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Article

Certain Properties of a Class of Functions Defined by Means of a Generalized Differential Operator

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Abstract: In this article, we construct a new subclass of analytic functions involving a generalized differential operator and investigate certain properties including the radius of starlikeness, closure properties and integral means result for the class of analytic functions with negative coefficients. Further, the relationship between the results and some known results in literature are also established.

Keywords: closure; integral transform; convex set; fractional calculus

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1. Introduction and Preliminaries

Let A be the class of analytic functions f in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0 = f'(0) - 1$ and $f^k(0) = 0$ $(2 \le k \le n)$

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \quad n \in \mathbb{N}. \tag{1}$$

Let T be the subclass of A consisting of functions of the form (see [1])

$$f(z) = z - \sum_{k=2}^{+\infty} a_k z^k, \quad a_k \ge 0, z \in \mathbb{U}. \tag{2}$$

We also define the identity function as

$$e(z) = z. \tag{3}$$

1.1. Generalized Differential Operator

Oluwayemi and Vijaya in [2], using differential operator $D^{n,\lambda}_{\alpha,\beta,\mu_1,\mu_2}$ defined in [3] as

$$D^{m,\lambda}_{\alpha,\beta,\mu_1,\mu_2} f(z) = z + \sum_{k=2}^{+\infty} \Upsilon^m a_k z^k \tag{4}$$

where

$$\Upsilon = \frac{a + (\alpha - \beta)(\lambda + \mu_2 - \mu_1)(k-1) + b}{a+b} \tag{5}$$

and $a, b \ge 0$, $a + b \ne 0$, $\alpha > \beta \ge 0$, $\lambda > \mu_2$, $\mu_2 \le \mu_1, m \in \mathbb{N}_0$, introduced the class $SI_m(\xi, \eta, \delta)$.

For the aforementioned differential operator, we can notice here that it is a generalized form of several differential operators introduced earlier by various researchers. For a function f in the form (1) with $a = 1, \mu_1 = \mu_2, b = 0, \alpha - \beta = 1$, one obtains the operator D_λ^m introduced and studied by Al-Oboudi [4] which also reduces to the Salagean differential operator [5] for $\lambda = 1$. Additionally, for $a = 1, \mu_1 = \mu_2, \lambda = 1, \alpha - \beta = 1, b \geq 0$, we find the operator I_l^m which was studied recently by Cho and Srivastava [6] and Cho and Kim [7]. For $a = 1, \mu_1 = \mu_2, \lambda = 1, b \geq 0$, we re-established the differential operator defined in [8].

1.2. Class $SI_m(\xi, \eta, \delta)$

Recently, Oluwayemi and Vijaya introduced the following class $SI_m(\xi, \eta, \delta)$ (see [2]):

$$SI_m(\xi, \eta, \delta) = \left\{ f \in A : \Re\left(\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right) + \xi z} \right) > \delta \right\}, \tag{6}$$

where $\xi \geq 1$, $0 \leq \eta \leq 1; 0 \leq \delta \leq 1; z \in \mathbb{U}$ and $m \in \mathbb{N}_0$. Further they obtained the necessary and sufficient conditions and closure properties for $f \in SI_m(\xi, \eta, \delta)$. Motivated by the work in [2,9,10], the authors investigated some geometric properties of the class of functions belonging to the class $SI_m(\xi, \eta, \delta)$. This class extends classes $K^m_{\alpha,\lambda,\gamma,\beta}(n, \delta)$ and $SI_m(\delta)$ investigated by [9,10], respectively.

