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Oscillation of Solutions to Third-Order Nonlinear Neutral Dynamic Equations on Time Scales

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Abstract: In this paper, we are concerned with the oscillation of solutions to a class of third-order nonlinear neutral dynamic equations on time scales. New oscillation criteria are presented by employing the Riccati transformation and integral averaging technique. Two examples are shown to illustrate the conclusions.

Keywords: third-order; neutral dynamic equation; time scale; oscillation; nonlinear

MSC: 34C10; 34K11; 34N05; 39A13; 39A21

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1. Introduction

In this paper, we investigate the oscillation of solutions to a class of third-order half-linear dynamic equations with a nonpositive neutral coefficient

$$\left(r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}\right)^{\Delta}+f(t,x(h(t)))=0 \tag{1}$$

on a time scale $\mathbb{T}$, which satisfies $\sup\mathbb{T}=\infty$, where $t\in[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$ with $t_0\in\mathbb{T}$ and $z(t)=x(t)-p(t)x(g(t))$. We assume that:

(A1) $r_1,r_2\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_1^{1/\gamma_1}(t)}=\int_{t_0}^{\infty}\frac{\Delta t}{r_2^{1/\gamma_2}(t)}=\infty;$$

(A2) γ,γ_1,γ_2 are quotients of odd positive integers, where $\gamma=\gamma_1\cdot\gamma_2$;
(A3) $p\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$ and there exists a constant p_0 with $0\le p_0<1$ such that $\lim_{t\to\infty}p(t)=p_0$;
(A4) $g,h\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $g(t)\le t$, $h^{\sigma}(t)\le t$, $h^{\Delta}(t)>0$, $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$, and there exists a sequence $\{c_k\}_{k\ge0}$ such that $\lim_{k\to\infty}c_k=\infty$ and $g(c_{k+1})=c_k$;
(A5) $f\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$ and there exists a function $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ such that $uf(t,u)\ge q(t)u^{\gamma+1}$;
(A6) if $0<\gamma<1$, then it satisfies that $r_2(t)=1$ for $t\in[t_0,\infty)_{\mathbb{T}}$ and

$$\int_{t_0}^{\infty}q(t)h^{\gamma}(t)\Delta t=\infty.$$

In what follows, we state some background details that motivate the analysis of (1). In recent years, numerous significant results for the oscillation of functional differential equations have been shown in [1–6]. Therein, Džurina et al. [3] and Santra et al. [6] studied the oscillation of half-linear/Emden–Fowler delay differential equations with a sublinear neutral term, whereas the papers [1,4,5] were concerned with the asymptotics and oscillation of solutions to (1) and its modifications in the continuous case (i.e., $\mathbb{T} = \mathbb{R}$). Chiu and Li [2] considered the oscillatory behavior of a class of scalar advanced and delayed differential equations with piecewise constant generalized arguments, which extended the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments.

To unify continuous and discrete analysis (i.e., the theories of differential equations and difference equations), Hilger introduced the time scale theory in [7,8]. Instead of repeating here the basic facts of time scales and time scale notation, we refer the reader to the papers [9,10] and monographs [11,12] for more details on this theory. Recently, there has been much attention to the study of oscillation of various classes of dynamic equations on time scales; see, for instance, the papers [13–16] concerning the analyses of Fite–Hille–Wintner-type criteria, comparison theorems, and Kamenev-type criteria for the half-linear dynamic equation with deviating argument

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)f(x(\tau(t)))=0, \quad t\in[t_0,\infty)_{\mathbb{T}},$$

respectively. In particular, Wu et al. [16] used the generalized Riccati substitution

$$\omega(t)=\Phi(t)r(t)\left(\left(\frac{x^{\Delta}(t)}{x(\tau(t))}\right)^{\gamma}+\phi(t)\right), \tag{2}$$

where $\Phi \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $\phi \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and obtained several oscillation criteria. In 2004, Mathsen et al. [17] presented some open problems for the study of qualitative properties of solutions to dynamic equations on time scales. Later on, numerous researchers analyzed the oscillation and asymptotic behavior of solutions to different classes of third-order dynamic equations. Agarwal et al. [18,19], Erbe et al. [20], and Hassan [21] investigated a third-order half-linear delay dynamic equation

$$\left(a(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta}+f(t,x(\tau(t)))=0,$$

whereas Yu and Wang [22] studied the third-order half-linear dynamic equation

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+q(t)f(x(t))=0$$

in the case when $\alpha_1\alpha_2 = 1$. Han et al. [23] investigated a third-order half-linear/Emden–Fowler neutral delay dynamic equation

$$\left(r(t)(x(t)-a(t)x(\tau(t)))^{\Delta\Delta}\right)^{\Delta}+p(t)x^{\gamma}(\delta(t))=0.$$

Qiu [24] considered (1) under the condition

$$h(t)\geq\begin{cases}\sigma(t), & 0<\gamma<1,\\ t, & \gamma\geq 1.\end{cases}$$

By employing the Riccati transformation

$$u(t) = A(t)\frac{r_1(t)\left(\left(r_2(t)(z^{\Delta}(t))^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}}{z^{\gamma}(t)} + B(t), \tag{3}$$

where $A \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $B \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, the author established several oscillation criteria for (1). As a matter of fact, it is not difficult to see that the functions ϕ in (2) and B in (3) can be deleted, respectively.

