)^{-1}F(z_n), \end{aligned} \quad (2)$$

where $A_n = A(x_n, y_n)$, $A : D \times D \rightarrow L(E, E_1)$, $A^{-1} \in L(E_1, E)$, $B_n = B(x_n, y_n)$, $B : D \times D \rightarrow L(E, E_1)$ and $B^{-1} \in L(E_1, E)$.

A plethora of schemes are special cases of (2).

Newton's Scheme [1–23]: Set $A_n = B_n = O$ to obtain

$$y_n = x_n - F'(x_n)^{-1}F(x_n).$$

This scheme is of order two.

Traub's Scheme [21,24–30]: Set $A_n = B_n = I$ to obtain

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n). \end{aligned}$$

This scheme is of order five.

Ostrowski Scheme [20]: Set $A_n = (2[y_n, x_n; F] - F'(x_n))F'(x_n)$ and $B_n = O$ to obtain

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - (2[y_n, x_n; F] - F'(x_n))^{-1}F(y_n). \end{aligned}$$

This scheme is of order four.

Ostrowski-Type Scheme [20]: Set $A_n = ([y_n, x_n; F]^{-1} - F'(x_n)^{-1})F'(x_n)$ and $B_n = O$ to obtain

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - ([y_n, x_n; F]^{-1} - F'(x_n)^{-1})F(y_n). \end{aligned}$$

This scheme is of order four.

Traub, Newton-type, Runge–Kutta and other three step schemes can be seen as special cases of (2) [7,14,20,30]. The local convergence of these schemes and their stability has been given in the aforementioned references using derivatives of order one more than the convergence order. As an example, the existence of the fifth derivative is required for the convergence of the Ostrowski scheme. Notice that convergence can be obtained without the fifth derivative, which does not appear on the scheme. Clearly there is a need to study the convergence of these schemes using a uniform set of criteria. That is why we developed scheme (2).

Let $E = E_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Consider λ on D by

$$\lambda(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then, we get $x_* = 1$, and

$$\lambda'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

But, $\lambda'''(x)$ is unbounded on D . So, the convergence of scheme (2) is not assured by the previous analyses. In order to extend the utilization of these schemes, we study the local convergence of scheme (2): in a Banach space, under generalized continuity conditions and using hypotheses only on the operators appearing on these schemes. Notice also that the convergence of most high convergence order schemes was given only on the finite dimensional Euclidean space with no error bounds on $\|x_n - x_*\|$ or information about the uniqueness of the solution. However, we give such results in this paper. Moreover, we present the semi-local convergence of these schemes not given before using majorizing sequences. The technique can apply on other single, multipoint, and multistep schemes along the same lines, since it is so general [8,9,16,27–29].

In Sections 2–4, we present results on majorizing sequences, semi-local and local convergence, respectively. The applications are conclusions can be found in Sections 4 and 5, respectively.

The analysis is provided in the next three sections followed by the examples and conclusions in the last two sections.

2. Convergence for Majorizing Sequence

We shall use the following definition of a majorizing sequence.

Definition 1. Let $\{\bar{v}_n\}$ be a sequence in E . Then, a sequence $\{v_n\}$ majorizes $\{\bar{v}_n\}$ if

$$\|\bar{v}_{n+1} - \bar{v}_n\| \leq v_{n+1} - v_n \text{ for each } n = 0, 1, 2, \dots$$

In this case, the study of the convergence for sequence $\{\bar{v}_n\}$ reduces to that of $\{v_n\}$ [21].

Next, we shall define sequences to be shown to be majorizing for scheme (2). Set $T = [0, \infty)$.

Let $a : T \times T \rightarrow T$, $\bar{a} : T \times T \rightarrow T$, $b : T \times T \times T \rightarrow T$ and $c : T \times T \times T \times T \rightarrow T$ be continuous and nondecreasing functions. We denote $a(n) = a_n, b(n) = b_n$ and $c(n) = c_n$. Let $\psi_0 : T \rightarrow T$ and $\psi : T_0 \subseteq T \rightarrow T$ be continuous and nondecreasing functions for some subset T_0 of T . Let also $\eta \geq 0$. Define scalar sequences $\{t_n\}, \{s_n\}, \{u_n\}$ by

$$\begin{aligned} t_0 &= 0, t_1 = \eta, \\ u_n &= s_n + \bar{a}_n(s_n - t_n), \\ t_{n+1} &= u_n + b_n(u_n - s_n), \end{aligned} \tag{3}$$

$$s_{n+1} = t_{n+1} + \frac{c_n(t_{n+1} - u_n)}{1 - \psi_0(t_{n+1})}, \tag{4}$$

where $\bar{\psi} = \begin{cases} \psi_0, & n = 0 \\ \psi, & n = 1, 2, \dots \end{cases}$.

Next, we present two general results on the convergence of majorant sequence (3).

Lemma 1. *Suppose that there exists $d > 0$ such that for each $n = 0, 1, 2, \dots$*

$$t_n \leq d \tag{5}$$

and

$$\psi_0(d) < 1. \tag{6}$$

Then, sequences $\{t_n\}, \{s_n\}, \{u_n\}$ converge to their unique least upper bound $t_* \in [0, d]$.