We note that by specializing the parameter ξ we state the following subclasses:

$$SI^*_m(\eta, \delta) = SI_m(1, \eta, \delta) = \left\{ f \in A : \Re\left(\frac{z\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)'}{\eta\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right) + z} \right) > \delta \right\}, \tag{7}$$

and

$$RI_m(\delta) = SI_m(1, 0, \delta) = \left\{ f \in A : \Re\left(\left(D^{m,\lambda}_{\alpha,\beta,\mu_2,\mu_1} f(z)\right)' \right) > \delta \right\}. \tag{8}$$

We now recall the coefficient estimate for $f \in SI_m(\xi, \eta, \delta)$:

Lemma 1 ([2])**.** *Let $0 \leq \delta \leq 1$, $\xi \geq 1$ and $0 \leq \eta \leq 1$. Suppose the function $f(z)$ is defined by (1). Then, $f \in SI_m(\xi, \eta, \delta)$ if and only if*

$$\sum_{k=2}^{+\infty} [k - \eta(2 - \delta)]\Upsilon^m a_k \leq \eta(2 - \delta) + \xi - 1. \tag{9}$$

The result is sharp for

$$f(z) = z - \frac{[2 - \eta(2 - \delta)]\Upsilon^m}{[\eta(2 - \delta) + \xi - 1]} z^2. \tag{10}$$

Making use of Lemma 1, in our present article, we investigate the radius of starlikeness, closure properties and integral means results for $f \in SI_m(\xi, \eta, \delta)$.

2. Main Results

2.1. Radius Properties for Class $SI_m(\xi, \eta, \delta)$

In this section we provide the radius properties for the starlike function of order ϖ, the convex function of order ϖ and the closed-to-convex function of order ϖ, $0 \leq \varpi < 1$, respectively.

Theorem 1. *Let the function $f(z)$ be in the class $SI_m(\xi, \eta, \delta)$. Then, $f(z)$ is starlike of the order $\varpi(0 \leq \varpi < 1)$ in $|z| < R_1$, where*

$$R_1 = \inf_k \left\{ \frac{(1 - \varpi)[k - \eta(2 - \delta)]\Upsilon^m}{[\eta(2 - \delta) + \xi - 1](k - \varpi)} \right\}^{\frac{1}{k-1}}, \qquad k \geq 2.$$

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \varpi$. That is,

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &= \left|\frac{\sum_{k=2}^{+\infty}(k-1)a_k z^{k-1}}{1 + \sum_{k=2}^{+\infty} a_k z^{k-1}}\right| \\ &\le \frac{\sum_{k=2}^{+\infty}(k-1)a_k |z|^{k-1}}{\left(1 + \sum_{k=2}^{+\infty} a_k |z|^{k-1}\right)} \\ &= 1 - \varpi\end{aligned}$$

which implies

$$|z| \le \left\{\frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1](k - \varpi)}\right\}^{\frac{1}{k-1}}; \ |z| < R_1.$$

Thus,

$$R_1 = \inf_k \left\{\frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1](k - \varpi)}\right\}^{\frac{1}{k-1}}; \qquad k \ge 2.$$

as required. □

The result is sharp for the univalent function $f(z) = z + \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1](k-\varpi)} z^k, \quad k \ge 2.$

Theorem 2. *Let the function $f(z)$ be in the class $SI_m(\xi, \eta, \delta)$. Then, $f(z)$ is convex of the order ϖ $(0 \le \varpi < 1)$ in $|z| < R_2$, where*

$$R_2 = \inf_k \left\{\frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1]k(k - \varpi)}\right\}^{\frac{1}{k-1}}; \quad k \ge 2. \tag{11}$$

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \varpi, \ |z| < R_2$.

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{k=2}^{+\infty} k(k-1)a_k z^{k-1}}{1 + \sum_{k=2}^{+\infty} k a_k z^{k-1}}\right| \le \frac{\sum_{k=2}^{+\infty} k(k-1)a_k |z|^{k-1}}{1 + \sum_{k=2}^{+\infty} k a_k |z|^{k-1}} < 1 - \varpi.$$

