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Characterizing Base in Warped Product Submanifolds of Complex Projective Spaces by Differential Equations

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Abstract: In this study, a link between the squared norm of the second fundamental form and the Laplacian of the warping function for a warped product pointwise semi-slant submanifold M^n in a complex projective space is presented. Some characterizations of the base N_T of M^n are offered as applications. We also look at whether the base N_T is isometric to the Euclidean space $\mathbb{R}^p$ or the Euclidean sphere $\mathbb{S}^p$, subject to some constraints on the second fundamental form and warping function.

Keywords: warped products; complex projective spaces; Dirichlet energy; Ricci curvature; ordinary differential equations

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1. Introduction and Statement of Results

Throughout the article, we shall utilize the acronyms listed as: 'WP' for warped product, 'WF' for warping function, 'CPS' for complex projectve spaces, "WPPSS" for warped product pointwise semi-slant submanifold, and 'SFF' for second fundamental form. An essential goal in Riemannian geometry is to find the relationship between extrinsic and intrinsic invariants on some given warped product manifolds. One way is to study the warping functions which arise as solutions of the Euler–Lagrange equations and partial differential equations for conditions on curvature functions. The philosophy of finding some Riemannian invariants to search the best relationship between intrinsic and extrinsic invariant for a given Riemannian manifold. In this respect, B.Y. Chen [1,2] provided the inequality for the second fundamental form as a main intrinsic invariant and characterized the Laplacian of the warping function as a main extrinsic invariant for CR-warped products in complex space forms. He also demonstrated the complete classification, that satisfied the equality case of this inequality. Many achievements in warped product submanifolds theory acquired for some different space forms (see [3–6]). Another critical concept in differential geometry is the theory of warped product manifolds. Robertson-Walker spacetime, asymptotically flat spacetime, Schwarzschild spacetime, and Reissner-Nordstrom spacetime are applications of warped product manifolds found in general relativity theory in physics. Besides, the spacetime, as mentioned earlier, models can be viewed as examples of the warped product manifolds theory, (for more details see [5–8]).

On the other hand, Sahin [9] derived both types of WPSS's, $N_T \times_f N_\vartheta$ and $N_\vartheta \times_f N_T$, in a Kaehler manifold are trivial where N_T and N_ϑ are holomorphic and slant submanifolds. By considering the slant angle ϑ as a function $\vartheta : M^n \longrightarrow \mathbb{R}$, Chen-Gray [10] studied pointwise slant submanifolds of almost Hermitian manifolds. Applying this notion to warped product submanifolds in Kaehler manifolds, Sahin [11] discussed pointwise semi-slant submanifolds and WPPSS in a Kaehler manifold. He also classified that a WPPSS of type $N_T \times_f N_\vartheta$ is nontrivial with examples. In this case, such a class of WPPSS's succeeds to generalize the class of CR-warped product submanifolds [12,13]. Ali et al. [3] studied WPPSS in complex space form and acquired an inequality for the squared norm of the SFF in terms of holomorphic constant section curvature by using Gauss equation.

In the present study, the WPPSS of complex projective spaces with positive constant sectional curvature is considered. In this case, $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mathbb{C}_*^{m+1} = \mathbb{C}^{m+1} - \{0\}$ and assume the action $\mathbb{C}^*$ on $\mathbb{C}_*^{m+1}$ is expressed by γ, that is $(z_0, z_1, \dots, z_m) = (\gamma z_0, \gamma z_1, \dots, \gamma z_m)$. The set of all equivalent classes derived from this action are represented by $\mathbb{C}P^m$. If $\pi(z)$ denotes the equivalent classes contained z, then $\mathbb{C}_*^{m+1} \to \mathbb{C}P^m$ is a surjection, and it is known that $\mathbb{C}P^m$ admits a complex structure induced from the complex structure on $\mathbb{C}^{m+1}$ with a Kaehler metric such that the constant holomorphic sectional curvature equal to 4 [2,13]. It may be remarked that the almost complex J on $\mathbb{C}P^m(4)$ is induced by the almost complex structure on $\mathbb{C}^{m+1}$ *via* the Hopf fibration $\ell : \mathbb{S}^{2m+1} \to \mathbb{C}P^m(4)$ [2,13]. Hence, $\mathbb{C}P^m(4)$ is a Kaehler manifold with constant holomorphic sectional curvature is equal to 4. Inspired by this notion, our method is to derive the extrinsic condition for the SFF, squared norm and Laplacian of the WF in a warped product pointwise semi-slant submanifold of complex projective space $\mathbb{C}P^m(4)$. In this respect, we use the equation of Gauss instead the equation of Codazzi in [2] and announce our first result.