Proof. By the definition of these sequences, it follows that they are nondecreasing bounded from above by d , with $t_k \leq s_k \leq u_k \leq t_{k+1}$, and as such they converge to t_* . \square

Lemma 2. *If function ψ_0 is increasing, then conditions (5) and (6) can be replaced by*

$$t_n \leq \psi_0^{-1}(1). \tag{7}$$

Proof. Set $d = \psi_0^{-1}(1)$ in Lemma 1. \square

Remark 1. *Convergence criteria (5)–(7) can be verified only in some special cases, since they are very general. That is why we also provide stronger conditions, but easier to be verify.*

Define functions $f_i, i = 1, 2, 3$ on the interval $[0, 1)$ by

$$f_1(t) = a(t) \int_0^1 \psi(\theta t^2 \eta) d\theta - t$$

$$f_2(t) = b(t) - t,$$

and

$$f_3(t) = c(t) + t\psi_0\left(\frac{\eta}{1-t}\right) - t.$$

Suppose that these functions have smallest zeros $\mu_i \in (0, 1)$, respectively. Set

$$\mu = \min\{\mu_i\} \text{ and } \mu_0 = \max\{\bar{a}_0, b_0, \frac{c_0}{1 - \psi_0(t_1)}\}. \tag{8}$$

Then, we can show the third convergence result.

Lemma 3. *Suppose that*

$$0 \leq \mu_0 \leq \mu. \tag{9}$$

Then, sequences $\{t_n\}, \{s_n\}, \{u_n\}$ are bounded from above by $t_{**} = \frac{\eta}{1-\mu}$, nondecreasing and converge to $t_* \in [\eta, t_{**}]$, so that

$$0 \leq s_n - t_n \leq \mu(t_n - s_{n-1}) \leq \mu^{2n}\eta, \tag{10}$$

$$0 \leq u_n - s_n \leq \mu(s_n - t_n) \leq \mu^{2n+1}\eta, \tag{11}$$

$$0 \leq t_{n+1} - u_n \leq \mu(u_n - s_n) \leq \mu^{2n+2}\eta, \tag{12}$$

and

$$0 \leq t_{n+1} \leq \frac{1 - \mu^{2n+3}}{1 - \mu}\eta. \tag{13}$$

Proof. Items (10)–(12) hold, provided

$$0 \leq \bar{a}_k \leq \mu, \tag{14}$$

$$0 \leq b_k \leq \mu, \tag{15}$$

and

$$0 \leq \frac{c_k}{1 - \psi_0(t_{k+1})} \leq \mu \tag{16}$$

for each $k = 0, 1, 2, \dots$. However, these items are true for $k = 0$, by (9). Let us suppose that (14)–(16) are true for $k = 0, 1, 2, \dots, n$. By induction hypotheses (10)–(12), we get

$$\begin{aligned} t_{k+1} &\leq u_k + \mu^{2k+2}\eta \leq s_k + \mu^{2k+1}\eta + \mu^{2k+2}\eta \\ &\leq t_k + \mu^{2k}\eta + \mu^{2k+1}\eta + \mu^{2k+2}\eta \\ &\leq \eta + \mu\eta + \dots + \mu^{2k+2}\eta. \\ &= \frac{1 - \mu^{2k+3}}{1 - \mu}\eta \leq \frac{\eta}{1 - \mu} = t_{**}. \end{aligned}$$

Then, (14)–(16) shall be true if

$$f_i(\mu_i) \leq 0,$$

which are true by the definition of f_i . So, we deduce $\lim_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} u_k = t_*$. \square

3. Semi-Local Convergence

We shall use some functions and parameters.

Suppose that there exists function $\psi_0 : T \rightarrow T$ continuous and nondecreasing such that $\psi_0(t) - 1 = 0$ has a minimal solution $\rho \in T - \{0\}$. Set $T_0 = [0, \rho]$. Suppose function $\psi : T_0 \rightarrow T$ is continuous and nondecreasing.

The following conditions (A) shall be used.

Suppose:

(A1) There exists $x_0 \in D, \eta \geq 0$ such that $F'(x_0)^{-1} \in L(E_1, E)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta.$$

(A2) For each $y \in D$

$$\|F'(x_0)^{-1}(F'(y) - F'(x_0))\| \leq \psi_0(\|y - x_0\|).$$

Set $D_0 = B[x_0, s] \cap D$, where $B(x_0, s) = \{y \in E : \|y - x_0\| < s\}$ and $B[x_0, s]$ is the closure of $B(x_0, s)$.

(A3)

$$\|F'(x_0)^{-1}(F'(w) - F'(w_1))\| \leq \psi(\|w - w_1\|) \text{ for all } w_1, w \in D_0.$$

(A4) For each $n = 0, 1, 2, \dots$

$$\begin{aligned} \|A_n F'(x_n)^{-1} F'(x_0)\| &= \bar{a}_n \leq a_n \leq a, \\ \|B_n F'(x_n)^{-1} \int_0^1 (F'(y_n + \theta(z_n - y_n)) - F'(x_n)A_n^{-1})d\theta\| &= \bar{b}_n \leq b_n \leq b, \\ \|F'(x_0)^{-1} \int_0^1 (F'(z_n + \theta(x_{n+1} - z_n)) - F'(x_n)B_n^{-1})d\theta\| &= \bar{c}_n \leq c_n \leq c. \end{aligned}$$

(A5) Conditions of Lemma 1 or Lemma 2 or Lemma 3 hold.

and

(A6) $B[x_0, t_*] \subset D$ (or $B[x_0, t_{**}] \subset D$).