To prove this theorem, we must show that

$$\frac{\sum_{k=2}^{+\infty} k(k-1)a_k |z|^{k-1}}{1 + \sum_{k=2}^{+\infty} k a_k |z|^{k-1}} < 1 - \varpi$$

$$\left(\frac{k(k-\varpi)}{1-\varpi}\right) a_k |z|^{k-1} \le 1.$$

By Lemma 1, we obtain

$$|z|^{k-1} \le \frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1]k(k - \varpi)}.$$

In other words,

$$R_2 = \inf_k \left\{\frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1]k(k - \varpi)}\right\}^{\frac{1}{k-1}},$$

which completes the proof. □

The result is sharp for

$$f(z) = z - \frac{(1-\varpi)[k - \eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta) + \xi - 1]k(k - \varpi)} z^k, \quad k \ge 2.$$

Theorem 3. *Let the function $f(z)$ be in the class $SI_m(\xi,\eta,\delta)$. Then, $f(z)$ is closed-to-convex of the order ϖ $(0 \le \varpi < 1)$ in $|z| < R_3$, where*

$$R_3 = inf_k \left\{ \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{k[\eta(2-\delta)+\xi-1]} \right\}^{\frac{1}{k-1}} k \ge 2. \tag{12}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{k[\eta(2-\delta)+\xi-1]} z^k, \ \ k \ge 2.$$

Proof. It suffices to show that $|f'(z) - 1| = 1 - \varpi$ $(0 \le \varpi < 1$ for $|z| < R_3)$.

Thus,

$$|f'(z) - 1| = \left| 1 + \sum_{k=2}^{+\infty} k a_k z^{k-1} - 1 \right| = \left| \sum_{k=2}^{+\infty} k a_k z^{k-1} \right| \le \sum_{k=2}^{+\infty} k a_k |z|^{k-1}.$$

Since

$$|f'(z) - 1| \le \sum_{k=2}^{+\infty} k a_k |z|^{k-1} \le 1 - \varpi$$

then,

$$\sum_{k=2}^{+\infty} \left(\frac{k}{1-\varpi} \right) a_k |z|^{k-1} \le 1. \tag{13}$$

Since $f \in SI_m(\xi,\eta,\delta)$, using Lemma 1 and (13) holds if

$$\frac{k|z|^{k-1}}{(1-\varpi)} \le \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{k[\eta(2-\delta)+\xi-1]} \ k \ge 2.$$

Then by further simplification, we have that

$$|z| \le \left\{ \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{k[\eta(2-\delta)+\xi-1]} \right\}^{\frac{1}{k-1}}.$$

Hence,

$$R_3 = \inf_k \left\{ \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{k[\eta(2-\delta)+\xi-1]} \right\}^{\frac{1}{k-1}}; \quad k \ge 2.$$

□

2.2. Application of Integral Operators

In this section, the authors applied some integral operators in geometric functions theory associated with class $SI_m(\xi,\eta,\delta)$. This is motivated by the work of Jadhav in [11].

Definition 1 ([11])**.** *Let $f(z)$ be defined by (1). Then*

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{x} \right)^{\sigma-1} f(x) dx, \quad \sigma > 0$$

$$= z + \sum_{k=2}^{+\infty} \left(\frac{2}{1+k} \right)^\sigma a_k z^k.$$

Theorem 4. *Let $f \in SI_m(\xi,\eta,\delta)$. Then, the Jung–Kim Stravastava integral operator defined by*

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{x} \right)^{\sigma-1} f(x) dx, \ \ \sigma > 0$$

also belongs to the class $SI_m(\xi,\eta,\delta)$.

Proof. Since f is given by (1), we have

$$I^{\sigma}f(z)=\sum_{k=2}^{+\infty}\left(\frac{2}{1+k}\right)^{\sigma}a_k z^k$$

Since $\sigma>0$, then

$$\left(\frac{2}{1+k}\right)^{\sigma}\leq 1.$$

Thus, by Lemma 1, we get

$$\sum_{k=2}^{+\infty}[k-\eta(2-\delta)]\Upsilon^m\left(\frac{2}{1+k}\right)^{\sigma}a_k\leq[\eta(2-\delta)+\xi-1].$$

Hence $I^{\sigma}f(z)\in SI_m(\xi,\eta,\delta)$. □

Theorem 5. *Let $f\in SI_m(\xi,\eta,\delta)$. Then, the Jung–Kim Stravastava integral operator defined by*

$$L_x(z)=(1-x)f(z)+x\int_0^z\frac{f(y)}{y}dy$$

is also in the class $SI_m(\xi,\eta,\delta)$.