Theorem 1. *Let $\Psi : M^n = N_T \times_f N_\vartheta \longrightarrow \mathbb{C}P^{2m}(4)$ be an isometric immersion from a WPPSS $N_T \times_f N_\vartheta$ into the CPS $CP^{2m}(4)$ with constant holomorphic sectional curvature is equal to 4. Then the following equality is satisfied*

$$\mathbf{S} = q\Big(||\nabla\chi||^2 + p - \Delta\chi\Big), \tag{1}$$

where $\nabla\chi$ and $\Delta\chi$ are the gradient and the Laplacian of the WF $\chi = \ln f$ on N_T, respectively. Moreover, $\mathbf{S}$ is the squared norm of the SFF of components N_T and N_ϑ, respectively.

A relevant observation is that the second fundamental form in the left-hand side in (1) has the relation with pointwise slant function ϑ. We reach the following result as a result of Theorem 1.

Theorem 2. *Let $\Psi : M^n = N_T \times_f N_\vartheta \longrightarrow \mathbb{C}P^{2m}(4)$ be an isometric immersion from a WPPSS $N_T \times_f N_\vartheta$ into CPS $\mathbb{C}P^{2m}(4)$. Then the following equality is satisfied*

$$\|h_\nu\|^2 + 2q\cot^2\vartheta||\nabla\chi||^2 = q(p - \Delta\chi), \tag{2}$$

where h_ν is a component of h in $\Gamma(\nu)$ and ϑ is regarded as pointwise slant function. Moreover $\Gamma(\nu)$ set of tangent vectors under invariant subspace ν.

Immediately as a result of Theorem 2, we consider the warping function $\ln f$ to be a harmonic function and get the following:

Corollary 1. *Let $\Psi : M^n = N_T \times_f N_\vartheta \longrightarrow \mathbb{C}P^{2m}(4)$ be an isometric immersion from a compact WPPSS $N_T \times_f N_\vartheta$ into CPS $\mathbb{C}P^{2m}(4)$ such that $\ln f$ is a harmonic function. Then we have*

$$\|h_\nu\|^2 + 2q\cot^2\vartheta||\nabla\chi||^2 = pq. \tag{3}$$

In Geometry and Physics, boundary estimations are well-studied topics. Calin-Chang [14] presented the geometrical approach to Riemannian manifolds and derived applications to partial differential equations such as Lagrangian formalism on Riemannian manifolds.

A Riemannian manifold can be thought of as a compact Riemannian submanifold with boundary, i.e., $\partial M \neq \emptyset$. Following that, we demonstrate the following theorem:

Theorem 3 ([15]). *Let M^n be a connected and compact Riemannian manifold and ω is a positive differentiable function defined on M^n such that $\Delta\omega = 0, \ \ and \ \ \omega/\partial M = 0 \ \ on \ \ M^n$. Then $\omega = 0$.*