Half-linear equations, as the classical nonlinear equations, arise in the analyses of p-Laplace equations, non-Newtonian fluid theory, porous medium problems, chemotaxis models, and so forth; see, for instance, the papers [13,14,25–27] for more details. On the basis of the above discussion, we will establish integral criteria and Kamenev-type criteria (see, e.g., [15]) for the oscillation of (1) by employing a similar Riccati transformation as (2). Finally, two examples are presented to show the significance of the conclusions.

2. Auxiliary Results

To establish oscillation criteria for (1), we give the following lemmas in this section; Lemmas 1–3 are also used in [24].

Lemma 1. *Let* (A1)–(A5) *be satisfied. Suppose that x is an eventually positive solution to* (1) *and there exists a constant $a \geq 0$ such that* $\lim_{t\to\infty} z(t) = a$. *Then,*

$$\lim_{t\to\infty} x(t) = \frac{a}{1-p_0}.$$

Lemma 2. *Let* (A1)–(A5) *be satisfied and assume that x is an eventually positive solution to* (1). *Then, there exists a sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ such that, for $t \in [T,\infty)_{\mathbb{T}}$,*

$$\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta} > 0,$$

and either $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$.

Lemma 3. *Let* (A1)–(A5) *be satisfied and assume that x is an eventually positive solution to* (1). *Then, z is eventually positive or* $\lim_{t\to\infty} x(t) = 0$.

Lemma 4. *For $0 < \gamma < 1$, assume that* (A1)–(A6) *hold. Suppose that x is an eventually positive solution to* (1) *and z, z^{Δ} are both eventually positive. Then, there exists a sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ such that, for any $t \in [T,\infty)_{\mathbb{T}}$,*

$$\frac{1}{z(h^{\sigma}(t))} \geq \frac{h(t)}{h^{\sigma}(t)}\frac{1}{z(h(t))}. \tag{4}$$

Proof. For $0 < \gamma < 1$, suppose that x is an eventually positive solution to (1), and z, z^{Δ} are both eventually positive. Then, there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $z(t) > 0$ and $z^{\Delta}(t) > 0$, $t \in [t_1,\infty)_{\mathbb{T}}$. In view of (A6) and Lemma 2, we have

$$\left(\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta} > 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Using the Pötzsche chain rule

$$\left(\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta} = \gamma_2\int_0^1 [hz^{\Delta}(\sigma(t)) + (1-h)z^{\Delta}(t)]^{\gamma_2-1}dh\cdot z^{\Delta\Delta}(t),$$

we deduce $z^{\Delta\Delta}(t) > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. For $t \in [t_1, \infty)_{\mathbb{T}}$, define

$$y(t) = z(t) - tz^{\Delta}(t).$$

It follows that $y^{\Delta}(t) = -\sigma(t)z^{\Delta\Delta}(t) < 0$. We can claim that y is eventually positive. Assume not; then there exists a sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $y(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Therefore, we have

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{y(t)}{t\sigma(t)} > 0, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

It is clear that $z(t)/t$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty} h(t) = \infty$, there exists a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $h(t) \geq h(t_3) \geq t_2$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, we obtain

$$\frac{z(h(t))}{h(t)} \geq \frac{z(h(t_3))}{h(t_3)}.$$

By virtue of (A5), we have $f(t, x(h(t))) \geq q(t)x^{\gamma}(h(t)) \geq q(t)z^{\gamma}(h(t))$. According to (1), for $t \in [t_3, \infty)_{\mathbb{T}}$, we conclude that

$$\left(r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}\right)^{\Delta} = -f(t, x(h(t))) \leq -q(t)z^{\gamma}(h(t)). \tag{5}$$

Note that $r_2(t) = 1$. Integrating (5) from t_3 to t, $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, we arrive at

$$\begin{aligned} r_1(t)\left(\left(\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1} &- r_1(t_3)\left(\left(\left(z^{\Delta}(t_3)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1} \\ &\leq -\int_{t_3}^{t} q(s)z^{\gamma}(h(s))\Delta s \leq -\frac{z^{\gamma}(h(t_3))}{h^{\gamma}(t_3)}\int_{t_3}^{t} q(s)h^{\gamma}(s)\Delta s. \end{aligned}$$