Proof. Since f is given by (1), we have

$$L_x(z)=(1-x)\left(z+\sum_{k=2}^{+\infty}a_k z^k\right)+x\int_0^z\frac{f(y)}{y}dy$$

$$=z-xz+\sum_{k=2}^{+\infty}a_k z^k-\sum_{k=2}^{+\infty}xa_k z^k+xz+\sum_{k=2}^{+\infty}a_k\frac{x}{k}z^k$$

$$=z+\sum_{k=2}^{+\infty}\left(1-x+\frac{x}{k}\right)a_k z^k.$$

Since $\left(1-x+\frac{x}{k}\right)<1$ for all $x\geq 0$ and $k\geq 2$. Then, by Lemma 1

$$\sum_{k=2}^{+\infty}\frac{[k-\eta(2-\delta)]\Upsilon^m\left(1-x+\frac{x}{k}\right)}{\eta(2-\delta)+\xi-1}a_k\leq 1.$$

Thus $f\in SI_m(\xi,\eta,\delta)$. □

2.3. Integral Transformation Properties for Class $SI_m(\xi,\eta,\delta)$

Following the works of Murugusundaramoorthy et al. [12,13], we discuss integral transformation results for a function $f(z)\in SI_m(\xi,\eta,\delta)$.

Definition 2. *For $f\in A$, we define the integral transform*

$$V_{\sigma}(f)(z)=\int_0^1\sigma(t)\frac{f(tz)}{t}dt$$

for a real valued, non-negative weight function normalized σ so that $\int_0^1 \sigma(t)dt = 1$. Since special cases of $\sigma(t)$ are particularly interesting, such as $\sigma(t) = (1+c)t^c$, $c > -1$, for which V_σ is known as the Bernadi operator, and

$$\sigma(t) = \frac{(c+1)^\lambda}{v(\lambda)} t^c \left(\log \frac{1}{t}\right)^{\lambda-1}, \; c > -1, \;\; \lambda > 0,$$

which gives the Komatu operator (for details, see [13]).

We now show that the class $SI_m(\xi,\eta,\delta)$ is closed under $V_\sigma(f)(z)$.

Theorem 6. *Let $f \in SI_m(\xi,\eta,\delta)$. Then, $V_\sigma(f)(z)$ also belongs to the class $SI_m(\xi,\eta,\delta)$.*

Proof. From Definition 2, it follows that

$$\begin{aligned} V_\sigma(f)(z) &= \frac{(c+1)^\lambda}{\Gamma(\lambda)} \int_0^1 (-1)^{\lambda-1} t^c (\log t)^{\lambda-1} \left(z + \sum_{k=2}^{+\infty} |a_k| t^{k-1}\right) dt \\ &= \frac{(-1)^{\lambda-1}(c+1)^\lambda}{\Gamma(\lambda)} \lim_{\to 0^+} \left[\int_0^1 (-1)^{\lambda-1} t^c (\log t)^{\lambda-1} \left(z + \sum_{k=2}^{\infty} |a_k| t^{k-1}\right) dt\right] \\ &= z + \sum_{k=2}^{+\infty} \left(\frac{c+1}{c+n}\right)^\lambda a_k z^k. \end{aligned}$$

We now show that $V_\sigma(f(z)) \in SI_m(\xi,\eta,\delta)$.

$$\sum_{k=2}^{+\infty} \frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1} \left(\frac{c+1}{c+k}\right)^\lambda a_k \le 1. \tag{14}$$