The gradient $\vec{\nabla}\omega$ is given by

$$g(\vec{\nabla}\omega, X) = X\omega, \quad and \ \ \vec{\nabla}\omega = \sum_{i=1}^{n} e_i(\omega)e_i, \tag{4}$$

and the Laplacian $\Delta\omega$ of ω is defined as:

$$\begin{aligned}\Delta\omega &= \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)\omega - e_i(e_i(\omega))\} \\ &= -\sum_{i=1}^{n} g(\nabla_{e_i}grad\omega, e_i) = -trHess(\omega).\end{aligned} \tag{5}$$

Similarly, assume that M^n is a compact Riemannian manifold and ω is a positive differentiable function on M^n, the energy function of Dirichlet is defined as [15];

$$\mathbf{E}(\omega) = \frac{1}{2}\int_M ||\nabla\omega||^2 \mathrm{d}V, \tag{6}$$

where dV denotes the volume element of M^n. Involving the pointwise slant function $\vartheta : M^n \longrightarrow \mathbb{R}$ in a WPPSS $M^n = N_T \times_f N_\vartheta$, and taking into account Theorem 3, and also the Dirichlet energy formulae (6). More precisely, we consider the Dirichlet energy function approach to warped product submanifold, and we establish the following result.

Theorem 4. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a connected and compact WPPSS into CPS $\mathbb{C}P^{2m}(4)$. Then the warped product $N_T \times_f N_\vartheta$ is a simply Riemannian product of N_T and N_ϑ if the Dirichlet energy function of the warped function satisfies:*

$$\mathbf{E}(\chi) = \frac{1}{4q}\tan^2\vartheta \int_M \Big(pq - \|h_\nu\|^2\Big)\mathrm{d}V, \tag{7}$$

where $0 < \mathbf{E}(\chi) < \infty$ represents the Dirichlet energy of the WF $\chi = \ln f$ and $\mathrm{d}V$ is the volume element of M^n.

Another goal of our equality (2) is to provide potential applications to the gradient Ricci curvature by considering a compact Riemannian manifold, and taking into account the Green's Theorem (see [16] for more detail). As a consequence, we give the following:

Theorem 5. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a compact WPPSS $N_T \times_f N_\vartheta$ into a CSP $\mathbb{C}P^{2m}(4)$. If the following equality is satisfied for the warped product submanifold M^n*

$$\|h_\nu\|^2 = q\Big\{p + \int_M \mathcal{R}ic(\nabla\chi, \bullet)\mathrm{dV}\Big\}, \tag{8}$$

then, the following conclusion is true for M^n:

(i) *The WPPSS $N_T \times_f N_\vartheta$ is a CR-warped product into the CPS $\mathbb{C}P^{2m}(4)$.*
(ii) *The WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$ is a simply Riemannian product of N_T and N_ϑ.*

The following implication follows directly from Theorem 5.

Corollary 2. *Assume that $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a compact WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$. If M^n is Ricci flat and having following equality*

$$\|h_v\|^2 = pq, \tag{9}$$

then, the following statements are hold for M^n:

(i) *The WPPSS $N_T \times_f N_\vartheta$ is a CR-warped product, which isometrically immersed into CPS $\mathbb{C}P^{2m}(4)$.*
(ii) *The WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$ is simply a Riemannian product of N_T and N_ϑ.*

The next observation is devoted to Obata [17], which is characterized a specific Riemannian manifolds by second-order ordinary differential equations. He derived the necessary and sufficient conditions for an n-dimensional complete and connected Riemannian manifold (M^n, g) to be isometric to the n-sphere $\mathbb{S}^n(c)$ if there exists a non-constant smooth function ω on M^n that satisfies the second-order differential equation $H_\omega = -c\omega g$, where H_ω is stand for Hessian of ω and c is a constant sectional curvature. A number of investigations devoted to this subject and, therefore, characterizations of spaces, the Euclidean space $\mathbb{R}^n$, the Euclidean sphere $\mathbb{S}^n$ and the CPS $\mathbb{C}P^n$, are important topics in geometric analysis.