It means that

$$r_1(t_3)\left(\left(\left(z^{\Delta}(t_3)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1} \geq \frac{z^{\gamma}(h(t_3))}{h^{\gamma}(t_3)}\int_{t_3}^{t} q(s)h^{\gamma}(s)\Delta s \to \infty$$

as $t \to \infty$, which is a contradiction. Therefore, y is eventually positive, and so there exists a sufficiently large $T \in [t_1, \infty)_{\mathbb{T}}$ such that for any $t \in [T, \infty)_{\mathbb{T}}$, we have

$$\left(\frac{z(t)}{t}\right)^{\Delta} = -\frac{y(t)}{t\sigma(t)} < 0,$$

which implies that $z(t)/t$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$. Since $h^{\Delta}(t) > 0$, $t \in [T, \infty)_{\mathbb{T}}$, we deduce

$$\frac{z(h(t))}{h(t)} \geq \frac{z(h^{\sigma}(t))}{h^{\sigma}(t)},$$

which means that (4) holds. The proof is complete. □

3. Main Results

In this section, we establish oscillation criteria for (1) by the Riccati transformation and integral averaging technique.

Theorem 1. *When* $\gamma \geq 1$, *assume that* (A1)–(A5) *hold. For any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$, *if there exist a sufficiently large* $T \in [t_1, \infty)_{\mathbb{T}}$ *and a function* $A \in C^1_{\mathrm{rd}}([T, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\limsup_{t\to\infty}\left\{A(t)\int_t^\infty q(s)\Delta s+\int_T^t\left[A(s)q(s)-\frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1}\frac{\left(A^\Delta(s)\right)^{1+\gamma}}{\left(A(s)h^\Delta(s)\right)^\gamma}\right]\Delta s\right\}=\infty, \tag{6}$$

where $r_2^*(t)=\max\{r_2(\xi):\xi\in[h(t),h^\sigma(t))_{\mathbb{T}}\}$ *and* $\delta(t)=\int_{t_1}^t 1/r_1^{1/\gamma_1}(s)\Delta s$, *then every solution* x *of* (1) *is oscillatory or* $\lim_{t\to\infty}x(t)$ *exists.*

Proof. Suppose that (1) is not oscillatory. Without loss of generality, assume that x is an eventually positive solution to (1). In view of Lemma 3, we deduce that z is eventually positive or $\lim_{t\to\infty}x(t)=0$. If $\lim_{t\to\infty}x(t)=0$, then the theorem is proved. When z is eventually positive, we know that z^Δ is eventually positive or eventually negative according to Lemma 2. If z^Δ is eventually negative, then $\lim_{t\to\infty}z(t)$ exists, and $\lim_{t\to\infty}x(t)$ exists on the basis of Lemma 1. The theorem is also proved.

If z^Δ is eventually positive, then there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that for any $t\in[t_1,\infty)_{\mathbb{T}}$, we have $x(t)>0$, $x(g(t))>0$, $x(h(t))>0$, $z(t)>0$, and $z^\Delta(t)>0$. Moreover, there exists a sufficiently large $T\in[t_1,\infty)_{\mathbb{T}}$ such that $h(t)\geq t_1$, $t\in[T,\infty)_{\mathbb{T}}$. Define

$$w(t)=A(t)\frac{r_1(t)\left(\left(r_2(t)\left(z^\Delta(t)\right)^{\gamma_2}\right)^\Delta\right)^{\gamma_1}}{z^\gamma(h(t))},\quad t\in[T,\infty)_{\mathbb{T}}, \tag{7}$$

where $A\in C^1_{\mathrm{rd}}([T,\infty)_{\mathbb{T}},(0,\infty))$. In view of (1) and (A5), for $t\in[T,\infty)_{\mathbb{T}}$ we have (5). Integrating (5), by Lemma 2 we obtain

$$r_1(t)\left(\left(r_2(t)\left(z^\Delta(t)\right)^{\gamma_2}\right)^\Delta\right)^{\gamma_1}\geq\int_t^\infty q(s)z^\gamma(h(s))\Delta s\geq z^\gamma(h(t))\int_t^\infty q(s)\Delta s,$$

which implies that

$$w(t)\geq A(t)\int_t^\infty q(s)\Delta s>0,\quad t\in[T,\infty)_{\mathbb{T}}. \tag{8}$$