In view of Lemma 1, $f \in SI_m(\xi,\eta,\delta)$ if and only if

$$\sum_{k=2}^{+\infty} \frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1} a_k \le 1.$$

Obviously, $c+1 < c+k$ for all $k \ge 2$ and $c > -1$, which implies that $\frac{c+1}{c+k} < 1$. Thus,

$$\sum_{k=2}^{+\infty} \frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1} \left(\frac{c+1}{c+k}\right)^\lambda a_k \le 1.$$

□

Theorem 7. *Let $f \in SI_m(\xi,\eta,\delta)$. Then, $V_\sigma(f)(z)$ is starlike of the order ϖ $(0 \le \varpi < 0)$ in $|z| < R_1$, where*

$$R_1 = \inf\left[\left(\frac{c+k}{c+1}\right)^\lambda \frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1](k-\varpi)}\right]^{\frac{1}{k-1}}, \;\; k \ge 2 \tag{15}$$

Proof. We need to show that

$$\left|\frac{z(V_\sigma f(z))'}{V_\sigma f(z)} - 1\right| < 1 - \varpi.$$

Thus,

$$\left|\frac{\sum_{k=2}^{+\infty}(k-1)\left(\frac{c+1}{c+k}\right)^{\lambda}a_kz^{k-1}}{1+\sum_{k=2}^{+\infty}\left(\frac{c+1}{c+k}\right)^{\lambda}a_kz^{k-1}}\right| \leq \frac{\sum_{k=2}^{+\infty}(k-1)\left(\frac{c+1}{c+k}\right)^{\lambda}|a_k||z|^{k-1}}{1+\sum_{k=2}^{+\infty}\left(\frac{c+1}{c+k}\right)^{\lambda}|a_k||z|^{k-1}}.$$

That is,

$$\sum_{k=2}^{+\infty}(k-\varpi)\left(\frac{c+1}{c+k}\right)^{\lambda}|a_k||z|^{k-1} < [1-\varpi](1-\sum_{k=2}^{+\infty}\left(\frac{c+1}{c+k}\right)^{\lambda}|a_k||z|^{k-1}).$$

On further simplification, we have the required result (15):

$$|z|^{k-1} < \left(\frac{c+k}{c+1}\right)^{\lambda}\frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1](k-\varpi)},$$

which completes the proof. □

Remark 1. *It is known that $f(z)$ is convex if and only if $zf'(z)$ is starlike. Hence, we have the following theorem.*

Theorem 8. *Let $f \in SI_m(\xi,\eta,\delta)$. Then, $V_\sigma(f)(z)$ is convex of the order ϖ $(0 \leq \varpi < 0)$ in $|z| < R_2$, where*

$$R_2 = \inf\left[\left(\frac{c+k}{c+1}\right)^{\lambda}\frac{(1-\varpi)[k-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1]k(k-\varpi)}\right]^{\frac{1}{k-1}}, \quad k \geq 2 \tag{16}$$

Proof. The proof follows from Theorem 7 and Remark 1. □

Remark 2. *By fixing $\lambda = 1$, one can easily prove that the class is closed under the Bernardi operator.*

2.4. Convolution Properties

Following the work of Murugusundaramoorthy et al. [13], we determine the convolution properties for functions belonging to the class $SI_m(\xi,\eta,\delta)$.