The semi-local convergence of scheme (2) is shown next.

Theorem 1. *Suppose conditions (A) hold. Then, sequences $\{x_n\}, \{y_n\}, \{z_n\}$ generated by scheme (2) are well defined in $B[x_0, t_*]$, stays in $B[x_0, t_*]$ for each $n = 0, 1, 2, \dots$ and converge to a solution $x_* \in B[x_0, t_*]$ of equation $F(x) = 0$, so that for each $n = 0, 1, 2, \dots$*

$$\|x_n - x_*\| \leq t_* - t_n. \tag{17}$$

Proof. We shall prove assertions

(I_k) $\|y_k - x_k\| \leq s_k - t_k$

(II_k) $\|z_k - y_k\| \leq u_k - s_k$

(III_k) $\|x_{k+1} - z_k\| \leq t_{k+1} - u_k$ using mathematical induction on k . By scheme (2) for $n = 0$ and (A1), we have

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 \leq t_*,$$

so $y_0 \in B[x_0, t_*]$ and (I₀) holds. By (A3), (A4) and the second substep of scheme (2), we have

$$\begin{aligned} \|z_k - y_k\| &= \|A_k F'(x_k)^{-1}(F(y_k) - F(x_k) - F'(x_k)(y_k - x_k))\| \\ &\leq \|A_k F'(x_k)^{-1}F'(x_0)\| \\ &\quad \times \left\| \int_0^1 F'(x_0)^{-1}(F'(x_k + \theta(y_k - x_k)) - F'(x_k))d\theta \right\| \|y_k - x_k\| \\ &\leq \|A_k F'(x_k)^{-1}F'(x_0)\| \int_0^1 \bar{\psi}(\theta\|y_k - x_k\|)d\theta \|y_k - x_k\| \\ &\leq \|A_k F'(x_k)^{-1}F'(x_0)\| \int_0^1 \bar{\psi}(\theta(s_k - t_k))d\theta (s_k - t_k) = u_k - s_k, \end{aligned}$$

showing (II_k). Similarly by the second substep of scheme (2), we obtain in turn

$$\begin{aligned} \|x_{k+1} - z_k\| &= \|B_k F'(x_k)^{-1}F(z_k)\| \\ &\leq \|B_k F'(x_k)^{-1} \int_0^1 (F'(y_k + \theta(z_k - y_k)) - F'(x_k)A_k^{-1})d\theta\| \|z_k - y_k\| \\ &= \bar{b}_k \|z_k - y_k\| \leq b_k \|u_k - s_k\| = t_{k+1} - u_k, \end{aligned}$$

and

$$\begin{aligned} \|y_k - x_k\| &\leq \|F'(x_k)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_k)\| \\ &\leq \frac{\| \int_0^1 F'(x_0)^{-1}(F'(z_{k-1} + \theta(x_k - z_{k-1})) - F'(x_{k-1})B_{k-1}^{-1})d\theta \|}{1 - \psi_0(\|x_k - x_0\|)} \|x_k - z_{k-1}\| \\ &= \frac{\bar{c}_k \|x_k - z_{k-1}\|}{1 - \psi_0(\|x_k - x_0\|)} \\ &\leq \frac{c_k(t_k - u_{k-1})}{1 - \psi_0(t_k)} = s_k - t_k, \end{aligned}$$

showing (I_k) and (III_k) , where we also used

$$\begin{aligned} F(x_k) &= F(x_k) - F(z_{k-1}) + F(z_{k-1}) \\ &= F(x_k) - F(z_{k-1}) - F'(x_{k-1})B_{k-1}^{-1}(x_k - z_{k-1}), \end{aligned}$$

(by the third substep of scheme (2))

$$F(z_k) = F(z_k) - F(y_k) + F(y_k) = F(z_k) - F(y_k) - F'(x_k)A_k^{-1}(z_k - y_k),$$

(by the second substep of scheme (2)),

$$\|F'(x_0)^{-1}(F'(x_k) - F'(x_0))\| \leq \psi_0(\|x_k - x_0\|) \leq \psi_0(t_*) < 1,$$

so

$$F'(x_k)^{-1}F'(x_0) \leq \frac{1}{1 - \psi_0(\|x_k - x_0\|)}$$

by the Banach lemma on invertible operators [17,20]. Suppose (I_k) , (II_k) , (III_k) hold, $y_k, z_k, x_{k+1} \in B[x_0, t_*]$. Then, by repeating these calculations with $x_{k+1}, y_{k+1}, z_{k+1}, x_{k+2}$ replacing x_k, y_k, z_k, x_{k+1} , respectively, we complete the induction. Therefore, sequence $\{x_k\}$ is fundamental in a Banach space E so $\lim_{k \rightarrow \infty} x_k = x_*$ for some $x_* \in B[x_0, t_*]$. If $k \rightarrow \infty$ then by the estimate

$$\|F'(x_0)^{-1}F(x_k)\| \leq c_k(t_{k+1} - u_k) \leq c(t_{k+1} - u_k) \rightarrow 0,$$

and using the continuity of F , we get $F(x_*) = 0$. \square

Next, the uniqueness result of x_* is discussed.