Then, Δ-differentiating (7) and using (1), we deduce

$$\begin{aligned}
w^\Delta(t)&=\frac{A(t)}{z^\gamma(h(t))}\left(r_1(t)\left(\left(r_2(t)\left(z^\Delta(t)\right)^{\gamma_2}\right)^\Delta\right)^{\gamma_1}\right)^\Delta\\
&\quad+\left(\frac{A(t)}{z^\gamma(h(t))}\right)^\Delta\left(r_1(t)\left(\left(r_2(t)\left(z^\Delta(t)\right)^{\gamma_2}\right)^\Delta\right)^{\gamma_1}\right)^\sigma\\
&=-\frac{A(t)}{z^\gamma(h(t))}\cdot f(t,x(h(t)))\\
&\quad+\frac{A^\Delta(t)z^\gamma(h(t))-A(t)(z^\gamma(h(t)))^\Delta}{z^\gamma(h(t))z^\gamma(h^\sigma(t))}\left(r_1(t)\left(\left(r_2(t)\left(z^\Delta(t)\right)^{\gamma_2}\right)^\Delta\right)^{\gamma_1}\right)^\sigma\\
&\leq -A(t)q(t)+A^\Delta(t)\left(\frac{w(t)}{A(t)}\right)^\sigma-A(t)\frac{(z^\gamma(h(t)))^\Delta}{z^\gamma(h(t))}\left(\frac{w(t)}{A(t)}\right)^\sigma.
\end{aligned} \tag{9}$$

Since $\gamma\geq 1$, we have

$$(z^\gamma(h(t)))^\Delta\geq\gamma z^{\gamma-1}(h(t))(z(h(t)))^\Delta,$$

which yields

$$w^\Delta(t)\leq -A(t)q(t)+A^\Delta(t)\left(\frac{w(t)}{A(t)}\right)^\sigma-\gamma A(t)\frac{(z(h(t)))^\Delta}{z(h(t))}\left(\frac{w(t)}{A(t)}\right)^\sigma.$$

If $\sigma(t) = t$, then $(z(h(t)))^{\Delta} = z'(h(t))h'(t)$. If $\sigma(t) > t$, then by Mean Value Theorem (see [12]), there exists a $\xi \in [h(t), h^{\sigma}(t))_{\mathbb{T}}$ such that

$$(z(h(t)))^{\Delta} = \frac{z(h^{\sigma}(t)) - z(h(t))}{\sigma(t) - t} = \frac{z(h^{\sigma}(t)) - z(h(t))}{h^{\sigma}(t) - h(t)} \cdot h^{\Delta}(t) \geq z^{\Delta}(\xi)h^{\Delta}(t).$$

Therefore, for $t \in [T, \infty)_{\mathbb{T}}$, we obtain

$$w^{\Delta}(t) \leq -A(t)q(t) + A^{\Delta}(t)\left(\frac{w(t)}{A(t)}\right)^{\sigma} - \gamma A(t)h^{\Delta}(t)\frac{z^{\Delta}(\xi)}{z(h(t))}\left(\frac{w(t)}{A(t)}\right)^{\sigma}, \tag{10}$$

where $\xi \in [h(t), h^{\sigma}(t))_{\mathbb{T}}$. From (A5), we have

$$\left(r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}\right)^{\Delta} = -f(t, x(h(t))) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

which means that $r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. For $t \in [t_1, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2} &= r_2(t_1)\left(z^{\Delta}(t_1)\right)^{\gamma_2} + \int_{t_1}^{t} \frac{r_1^{1/\gamma_1}(s)\left(r_2(s)\left(z^{\Delta}(s)\right)^{\gamma_2}\right)^{\Delta}}{r_1^{1/\gamma_1}(s)}\Delta s \\ &\geq r_1^{1/\gamma_1}(t)\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\int_{t_1}^{t}\frac{\Delta s}{r_1^{1/\gamma_1}(s)}. \end{aligned}$$

Since $\xi \in [h(t), h^{\sigma}(t))_{\mathbb{T}}$, $h(t) \geq t_1$, and $h^{\sigma}(t) \leq t$ for $t \in [T, \infty)_{\mathbb{T}}$, it follows that

$$\begin{aligned} r_2(\xi)\left(z^{\Delta}(\xi)\right)^{\gamma_2} &\geq r_1^{1/\gamma_1}(\xi)\left(r_2(\xi)\left(z^{\Delta}(\xi)\right)^{\gamma_2}\right)^{\Delta}\int_{t_1}^{h(t)}\frac{\Delta s}{r_1^{1/\gamma_1}(s)} \\ &\geq r_1^{1/\gamma_1}(t)\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\delta(h(t)). \end{aligned}$$

Then, we have

$$\left(z^{\Delta}(\xi)\right)^{\gamma} \geq \frac{r_1(t)\left(\left(r_2(t)(z^{\Delta}(t))^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}}{\left(r_2^*(t)\right)^{\gamma_1}}\delta^{\gamma_1}(h(t)),$$

and so

$$\frac{z^{\Delta}(\xi)}{z(h(t))} \geq \left(\frac{\delta(h(t))}{r_2^*(t)}\right)^{1/\gamma_2}\left[\left(\frac{w(t)}{A(t)}\right)^{\sigma}\right]^{1/\gamma}. \tag{11}$$