Theorem 9. *If $f(z) = z+\sum_{k=2}^{\infty}a_kz^k$ and $g(z) = z+\sum_{k=2}^{\infty}b_kz^k$ belong to $SI_m(\xi,\eta,\delta)$, then the convolution of f and g given by $(f*g)(z) = z+\sum_{k=2}^{\infty}a_kb_kz^k$, which also belongs to $SI_m(\xi,\eta,\delta)$.*

Proof. Let $f(z)$ and $g(z)$ belong to $SI_m(\xi,\eta,\delta)$; then

$$\sum_{k=2}^{\infty}\frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1}a_k \leq 1 \ \ and \ \ \sum_{k=2}^{\infty}\frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1}b_k \leq 1.$$

By the Cauchy–Schwartz inequality, we have

$$\sum_{k=2}^{\infty}\frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1}a_kb_k = \sum_{k=2}^{\infty}\left\{\frac{[k-\eta(2-\delta)]\Upsilon^m}{\eta(2-\delta)+\xi-1}\sqrt{a_kb_k}\right\}\sqrt{a_kb_k}$$

$$\leq \left(\sum_{k=2}^{\infty}\left\{\frac{[k-\eta(2-\delta)]\Upsilon^m b_k}{\eta(2-\delta)+\xi-1}\right\}a_k\right)^{\frac{1}{2}}.$$

which implies that

$$\left(\sum_{k=2}^{\infty}\left\{\frac{[k-\eta(2-\delta)]\Upsilon^m a_k}{\eta(2-\delta)+\xi-1}\right\}b_k\right)^{\frac{1}{2}} \leq 1.$$

□

2.5. Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $SI_m(\xi, \eta, \delta)$.

Lemma 2 ([14]). *If the functions f and g are analytic in $\mathbb{U}$ with $g \prec f$, then for $\kappa > 0$, and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta. \tag{17}$$

In [1], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality and conjectured in [15] and settled in [16], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\kappa} d\theta$$

for all $f \in T$, $\kappa > 0$ and $0 < r < 1$. In [16], he also proved his conjecture for the subclasses of starlike functions of order α and convex functions of order α.

Applying Lemma 2 and Lemma 1, we prove the following result.

Theorem 10. *Suppose $f(z) \in SI_m(\xi, \eta, \delta)$ and $f_2(z)$ is defined by $f_2(z) = z - \frac{[2-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1]} z^2$. Then, for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} |f(z)|^{\kappa} d\theta \leq \int_0^{2\pi} |f_2(z)|^{\kappa} d\theta. \tag{18}$$

Proof. For $f(z) = z - \sum_{k=2}^{+\infty} a_k z^k$, (18) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{+\infty} a_k z^{k-1}\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|1 - \frac{[2-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1]} z\right|^{\kappa} d\theta.$$

By Lemma 2, it suffices to show that

$$1 - \sum_{k=2}^{+\infty} a_k |z|^{k-1} \prec 1 - \frac{[2-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1]} |z|.$$

Setting

$$1 - \sum_{k=2}^{+\infty} a_k |z|^{k-1} = 1 - \frac{[2-\eta(2-\delta)]\Upsilon^m}{[\eta(2-\delta)+\xi-1]} w(z), \tag{19}$$

and using (9), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=2}^{+\infty} \frac{[\eta(2-\delta)+\xi-1]}{[k-\eta(2-\delta)]\Upsilon^m} a_k z^{k-1} \right| \\ &\leq |z| \sum_{k=2}^{+\infty} \frac{[\eta(2-\delta)+\xi-1]}{[k-\eta(2-\delta)]\Upsilon^m} |a_k| \\ &\leq |z|. \end{aligned}$$

This completes the proof Theorem 10. □

3. Conclusions

Several results in literature describe the characteristics of univalent (or multivalent) analytic functions involving various types of linear operators associated with the operations of integration as well as differentiation; see, for instance [4–8,11,13]. Due to the generalized nature of the class of operators defined by (4), our results (Theorems 1–8) would include (in view of the special cases discussed in Section 1) the known (or new) results pertaining to the univalent case of recent developments. The class of function studied in the work generalizes some known classes of functions. For examples, see class $SI_m(1,0,\delta) \equiv SI_m(\delta)$ investigated by [10] and $SI_m(0,1,\delta) \equiv K^m_{\alpha,\lambda,\gamma,\beta}(n,\delta)$ studied by [9]. Further, by taking ξ and η, one can deduce the results for the function class given in (7) and (8). Following the works of Murugusundaramoorthy et al. [13], one can extend the study for results on Hölder inequalities, partial sums and subordination for $f \in SI_m(\xi,\eta,\delta)$. We also consider for future study the papers [17–20].