For example, Deshmukh-Al-Solamy [18] demonstrated that an n-dimensional compact connected Riemannian manifold whose Ricci curvature satisfies the bound $0 < Ric \leq (n-1)(2 - \frac{nc}{\chi_1})c$ for a constant c and χ_1 is the first non-zero eigenvalue of the Laplace operator, then M^n is isometric to $\mathbb{S}^n(c)$ if M^n admitted a non-zero conformal gradient vector field. They also demonstrated that if M^n is Einstein manifold with Einstein constant $\chi = (n-1)c$, then M^n is isometric to $\mathbb{S}^n(c)$ with $c > 0$ if it is admitted conformal gradient vector field. Taking into consideration the Obata equation [17], Barros, et al. [19] demonstrated that a compact gradient almost Ricci soliton $(M^n, g, \nabla\omega, \pi)$, whose Ricci tensor is Codazzi with constant sectional curvature, is isometric to a Euclidean sphere and ω is a height function in this case. Similar results have acquired in [8,18,20–23]. After these observations, we state following next result, which is a version of Theorem 1 employing the partial differential equation.

Theorem 6. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a WPPSS $N_T \times_f N_\vartheta$ into the CPS $\mathbb{C}P^{2m}(4)$. Then a connected, compact base N_T is isometric to the sphere $\mathbb{S}^p(\sqrt{\frac{\pi_1}{p}})$ if the following equality is satisfied,*

$$\|\nabla^2\chi\|^2 = \frac{\pi_1}{pq}(\mathbf{S} - pq), \tag{10}$$

where $\pi_1 > 0$ is a positive eigenvalue linked to the eigenfunction $\chi = \ln f$ and $\nabla^2\chi$ is a Hessian tensor of the function χ. Moreover, in this case a constant curvature c is equal to $\sqrt{\frac{\pi_1}{p}}$.

Following result is motivated by the Bochner formula.

Theorem 7. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$ with connected and compact base N_T. Then N_T is isometric to the sphere $\mathbb{S}^p(c)$ if the following relation holds:*

$$Ric(\nabla\chi, \nabla\chi) = \pi_1\Big(\frac{p+1}{pq}\Big)\Big\{pq - \mathbf{S}\Big\}, \tag{11}$$

where $\pi_1 > 0$ is a positive eigenvalue linked to the eigenfunction $\chi = \ln f$.

Rio, Kupeli, and Unal [24] use a standard differential equation, which is a variant of Obata's differential equation, to describe the Euclidean sphere. If a complete Riemannian manifold M^n admits a real-valued non-constant function ω with the formula $\Delta\omega + \pi_1\omega = 0$, then M^n is isometric to a warped product of the Euclidean line and a complete Riemannian manifold with the equation $\frac{d^2\phi}{dt^2} + \pi_1\phi = 0$ as ϕ is warping function. In this regard, we arrive to the following conclusion:

Theorem 8. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$ and the base N_T is a connected, compact manifold. If the following equality is satisfied,*

$$Ric(\nabla\chi, \nabla\chi) = \pi_1\Big(\frac{p+1}{pq}\Big)\Big\{pq - \mathbf{S}\Big\}, \tag{12}$$

where $\pi_1 < 0$ is a negative eigenvalue linked to the eigenfunction $\chi = \ln f$, then N_T is isometric to a warped product of the Euclidean line and a complete Riemannian manifold with the equation $\frac{d^2\phi}{dt^2} + \pi_1\phi = 0$ as ϕ is warping function.