In view of (10) and (11), for $t \in [T, \infty)_{\mathbb{T}}$ we deduce that

$$w^{\Delta}(t) \leq -A(t)q(t) + A^{\Delta}(t)\left(\frac{w(t)}{A(t)}\right)^{\sigma} - \gamma A(t)h^{\Delta}(t)\left(\frac{\delta(h(t))}{r_2^*(t)}\right)^{1/\gamma_2}\left[\left(\frac{w(t)}{A(t)}\right)^{\sigma}\right]^{1+1/\gamma}. \tag{12}$$

For (12), applying the inequality $\lambda ab^{\lambda-1} - a^{\lambda} \leq (\lambda - 1)b^{\lambda}$, with $\lambda = 1 + 1/\gamma$,

$$a^{\lambda} = \gamma A(t)h^{\Delta}(t)\left(\frac{\delta(h(t))}{r_2^*(t)}\right)^{1/\gamma_2}\left[\left(\frac{w(t)}{A(t)}\right)^{\sigma}\right]^{1+1/\gamma},$$

and

$$b^{\lambda-1} = \frac{\gamma}{1+\gamma}\left(\frac{r_2^*(t)}{\delta(h(t))}\right)^{\gamma_1/(1+\gamma)} \frac{A^\Delta(t)}{(\gamma A(t)h^\Delta(t))^{\gamma/(1+\gamma)}},$$

we conclude that

$$w^\Delta(t) \le -A(t)q(t) + \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(t)}{\delta(h(t))}\right)^{\gamma_1} \frac{\left(A^\Delta(t)\right)^{1+\gamma}}{(A(t)h^\Delta(t))^\gamma}. \tag{13}$$

Letting t be replaced by s, and integrating (13) with respect to s from T to $t \in [\sigma(T),\infty)_{\mathbb{T}}$, we obtain

$$w(t) \le w(T) - \int_T^t \left[A(s)q(s) - \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1} \frac{\left(A^\Delta(s)\right)^{1+\gamma}}{(A(s)h^\Delta(s))^\gamma}\right]\Delta s.$$

By (8), we arrive at

$$A(t)\int_t^\infty q(s)\Delta s + \int_T^t \left[A(s)q(s) - \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1} \frac{\left(A^\Delta(s)\right)^{1+\gamma}}{(A(s)h^\Delta(s))^\gamma}\right]\Delta s \le w(T),$$

which is a contradiction to (6). This completes the proof. □

Theorem 2. *When $0<\gamma<1$, assume that* (A1)–(A6) *hold. For any $t_1 \in [t_0,\infty)_{\mathbb{T}}$, if there exist a sufficiently large $T \in [t_1,\infty)_{\mathbb{T}}$ and a function $A \in C^1_{rd}([T,\infty)_{\mathbb{T}},(0,\infty))$ such that*

$$\limsup_{t\to\infty}\left\{A(t)\int_t^\infty q(s)\Delta s + \int_T^t \left[A(s)q(s) - \frac{\left(A^\Delta(s)\right)^{1+\gamma}(h^\sigma(s))^\gamma}{(1+\gamma)^{1+\gamma}\delta^{\gamma_1}(h(s))(A(s)h(s)h^\Delta(s))^\gamma}\right]\Delta s\right\} = \infty, \tag{14}$$

where δ is defined as in Theorem 1, then every solution x of (1) is oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Proof. As in the proof of Theorem 1, for $t \in [T,\infty)_{\mathbb{T}}$ we obtain (9). Since $0<\gamma<1$, in view of Lemma 4, we have

$$\begin{aligned}
(z^\gamma(h(t)))^\Delta &\ge \gamma z^{\gamma-1}(h^\sigma(t))(z(h(t)))^\Delta \\
&= \gamma \frac{z^\gamma(h^\sigma(t))}{z(h^\sigma(t))}(z(h(t)))^\Delta \\
&\ge \gamma \frac{h(t)}{h^\sigma(t)} \frac{z^\gamma(h^\sigma(t))}{z(h(t))}(z(h(t)))^\Delta \\
&\ge \gamma \frac{h(t)}{h^\sigma(t)} z^{\gamma-1}(h(t))(z(h(t)))^\Delta,
\end{aligned}$$

which means that

$$w^\Delta(t) \le -A(t)q(t) + A^\Delta(t)\left(\frac{w(t)}{A(t)}\right)^\sigma - \gamma A(t)\frac{h(t)}{h^\sigma(t)}\frac{(z(h(t)))^\Delta}{z(h(t))}\left(\frac{w(t)}{A(t)}\right)^\sigma.$$

The remainder of the proof is similar to that in Theorem 1 and so we omit it here. The proof is complete. □