Proposition 1. *Suppose*

- (a) Point x_* solves the equation $F(x) = 0$.
- (b) There exists $\bar{s} \geq t_*$ such that

$$\int_0^1 \psi_0((1 - \theta)\bar{s} + \theta t_*)d\theta < 1. \tag{18}$$

Set $D_1 = B[x_0, \bar{s}] \cap D$. Then, x_* is unique in D_1 .

Proof. Consider $\bar{x} \in D_1$ satisfying $F(\bar{x}) = 0$. Define $M_0 = \int_0^1 F'(x_* + \theta(\bar{x} - x_*))d\theta$. It then follows from (A1) and (18) that

$$\begin{aligned} \|F'(x_0)^{-1}(M_0 - F'(x_0))\| &\leq \int_0^1 \psi_0((1 - \theta)\|\bar{x} - x_0\| + \theta\|x_* - x_0\|)d\theta \\ &\leq \int_0^1 \psi_0((1 - \theta)\bar{s} + \theta t_*)d\theta < 1, \end{aligned}$$

so $\bar{x} = x_*$ follows from $M_0^{-1} \in L(E_1, E)$ and the approximation $M_0(\bar{x} - x_*) = F(\bar{x}) - F(x_*) = 0 - 0 = 0$. \square

Remark 2. We want to see how developed sequences are defined in some interesting cases. Runge–Kutta and other high convergence schemes can also be studied under this technique by also specializing the operators A_n and B_n . Let us choose $A_n = B_n = I$. That is, we consider Traub’s scheme. Then, we have in turn

$$\begin{aligned} \bar{a}_n &= \|A_n F'(x_n)^{-1} F'(x_0)\| = \|F'(x_n)^{-1} F'(x_0)\| \\ &\leq \frac{1}{1 - \psi_0(\|x_n - x_0\|)} \leq \frac{1}{1 - \psi_0(t_n)} =: a_n, \\ \bar{b}_n &= \|F'(x_n)^{-1} (\int_0^1 (F'(y_n + \theta(z_n - y_n)) - F'(z_n)) d\theta + (F'(z_n) - F'(x_n)))\| \\ &\leq \frac{\int_0^1 \psi((1 - \theta)\|z_n - y_n\|) d\theta + \psi(\theta\|z_n - x_n\|)}{1 - \psi_0(\|x_n - x_0\|)} \\ &\leq \frac{\int_0^1 \psi((1 - \theta)(u_n - s_n)) d\theta + \psi(u_n - t_n)}{1 - \psi_0(t_n)} =: b_n, \end{aligned}$$

$$\begin{aligned} \bar{c}_n &= \|F'(x_0)^{-1} (\int_0^1 (F'(z_{n-1} + \theta(x_n - z_{n-1})) - F'(x_n)) d\theta + (F'(x_n) - F'(x_{n-1})))\| \\ &\leq \frac{\int_0^1 \psi((1 - \theta)\|x_n - z_{n-1}\|) d\theta + \psi(\|x_n - x_{n-1}\|)}{1 - \psi_0(\|x_{n-1} - x_0\|)} \\ &\leq \frac{\int_0^1 \psi((1 - \theta)(t_n - u_{n-1})) d\theta + \psi(t_n - t_{n-1})}{1 - \psi_0(t_{n-1})} =: c_{n-1}. \end{aligned}$$

Under these choices, we shall present another result on majorizing sequences using our idea of recurrent functions, which is weaker than that of Lemma 3. We have in this case

$$\bar{a}_n = \frac{\int_0^1 \bar{\psi}((1 - \theta)(s_n - t_n)) d\theta}{1 - \psi_0(t_n)},$$

$$b_n = \frac{\int_0^1 \psi((1 - \theta)(u_n - s_n)) d\theta + \psi((u_n - s_n) + (s_n - t_n))}{1 - \psi_0(t_n)}$$

and

$$\frac{c_n}{1 - \psi_0(t_{n+1})} = \frac{\int_0^1 \psi((1 - \theta)(t_{n+1} - u_n)) d\theta + \psi((t_{n+1} - u_n) + (u_n - s_n) + (s_n - t_n))}{1 - \psi_0(t_{n+1})}.$$

Let us consider the interesting case when $\psi_0(t) = L_0 t$ and $\psi(t) = Lt$.