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References

1. Silverman, H. Univalent functions with negative coefficients. *Proc. Am. Math. Soc.* **1975**, *51*, 109–116. [CrossRef]
2. Oluwayemi, M.O.; Vijaya, K. On a study of univalent functions associated with fractional calculus. *Int. J. Math. Compu. Sci. (IJMCS)* **2022**, *17*, 110–116.
3. Alamoush, A.G.; Darus, M. On certain subclasses of analytic functions defined by generalized differential operators. *Romai J.* **2015**, *11*, 17–31.
4. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Int. J. Math. Math. Sci.* **2004**, *27*, 1429–1436. [CrossRef]
5. Sălăgean, G.S. Subclasses of univalent functions. In *Complex Analysis—Fifth Romanian-Finnish Seminar*; Springer: Berlin/Heidelberg, Germany, 1983; Volume 1013, pp. 362–372.
6. Cho, N.E.; Srivastava, H.M. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Math. Comput. Model.* **2003**, *37*, 39–49. [CrossRef]
7. Cho, N.E.; Kim, T.H. Multiplier transformations and strongly close-to-convex functions. *Bull. Korean Math. Soc.* **2003**, *40*, 399–410. [CrossRef]
8. Cătaş, A. On certain class of p-valent functions defined by a new multiplier transformations. In *Proceedings Book of the International Symposium GFTA*; Istanbul Kultur University: Istanbul, Turkey, 2007; pp. 241–250.
9. Alamri, M.; Darus, M. On Univalent Function With Negative Coefficients by Using Differential Operator. In *AIP Conference Proceedings*; American Institute of Physics: College Park, MD, USA, 2014; Volume 1614, p. 871. [CrossRef]
10. Alb Lupas, A. A new comprehensive class of analytic functions defined by multiplier transformation. *Math. Comput. Model.* **2011**, *54*, 2355–2362. [CrossRef]
11. Jadhav, P.G. Application of fractional calculus and other properties of univalent functions associated with subordination. *Int. J. Recent Sci. Res.* **2016**, *7*, 14819–14824.
12. Murugusundaramoorthy, G. Certain subclasses of univalent functions associated with a unification of the Srivastava-Attiya and Cho-Saigo-Srivastava operators. *Novi Sad J. Math.* **2015**, *45*, 59–76. [CrossRef]
13. Murugusundaramoorthy, G.; Vijaya, K.; Deepa, K. Holder Inequalities for a subclass of univalent functions involving Dziok-Srivastava Operator. *Glob. J. Math. Anal.* **2013**, *1*, 74–82. [CrossRef]
14. Littlewood, J.E. On inequalities in theory of functions. *Proc. Lond. Math. Soc.* **1925**, *23*, 481–519. [CrossRef]
15. Silverman, H. A survey with open problems on univalent functions whose coefficients are negative. *Rocky Mt. J. Math.* **1991**, *21*, 1099–1125. [CrossRef]

16. Silverman, H. Integral means for univalent functions with negative coefficients. *Houst. J. Math.* **1997**, *23*, 169–174.
17. Breaz, D.; Cotîrlă, L.I. The Study of the New Classes of m-Fold Symmetric bi-Univalent Functions. *Mathematics* **2022**, *10*, 75. [CrossRef]
18. Koskela, P.; Manojlović, V. Quasi-nearly subharmonic functions and quasiconformal mappings. *Potential Anal.* **2012**, *37*, 187–196. [CrossRef]
19. Oluwayemi, M.O.; Faisal, I. A new family of analytic functions associated with multiplier transformation. *Sci. Afr.* **2021**, *12*, e00754. [CrossRef]
20. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer Nature: Cham, Switzerland, 2019.