Remark 1. *The term $A(t)\int_t^\infty q(s)\Delta s$ can be deleted in* (6) *of Theorem* 1 *and* (14) *of Theorem* 2*, respectively. However, it is not difficult to see that conditions* (6) *and* (14) *are weaker than those without $A(t)\int_t^\infty q(s)\Delta s$ and easier to be satisfied.*

Next, we give a definition as follows. Let $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. Define

$$\mathscr{H} = \{H \in \mathrm{C}^1(D,[0,\infty)) : H(t,t) = 0, H(t,s) > 0, H_2^\Delta(t,s) \leq 0, t > s \geq t_0\},$$

where H_2^Δ is the partial derivative of H with respect to s. Then, the following results are obtained.

Theorem 3. *When $\gamma \geq 1$, assume that* (A1)–(A5) *hold. For any $t_1 \in [t_0,\infty)_\mathbb{T}$, if there exist a sufficiently large $T \in [t_1,\infty)_\mathbb{T}$ and two functions $H \in \mathscr{H}$ and $A \in \mathrm{C}^1_{\mathrm{rd}}([T,\infty)_\mathbb{T},(0,\infty))$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Bigg[H(t,s)A(s)q(s) - \frac{1}{(1+\gamma)^{1+\gamma}} \left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1} \frac{\left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)^{1+\gamma}}{(H(t,s)A(s)h^\Delta(s))^\gamma} \Bigg] \Delta s = \infty, \quad (15)$$

where r_2^* and δ are defined as in Theorem 1*, then every solution x of* (1) *is oscillatory or $\lim_{t\to\infty} x(t)$ exists.*

Proof. Suppose that (1) is not oscillatory. Similarly, assume that x is an eventually positive solution to (1). It is clear that z is eventually positive or $\lim_{t\to\infty} x(t) = 0$ by Lemma 3. If $\lim_{t\to\infty} x(t) = 0$, then the theorem is proved. If z is eventually positive, then in view of Lemma 2, we obtain that z^Δ is eventually positive or eventually negative. If z^Δ is eventually negative, then $\lim_{t\to\infty} z(t)$ and $\lim_{t\to\infty} x(t)$ exist by virtue of Lemma 1, which also completes the proof of the theorem.

If z^Δ is eventually positive, then there exists a $t_1 \in [t_0,\infty)_\mathbb{T}$ such that for any $t \in [t_1,\infty)_\mathbb{T}$, we have $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, $z(t) > 0$, $z^\Delta(t) > 0$, and there exists a sufficiently large $T \in [t_1,\infty)_\mathbb{T}$ such that $h(t) \geq t_1$, $t \in [T,\infty)_\mathbb{T}$. Define w as (7). Then, by the proof of Theorem 1, we arrive at (12).

Replace t by s in (12), and multiply it by $H(t,s)$. Integrating the resulting inequality from T to t, $t \in [\sigma(T),\infty)_\mathbb{T}$, we have

$$\begin{aligned}
\int_T^t H(t,s)A(s)q(s)\Delta s \leq & -\int_T^t H(t,s)w^\Delta(s)\Delta s + \int_T^t H(t,s)A^\Delta(s)\left(\frac{w(s)}{A(s)}\right)^\sigma \Delta s \\
& - \int_T^t \gamma H(t,s)A(s)h^\Delta(s)\left(\frac{\delta(h(s))}{r_2^*(s)}\right)^{1/\gamma_2} \left[\left(\frac{w(s)}{A(s)}\right)^\sigma\right]^{1+1/\gamma} \Delta s \\
= & H(t,T)w(T) + \int_T^t H_2^\Delta(t,s)w^\sigma(s)\Delta s + \int_T^t H(t,s)A^\Delta(s)\left(\frac{w(s)}{A(s)}\right)^\sigma \Delta s \\
& - \int_T^t \gamma H(t,s)A(s)h^\Delta(s)\left(\frac{\delta(h(s))}{r_2^*(s)}\right)^{1/\gamma_2} \left[\left(\frac{w(s)}{A(s)}\right)^\sigma\right]^{1+1/\gamma} \Delta s \\
= & H(t,T)w(T) + \int_T^t \Bigg\{ \left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)\left(\frac{w(s)}{A(s)}\right)^\sigma \\
& -\gamma H(t,s)A(s)h^\Delta(s)\left(\frac{\delta(h(s))}{r_2^*(s)}\right)^{1/\gamma_2} \left[\left(\frac{w(s)}{A(s)}\right)^\sigma\right]^{1+1/\gamma} \Bigg\}\Delta s.
\end{aligned}$$