Tashiro [25] also demonstrated more general results similar to the results of Obata [17]. The following theorem is also of interest from viewpoint of the characterization of the Euclidean space by a differential equation. We are now able to give the following:

Theorem 9. *Let $\Psi : M^n = N_T \times_f N_\vartheta$ be an isometric immersion of a WPPSS $N_T \times_f N_\vartheta$ into a CPS $\mathbb{C}P^{2m}(4)$ such that base N_T is a connected and compact manifold. Then N_T is isometric to the Euclidean space $\mathbb{R}^p$ if the following equality is satisfied:*

$$\pi_1\left(p + \frac{\pi_1}{p} - \frac{\mathbf{S}}{q}\right) = Ric(\nabla\chi, \nabla\chi) \tag{13}$$

where $\pi_1 > 0$ is positive eigenvalue of the non-constant warping function $\chi = \ln f$.

In the present paper, we consider only the non-trivial WPPSS of the type $M^n = N_T \times_f N_\vartheta$ to be isometrically immersed into a CPS because other types of warped products are trivial in Kaehler manifold. Then, we will consider connected, compact Riemannian submanifolds whose boundaries are non-empty and provide some new, necessary, and sufficient conditions for a WPPSS, which can be reduced to a Riemannian product manifold. We have following motivational example:

Example 1. *The pioneering work of Solomon [26] regarding the harmonic map from a compact Riemannian manifold into a sphere $\mathbb{S}^m$, the standard sphere with codimension two, totally geodesic subsphere removed, This sphere is isometric to the warped product $\mathbb{S}^{n-1}_+ \times_f \mathbb{S}^1$ of an open hemisphere and a circle, for warping function $f \in \mathbb{C}^\infty(\mathbb{S}^{n-1}_+)$. Zhang [27] also considered the warped product manifold $\mathbb{H}^n \times_f \mathbb{R}$ with n-dimensional hyperbolic space whose sectional curvature is -1 and Euclidean line $\mathbb{R}$, and demonstrated that if the warping function f of the warped product manifold $\mathbb{H}^n \times_f \mathbb{R}$ has a critical point, then $\mathbb{H}^n \times_f \mathbb{R}$ is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ if and only if there exists a real number $k > 0$ such that $f(x) = k\cosh r(x)$, where $r(x)$ denotes the hyperbolic distance from x to a fixed point $t \in \mathbb{H}^n$.*

2. Preliminaries

Let $\widetilde{M}$ be an $2m$-dimensional manifold and J be an almost complex structure with a Riemannian metric g that satisfies $J^2 = -I,$ *and* $g(J\mathbb{U}_1, J\mathbb{V}_1) = g(\mathbb{U}_1, \mathbb{V}_1)$, for all vector fields $U, V \in \mathfrak{X}(T\widetilde{M})$, The structure $(\widetilde{M}^{2m}, J, g)$ is then referred to as a Hermitian manifold. Yano and Kon [16] define a Kaehler manifold as a complex structure that satisfies $(\widetilde{\nabla}_{\mathbb{U}_1} J)\mathbb{V}_1 = 0$, for any $\mathbb{U}_1, \mathbb{V}_1 \in \mathfrak{X}(T\widetilde{M})$.

Let M^n be an isometrically immersed into an almost Hermitian manifold $\widetilde{M}^{2m}$ with induced metric g. Assume that ∇ and $\nabla^\perp$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M^n, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_{\mathbb{U}_1}\mathbb{V}_1 = \nabla_{\mathbb{U}_1}\mathbb{V}_1 + h(\mathbb{U}_1, \mathbb{V}_1), \tag{14}$$

$$\widetilde{\nabla}_{\mathbb{U}_1}N = -A_N\mathbb{U}_1 + \nabla^{\perp}_{\mathbb{U}_1}N, \tag{15}$$

for each $\mathbb{U}_1, \mathbb{V}_1 \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M^n into $\widetilde{M}^{2m}$. These are related as follows: $g(h(\mathbb{U}_1, \mathbb{V}_1), N) = g(A_N\mathbb{U}_1, \mathbb{V}_1)$, where g denotes the Riemannian metric on $\widetilde{M}^{2m}$ as well as the metric induced on M^n. Now, for any $U \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$, we have