Define sequences of functions $\{h_n^{(j)}\}, j = 1, 2, 3$ on the interval $[0, 1)$ by

$$\begin{aligned} h_n^{(1)} &= \frac{L}{2}t^{2n-1}\eta + L_0(1 + t + \dots + t^{2n})\eta - 1, \\ h_n^{(2)}(t) &= \frac{L}{2}t^{2n}\eta + L(t^{2n} + t^{2n-1})\eta + L_0(1 + t + \dots + t^{2n})\eta - 1, \\ h_n^{(3)} &= \frac{L}{2}t^{2n+1}\eta + L(t^{2n+1} + t^{2n} + t^{2n-1})\eta \\ &\quad + L_0(1 + t + \dots + t^{2n})\eta - 1, \\ g_1(t) &= L_0t^3 + (L_0 + \frac{L}{2})t^2 - \frac{L}{2}, \\ g_2(t) &= (\frac{3L}{2} + L_0)t^3 + (L_0 + L)t^2 - \frac{3}{2}Lt - L, \\ g_3(t) &= L_0t^5 + (\frac{3L}{2} + L_0)t^4 + Lt^3 - \frac{L}{2}t^2 - Lt - L, \\ h_\infty^{(1)}(t) &= h_\infty^{(2)}(t) = h_\infty^{(3)}(t) = \frac{L\eta}{1-t} - 1. \end{aligned}$$

Notice that $g_1(0) = -\frac{L}{2}, g_2(0) = g_3(0) = -L, g_1(1) = g_2(1) = g_3(1) = 2L_0$. It then follows (by the intermediate value theorem) that polynomial g_j have zeros in $(0, 1)$. Call by ρ_j the minimal such zeros, respectively. Set $\rho = \min\{\rho_j\}$.

Next, we show another result on majorizing sequences for scheme (2).

Lemma 4. Suppose that

$$\mu_0 \leq \rho < 1 - L_0\eta. \tag{19}$$

Then, the conclusions of Lemma 3 hold for sequences $\{s_n\}, \{t_n\}, \{u_n\}$ with ρ replacing μ .

Proof. We shall show this time

$$0 \leq \bar{a}_k \leq \rho, \tag{20}$$

$$0 \leq b_k \leq \rho, \tag{21}$$

$$0 \leq \frac{c_k}{1 - L_0t_{k+1}} \leq \rho \tag{22}$$

and

$$t_k \leq s_k \leq u_k \leq t_{k+1}. \tag{23}$$

These assertions hold for $k = 0$ by (48) and the definition of these sequences (see Remark 2). Then, as in Lemma 3, we can show instead of (20) that

$$\frac{L}{2}\rho^{2k-1}\eta + L_0(1 + \rho + \dots + \rho^{2k})\eta - 1 \leq 0. \tag{24}$$

This estimate motivates us to define recurrent functions $h_k^{(1)}$ and show instead

$$h_k^{(1)}(\rho) \leq 0. \tag{25}$$

We need a relationship between two consecutive functions $h_k^{(1)}$. Then, we can write

$$\begin{aligned} h_{k+1}^{(1)} &= \frac{L}{2}t^{2k+1}\eta + L_0(1 + t + \dots + t^{2k+2})\eta - 1 \\ &\quad - \frac{L}{2}t^{2k-1} - L_0(1 + t + \dots + t^{2k})\eta + 1 + h_k^{(1)}(t) \\ &= h_k^{(1)}(t) + g_1(t)t^{2k-1}\eta. \end{aligned} \tag{26}$$

In particular, we have

$$h_{k+1}^{(1)}(\rho_1) = h_k(\rho_1) \tag{27}$$

by the definition of ρ_1 . Define also function $h_\infty^{(1)}(t) = \lim_{k \rightarrow \infty} h_k^{(1)}(t)$. Then, we have

$$h_\infty^{(1)}(t) = \frac{L_0\eta}{1-t} - 1. \tag{28}$$

Hence, we can show instead of (20) that

$$h_\infty^{(1)}(\rho_1) \leq 0, \tag{29}$$

which is true by (48). The induction for (20) is completed. Moreover, similarly instead of (21) we can show that

$$\frac{L}{2}\rho^{2k}\eta + L(\rho^{2k} + \rho^{2k-1})\eta + L_0(1 + \rho + \dots + \rho^{2k})\eta - 1 \leq 0$$

or

$$h_k^{(2)}(\rho) \leq 0. \tag{30}$$

Then, again we have

$$\begin{aligned} h_{k+1}^{(2)} &= \frac{L}{2}t^{2k+2}\eta + L(t^{2k+2} + t^{2k+1})\eta \\ &\quad + L_0(1 + t + \dots + t^{2k+2})\eta - 1 \\ &\quad - \frac{L}{2}t^{2k}\eta - \frac{L}{2}t^{2k}\eta - L(t^{2k} + t^{2k-1})\eta \\ &\quad - L_0(1 + t + \dots + t^{2k})\eta + 1 + h_k^{(2)}(t) \\ &= h_k^{(2)}(t) + g_2(t)t^{2k-1}\eta. \end{aligned} \tag{31}$$