Using the inequality $\lambda a b^{\lambda-1} - a^\lambda \leq (\lambda-1)b^\lambda$, with $\lambda = 1 + 1/\gamma$,

$$a^\lambda = \gamma H(t,s)A(s)h^\Delta(s)\left(\frac{\delta(h(s))}{r_2^*(s)}\right)^{1/\gamma_2} \left[\left(\frac{w(s)}{A(s)}\right)^\sigma\right]^{1+1/\gamma},$$

and

$$b^{\lambda-1} = \frac{\gamma}{1+\gamma}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1/(1+\gamma)} \frac{H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)}{(\gamma H(t,s)A(s)h^\Delta(s))^{\gamma/(1+\gamma)}},$$

we conclude that

$$\begin{aligned}\int_T^t H(t,s)A(s)q(s)\Delta s &\le H(t,T)w(T) \\ &\quad + \int_T^t \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1} \frac{\left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)^{1+\gamma}}{(H(t,s)A(s)h^\Delta(s))^\gamma}\Delta s,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{H(t,T)}\int_T^t \Bigg[H(t,s)A(s)q(s) \\ &\quad - \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1} \frac{\left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)^{1+\gamma}}{(H(t,s)A(s)h^\Delta(s))^\gamma}\Bigg]\Delta s \le w(T) < \infty.\end{aligned}$$

This result contradicts (15). The proof is complete. □

By virtue of the proofs of Theorems 2 and 3, it is not difficult to obtain the following theorem, and so we omit the proof here.

Theorem 4. *When $0 < \gamma < 1$, assume that* (A1)–(A6) *hold. For any $t_1 \in [t_0,\infty)_{\mathbb{T}}$, if there exist a sufficiently large $T \in [t_1,\infty)_{\mathbb{T}}$ and two functions $H \in \mathscr{H}$ and $A \in \mathrm{C}^1_{\mathrm{rd}}([T,\infty)_{\mathbb{T}},(0,\infty))$ such that*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \Bigg[H(t,s)A(s)q(s) \\ &\quad - \frac{\left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)^{1+\gamma}(h^\sigma(s))^\gamma}{(1+\gamma)^{1+\gamma}\delta^{\gamma_1}(h(s))(H(t,s)A(s)h(s)h^\Delta(s))^\gamma}\Bigg]\Delta s = \infty,\end{aligned}$$

where δ is defined as in Theorem 1, then every solution x of (1) is oscillatory or $\lim_{t\to\infty} x(t)$ exists.

4. Examples and Discussion

Two examples are presented to show the applications of our results. The first example is given to demonstrate Theorems 1 and 2.

Example 1. *Let $\mathbb{T} = \bigcup_{n=1}^\infty [2n-1,2n]$. For $t \in [5,\infty)_{\mathbb{T}}$, consider*

$$\left(t^2\left(\left(r_2(t)\left(\left(x(t) - \frac{t-1}{2t}x(t-4)\right)^\Delta\right)^{1/5}\right)^\Delta\right)^{\gamma_1}\right)^\Delta + \frac{x^{\gamma_1/5}(t-2)}{t^{3/2}} = 0. \tag{16}$$

Here, $r_1(t) = t^2$, $\gamma_2 = 1/5$, $p(t) = (t-1)/(2t)$, $g(t) = t-4$, $h(t) = t-2$, and $f(t,x) = x^{\gamma_1/5}/t^{3/2}$. In view of (A5)*, we can take $q(t) = 1/t^{3/2}$.*

Case 1: $r_2(t) = 1/t$ and $\gamma_1 = 7$, which means that $\gamma = 7/5 > 1$. It is not difficult to see that the coefficients of (16) *satisfy* (A1)–(A5)*. Moreover, we have*

$$r_2^*(t) = \frac{1}{t-2}, \quad \delta(h(t)) = \int_{t_1}^{t-2}\frac{\Delta s}{s^{2/7}} = O(t^{5/7}).$$

Taking $A(t) = t$, it follows that

$$\begin{aligned}
&\limsup_{t\to\infty}\left\{A(t)\int_t^\infty q(s)\Delta s + \int_T^t \left[A(s)q(s) - \frac{1}{(1+\gamma)^{1+\gamma}}\left(\frac{r_2^*(s)}{\delta(h(s))}\right)^{\gamma_1}\frac{\left(A^\Delta(s)\right)^{1+\gamma}}{\left(A(s)h^\Delta(s)\right)^\gamma}\right]\Delta s\right\} \\
&= \limsup_{t\to\infty}\left\{t\int_t^\infty \frac{1}{s^{3/2}}\Delta s + \int_T^t \left[\frac{1}{s^{1/2}} - \left(\frac{5}{8}\right)^{8/5}\left(\frac{1/(s-2)}{O(s^{5/7})}\right)^7 \frac{1}{s^{7/5}}\right]\Delta s\right\} \\
&= \limsup_{t\to\infty}\left\{O(t^{1/2}) + \int_T^t \left[O(s^{-1/2}) - O(s^{-67/5})\right]\Delta s\right\} = \infty.
\end{aligned}$$