We have in particular

$$h_{k+1}^{(2)}(\rho_2) = h_k^{(2)}(\rho_2).$$

Define function $h_\infty^{(2)}(t) = \lim_{k \rightarrow \infty} h_k^{(2)}(t)$. Then, we have

$$h_\infty^{(2)}(t) = h_\infty^{(1)}(t). \tag{32}$$

Hence, we can show instead of (30) that $h_\infty^{(2)}(\rho_2) \leq 0$, which is true by (48). Furthermore, (22) holds if we show instead

$$\frac{L}{2}\rho^{2k+1}\eta + L(\rho^{2k+1} + \rho^{2k} + \rho^{2k-1})\eta + L_0(1 + \rho + \dots + \rho^{2k+2})\eta - 1 \leq 0$$

or

$$h_k^{(3)}(\rho) \leq 0. \tag{33}$$

However, we get

$$\begin{aligned} h_{k+1}^{(3)} &= \frac{L}{2}t^{2k+3}\eta + L(t^{2k+3} + t^{2k+2} + t^{2k+1})\eta \\ &\quad + L_0(1 + t + \dots + t^{2k+4})\eta - 1 \\ &\quad - \frac{L}{2}t^{2k+1}\eta - L(t^{2k+1} + t^{2k} + t^{2k-1})\eta \\ &\quad - L_0(1 + t + \dots + t^{2k+2})\eta + 1 + h_k^{(3)}(t) \\ &= h_k^{(3)}(t) + g_3(t)t^{2k-1}\eta. \end{aligned}$$

In particular, we have

$$h_{k+1}^{(3)}(\rho_3) = h_k^{(3)}(\rho_3).$$

Defined function $h_\infty^{(3)}(t) = \lim_{k \rightarrow \infty} h_k^{(3)}(t)$. Then, we also get

$$h_\infty^{(3)}(t) = h_\infty^{(1)}(t).$$

Hence, (33) holds if $h_\infty^{(3)}(\rho_3) \leq 0$. However, this is true by (48). Therefore, the induction is terminated for (20)–(23). The rest is as in the proof of Lemma 1. \square

4. Local Convergence

We shall introduce real parameter and functions to be used in the convergence analysis. Set $M = [0, \infty)$.

Suppose

- (i) $\varphi_0(t) - 1$ has a minimal zero $R_0 \in M - \{0\}$, where $\varphi_0 : M \rightarrow M$ is some continuous and nondecreasing function. Let $M_0 = [0, R_0)$.
- (ii) $\varphi_1(t) - 1$ has a minimal zero $R_1 \in M_0 - \{0\}$, where function $\varphi : M_0 \rightarrow M$ is continuous and nondecreasing and $\varphi_1 : M_0 \rightarrow M$ is defined by

$$\varphi_1(t) = \frac{\int_0^1 \varphi((1-\theta)t) d\theta}{1 - \varphi_0(t)}.$$

- (iii) $\varphi_0(\varphi_1(t)t) - 1$ has a minimal zero $\bar{R}_1 \in M_0 - \{0\}$. Let $\bar{R}_2 = \min\{R_0, \bar{R}_1\}$ and $M_1 = [0, \bar{R}_2)$.
- (iv) $\varphi_2(t) - 1$ has a smallest zero $R_2 \in M_1 - \{0\}$, where

$$\varphi_2(t) = \left[\varphi_1(\varphi_1(t)t) + \frac{(\varphi_0(t) + h(t, \varphi_1(t)t)) \int_0^1 \omega(\theta \varphi_1(t)t) d\theta}{(1 - \varphi_0(t))(1 - \varphi_0(\varphi_1(t)t))} \right] \varphi_1(t),$$

with functions $\omega : M_1 \rightarrow M$ and $h : M \times M_1 \rightarrow M$ are continuous and nondecreasing.

- (v) $\varphi_3(t) - 1$ has a smallest zero $R_3 \in M_1 - \{0\}$ where

$$\varphi_3(t) = \left[\varphi_1(\varphi_2(t)t) + \frac{(\varphi_0(t) + h_1(t, \varphi_1(t)t, \varphi_2(t)t)) \int_0^1 \omega(\theta \varphi_2(t)t) d\theta}{(1 - \varphi_0(t))(1 - \varphi_0(\varphi_2(t)t))} \right] \varphi_2(t),$$

where $h_1 : M \times M_1 \times M_1 \rightarrow M$ is nondecreasing and continuous. It shall be shown that

$$R = \min\{R_i\}, i = 1, 2, 3 \tag{34}$$

is a convergence radius for scheme (2). Set $M_2 = [0, R)$. These definitions imply that for each $t \in E_2$

$$0 \leq \varphi_0(t) < 1, \tag{35}$$

$$0 \leq \varphi_0(\varphi_1(t)t) < 1, \tag{36}$$

and

$$0 \leq \varphi_1(t) < 1, i = 1, 2, 3. \tag{37}$$

The conditions (H) shall be used.

Suppose

- (H1) For all $w \in D$

$$\|F'(x_*)^{-1}(F'(w) - F'(x_*))\| \leq \varphi_0(\|w - x_*\|).$$

Let $D_0 = B(x_*, R_0) \cap D$.

(H2) For all $w, w_1, z \in D_0$

$$\begin{aligned} \|F'(x_*)^{-1}(F'(w_1) - F'(w))\| &\leq \varphi(\|w_1 - w\|), \\ \|F'(x_*)^{-1}F'(w)\| &\leq \omega(\|w - x_*\|), \\ \|F'(x_*)^{-1}(F'(x_*) - F'(w_1)A(w, w_1))\| &\leq h(\|w - x_*\|, \|w_1 - x_*\|) \end{aligned}$$

and

$$\|F'(x_*)^{-1}(F'(x_*) - F'(z)B(w, w_1))\| \leq h_1(\|w - x_*\|, \|w_1 - x_*\|, \|z - x_*\|)$$

(H3) $B[x_*, R] \subset D$.