Therefore, by virtue of Theorem 1, we conclude that every solution x of (16) is oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Case 2: $r_2(t) = 1$ and $\gamma_1 = 3$, which means that $0 < \gamma = 3/5 < 1$. It is not difficult to see that the coefficients of (16) satisfy (A1)–(A6). *Furthermore, we have*

$$\delta(h(t)) = \int_{t_1}^{t-2} \frac{\Delta s}{s^{2/3}} = O(t^{1/3}).$$

Letting $A(t) = t$, it follows that

$$\begin{aligned}
&\limsup_{t\to\infty}\left\{A(t)\int_t^\infty q(s)\Delta s + \int_T^t \left[A(s)q(s) - \frac{\left(A^\Delta(s)\right)^{1+\gamma}(h^\sigma(s))^\gamma}{(1+\gamma)^{1+\gamma}\delta^{\gamma_1}(h(s))\left(A(s)h(s)h^\Delta(s)\right)^\gamma}\right]\Delta s\right\} \\
&= \limsup_{t\to\infty}\left\{t\int_t^\infty \frac{1}{s^{3/2}}\Delta s + \int_T^t \left[\frac{1}{s^{1/2}} - \frac{O(s^{3/5})}{(8/5)^{8/5}\cdot O(s)\cdot (s(s-2))^{3/5}}\right]\Delta s\right\} \\
&= \limsup_{t\to\infty}\left\{O(t^{1/2}) + \int_T^t \left[O(s^{-1/2}) - O(s^{-8/5})\right]\Delta s\right\} = \infty.
\end{aligned}$$

Therefore, we conclude that every solution x of (16) is oscillatory or $\lim_{t\to\infty} x(t)$ exists via Theorem 2.

Now, we give the second example to demonstrate Theorem 3.

Example 2. *Let $\mathbb{T} = \bigcup_{n=0}^\infty [2\cdot 3^n, 3^{n+1}]$. For $t \in [6,\infty)_\mathbb{T}$, consider*

$$\left(\frac{1}{t}\left(\left(t^3\left(\left(x(t) - \frac{1}{t^2}x\left(\frac{t}{3}\right)\right)^\Delta\right)^5\right)^\Delta\right)^{3/5}\right)^\Delta + t^\lambda x^3\left(\frac{t}{9}\right) = 0. \qquad (17)$$

Here, $r_1(t) = 1/t$, $r_2(t) = t^3$, $p(t) = 1/t^2$, $g(t) = t/3$, $h(t) = t/9$, $\gamma_1 = 3/5$, $\gamma_2 = 5$, $\gamma = 3 > 1$, and $f(t,x) = t^\lambda x^3$, where $\lambda \in \mathbb{R}$. In view of (A5), *we can take $q(t) = t^\lambda$. It is obvious that the coefficients of (17) satisfy* (A1)–(A5). *Moreover, we have*

$$r_2^*(t) = \left(\frac{t}{9}\right)^3, \quad \delta(h(t)) = \int_{t_1}^{t/9} \frac{\Delta s}{(1/s)^{5/3}} = O(t^{8/3}).$$

Taking $H(t,s) = (t-s)^2$ *and* $A(t) = 1$*, we conclude that*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Bigg[H(t,s)A(s)q(s) \\
&\qquad\qquad - \frac{1}{(1+\gamma)^{1+\gamma}} \left(\frac{r_2^*(s)}{\delta(h(s))} \right)^{\gamma_1} \frac{\left(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\right)^{1+\gamma}}{(H(t,s)A(s)h^\Delta(s))^\gamma} \Bigg] \Delta s \\
&= \limsup_{t\to\infty} \frac{1}{(t-T)^2} \int_T^t \left[(t-s)^2 s^\lambda - \frac{1}{256} \left(\frac{(s/9)^3}{O(s^{8/3})} \right)^{3/5} \frac{O(s^4)}{O(s^6)} \right] \Delta s \\
&= \limsup_{t\to\infty} \frac{1}{(t-T)^2} \int_T^t \left[O(s^{2+\lambda}) - O(s^{-9/5}) \right] \Delta s.
\end{aligned}$$

When $\lambda > -1$*, it follows that* (15) *holds. Then, by virtue of Theorem* 3*, we deduce that every solution* x *of* (17) *is oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Remark 2. *Due to the fact that the derivative* z^Δ *does not fixed, it is difficult to establish criteria which ensure oscillation of all solutions of* (1)*. It is interesting to suggest a different method to study* (1) *for future research. It would be of interest to investigate* (1) *with a damping term or a nonlinear neutral term; see, for instance, the papers [3,15] for more details, respectively.*

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