The local convergence of scheme (2) is given next based on the previous notation and conditions (H).

Theorem 2. *Suppose: $x_0 \in B(x_*, R) - \{x_*\}$ and conditions (H) hold. Then, we conclude $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = x_*$.*

Proof. Let $v \in B(x_*, R) - \{x_*\}$. Using (17), (18) and (H1), we have

$$\|F'(x_*)^{-1}(F'(v) - F'(x_*))\| \leq \varphi_0(\|v - x_*\|) \leq \varphi_0(R) < 1,$$

so

$$\|F'(v)^{-1}F'(x_*)\| \leq \frac{1}{1 - \varphi_0(\|v - x_*\|)}. \tag{38}$$

In particular, iterate is well defined by (38) for $v = x_0$ and the first substep of scheme (2), from which we can also write

$$\begin{aligned} y_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ &= (F'(x_0)^{-1}F'(x_*)) \\ &\quad \times \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*). \end{aligned} \tag{39}$$

By (34), (37) (for $i = 1$), (38) (for $v = x_0$), (39) and (H2), we find in turn that

$$\begin{aligned} \|y_0 - x_*\| &\leq \frac{\int_0^1 \bar{\varphi}((1 - \theta)\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{1 - \varphi_0(\|x_0 - x_*\|)} \\ &\leq \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &\leq \|x_0 - x_*\| < R, \end{aligned} \tag{40}$$

so $y_0 \in B(x_*, R)$, where $\bar{\varphi} = \begin{cases} \varphi_0, & k = 0 \\ \varphi, & k = 1, 2, \dots \end{cases}$. We also have that (38) holds for $v = y_0$, and iterate z_0 is well defined, from which we can write in turn that

$$\begin{aligned} z_0 - x_* &= y_0 - x_* - F'(y_0)^{-1}F(y_0) \\ &\quad + (F'(y_0)^{-1} - A_0F'(x_0)^{-1})F(y_0) \\ &= y_0 - x_* - F'(y_0)^{-1}F(y_0) \\ &\quad + F'(y_0)^{-1}(F'(x_0) - F'(y_0)A_0)F'(x_0)^{-1}F(y_0). \end{aligned} \tag{41}$$

In view of (34), (37) (for $i = 2$), (38) (for $v = x_0, y_0$), (40), (41) and (H2), we obtain in turn

$$\begin{aligned} \|z_0 - x_*\| &\leq [\varphi_1(\varphi_1(\|x_0 - x_*\|))\|x_0 - x_*\| \\ &+ \frac{(\varphi_0(\|x_0 - x_*\|) + h(\|x_0 - x_*\|, \|y_0 - x_*\|)) \int_0^1 \omega(\theta\|y_0 - x_*\|)d\theta}{(1 - \varphi_0(\|y_0 - x_*\|))(1 - \varphi_0(\|x_0 - x_*\|))}] \|y_0 - x_*\| \\ &\leq \varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R, \end{aligned} \tag{42}$$

so $z_0 \in B(x_*, R)$. Similarly, but using the third substep of scheme (2), we can write

$$\begin{aligned} x_1 - x_* &= z_0 - x_* - F'(z_0)^{-1}F(z_0) \\ &+ (F'(z_0)^{-1} - B_0F'(x_0)^{-1})F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) \\ &+ (F'(z_0)^{-1} - B_0F'(x_0)^{-1})F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}((F'(x_0) - F'(x_*)) \\ &+ (F'(x_*) - F'(z_0)B_0))F'(x_0)^{-1}F(z_0), \end{aligned}$$

so

$$\begin{aligned} \|x_1 - x_*\| &\leq [\varphi_1(\varphi_2(\|x_0 - x_*\|))\|x_0 - x_*\| \\ &+ \frac{H}{(1 - \varphi_0(\varphi_2(\|x_0 - x_*\|))\|x_0 - x_*\|)(1 - \varphi_0(\|x_0 - x_*\|))}] \\ &\times \varphi_2(\|x_0 - x_*\|) \\ &\leq \varphi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned}$$

for $H = (\varphi_0(\|x_0 - x_*\|) + h_1(\|x_0 - x_*\|, \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\|, \varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\|)) \int_0^1 \omega(\theta\varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\|)d\theta$. Simply, switch x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} , respectively, in the preceding calculations to get

$$\|y_k - x_*\| \leq \varphi_1(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| < R \tag{43}$$

$$\|z_k - x_*\| \leq \varphi_2(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| < R \tag{44}$$

and

$$\|x_{k+1} - x_*\| \leq \varphi_3(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\|. \tag{45}$$

Then, from the estimation

$$\|x_{k+1} - x_*\| \leq d\|x_k - x_*\| < R, \tag{46}$$

where $d = \varphi_3(\|x_k - x_*\|)$, we get $x_{k+1} \in B(x_*, R)$ and $\lim_{k \rightarrow \infty} x_k = x_*$. \square

A uniqueness result is given next.

Proposition 2. Suppose:

- (i) Point x_* is a simple solution of equation $F(x) = 0$.
- (ii)

$$\int_0^1 \varphi_0(\theta R_*)d\theta < 1, \text{ for some } R_* \geq R. \tag{47}$$

Set $D_1 = B[x_*, R_*] \cap D$. Then, x_* is unique in D_1 .

Proof. Consider $\tilde{x} \in D_1$ with $F(\tilde{x}) = 0$. Set $S = \int_0^1 F'(x_* + \theta(\tilde{x} - x_*))d\theta$. Using (H1) and (47), we obtain

$$\|F'(x_*)^{-1}(S - F'(x_*))\| \leq \int_0^1 \varphi_0(\theta)\|\tilde{x} - x_*\|d\theta \leq \int_0^1 \varphi_0(\theta R_*)d\theta < 1,$$

so $\tilde{x} = x_*$ follows by $S^{-1} \in L(E_1, E)$ and $S(\tilde{x} - x_*) = F(\tilde{x}) - F(x_*) = 0 - 0 = 0$. \square

5. Numerical Experiments

We consider four examples with $A_n = B_n = I$ (Multistep Newton method). Here, $\varphi_0(t) = a_0t, \varphi(t) = at, \omega(t) = a_1t$ and $h(t, t) = h_1(t, t) = \varphi_0(t)$.

Example 1. Define function

$$q(t) = \delta_0t + \delta_1 + \delta_2 \sin \delta_3t, t_0 = 0,$$

where $\delta_j, j = 0, 1, 2, 3$ are numbers. Then, for δ_3 large and δ_2 small, $\frac{a_0}{a}$ is arbitrarily small and $\frac{a_0}{a} \rightarrow 0$.

Example 2. If $E = E_1 = C[0, 1], D = B[0, 1]$, consider $Q : D \rightarrow E_1$ given as

$$Q(\lambda)(x) = \varphi(x) - 5 \int_0^1 x\theta\lambda(\theta)^3d\theta. \tag{48}$$

We obtain

$$Q'(\lambda(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\lambda(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D,$$

so for $x_* = 0$, we set $a_0 = 7.5, a = a_4 = K = 15, a_1 = \frac{L_0}{2}$. Then, the radii:

$$r_1 = 0.066667 = r, r_2 = 0.109818, \text{ and } \rho_R = \frac{2}{3K} = 0.0667.$$

Example 3. Consider the motion system

$$N'_1(w_1) = e^{w_1}, N'_2(w_2) = (e - 1)w_2 + 1, N'_3(w_3) = 1$$

with $N_1(0) = N_2(0) = N_3(0) = 0$. Let $N = (N_1, N_2, N_3)$. Let $E = E_1 = \mathbb{R}^3, D = B[0, 1], x_* = (0, 0, 0)^T$. Let function N on D for $w = (w_1, w_2, w_3)^T$ given as

$$N(w) = (e^{w_1} - 1, \frac{e - 1}{2}w_2^2 + w_2, w_3)^T.$$

Then, N' is given as

$$N'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we can set $a_0 = (e - 1), a = e^{\frac{1}{e-1}} = a_4, a_1 = \frac{a_0}{2}$ and $K = e$. Then, the radii:

$$r_1 = 0.382692 = r, r_2 = 0.417923, \text{ and } \rho_R = \frac{2}{3K} = 0.2453,$$

where K is the Lipschitz constant in D given in [26,30].

Example 4. Consider E, E_1 and D as in the second Example. Then, the (BVP) [13]

$$p(0) = 0, p(1) = 1, \\ p'' = -p - \sigma p^2$$

can be also given as

$$p(s) = s + \int_0^1 G(s, t)(p^3(t) + \sigma p^2(t))dt$$

where σ is a constant, Consider $F : D \rightarrow V_2$ as

$$[F(x)](s) = x(s) - s - \int_0^1 G(s, t)(x^3(t) + \sigma x^2(t))dt.$$

Let us set $p_0(s) = s$ and $D = B(p_0, \rho_0)$. Then, clearly $B(p_0, \rho_0) \subset B(0, \rho_0 + 1)$, since $\|p_0\| = 1$. Suppose $2\sigma < 5$. Then, conditions (A) are satisfied for

$$a_0 = \frac{2\sigma + 3\rho_0 + 6}{8}, a = \frac{\sigma + 6\rho_0 + 3}{4},$$

$a_1 = \frac{a_0}{2}$ and $\mu = \frac{1+\sigma}{5-2\sigma}$. Notice that $a_0 < a_1$.

6. Conclusions

We have provided a single sufficient criterion for the semi-local convergence of three step schemes. Upon specializing the parameters involved we show that although our majorizing sequence is more general than earlier ones: Convergence criteria are weaker (i.e., the utility of the schemes is extended); the upper error estimates are more accurate (i.e., at least as few iterates are required to achieve a predecided error tolerance) and we have an at least as large ball containing the solution. These benefits are obtained without additional hypotheses. According to our new technique we locate a more accurate domain than before containing the iterates resulting to more accurate (at least as small) Lipschitz condition.

Our theoretical results are further justified using numerical experiments. In the future we plan to extend these results by replacing the Lipschitz constants by generalized functions along the same lines.

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