$$(i)\ J\mathbb{U}_1 = T\mathbb{U}_1 + F\mathbb{U}_1, \quad (ii)\ \ JN = tN + fN, \tag{16}$$

where $T\mathbb{U}_1(tN)$ and $F\mathbb{U}_1(fN)$ are the tangential and normal components of $J\mathbb{U}_1(JN)$, respectively. If T is identically zero, then the submanifold M^n is called a totally real submanifold. The Gauss equation for a submanifold M^n is defined as:

$$\begin{aligned}\widetilde{R}(\mathbb{X}_1, \mathbb{Y}_1, \mathbb{Z}_1, \mathbb{W}_1) =& R(\mathbb{X}_1, \mathbb{Y}_1, \mathbb{Z}_1, \mathbb{W}_1) + g\big(h(\mathbb{X}_1, \mathbb{Z}_1), h(\mathbb{Y}_1, \mathbb{W}_1)\big) \\ &- g\big(h(\mathbb{X}_1, \mathbb{W}_1), h(\mathbb{Y}_1, \mathbb{Z}_1)\big),\end{aligned} \tag{17}$$

for any $\mathbb{X}_1, \mathbb{Y}_1, \mathbb{Z}_1, \mathbb{W}_1 \in \mathfrak{X}(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}^{2m}$ and M^n, respectively. If $\widetilde{M}^{2m}$ is a CPS form of a constant holomorphic sectional curvature is equal to 4 and it is denoted by $\mathbb{C}P^{2m}(4)$, then the curvature tensor $\widetilde{R}$ of $\mathbb{C}P^{2m}(4)$ is expressed as.

$$\begin{aligned}\widetilde{R}(\mathbb{X}_1, \mathbb{Y}_1)\mathbb{Z}_1 =& g(\mathbb{X}_1, \mathbb{Z}_1)\mathbb{Y}_1 - g(\mathbb{Y}_1, \mathbb{Z}_1)\mathbb{X}_1 + g(\mathbb{X}_1, J\mathbb{Z}_1)J\mathbb{Y}_1 - g(\mathbb{Y}_1, J\mathbb{Z}_1)J\mathbb{X}_1 \\ &+ 2g(\mathbb{X}_1, J\mathbb{Y}_1)J\mathbb{Z}_1.\end{aligned} \tag{18}$$

The mean curvature vector $\mathcal{H}$ for an orthonormal frame $\{e_1, e_2, \cdots e_n\}$ of the tangent space TM on M^n is defined by

$$\|\mathcal{H}\|^2 = \frac{1}{n}trace(h) = \frac{1}{n^2}\sum_{r=n+1}^{m}\Big(\sum_{i=1}^{n}h^r_{ii}\Big)^2, \tag{19}$$

where $n = \dim M$. Additionally, we set

$$(i)\quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \ \text{ and } \quad (ii)\quad \mathbf{S} = \sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2. \tag{20}$$

Now, an important Riemannian intrinsic invariant called the scalar curvature of M^n is defined by $\widetilde{\tau}(T_xM^n)$, that is

$$2\widetilde{\tau}(T_xM^n) = \sum_{1\leq\alpha<\beta\leq n} K(e_\alpha \wedge e_\beta). \tag{21}$$

The notations $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_\alpha, e_\beta\}$ at x, thus from Gauss Equation (18), we have

$$\begin{aligned}2\tau(T_xM^n) = K_{\alpha\beta} &= 2\widetilde{\tau}(T_xM^n) + \sum_{r=n+1}^{m}\Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big) \\ &= \widetilde{K}_{\alpha\beta} + \sum_{r=n+1}^{m}\Big(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2\Big)\end{aligned} \tag{22}$$

where $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ denote the sectional curvature of the plane section spanned and e_α at x in the submanifold M^n and at the Riemannian space form $\widetilde{M}^m(c)$, respectively. The following consequences are acquired from (18) and (22) as:

$$\tau(T_x N_1^p) = \sum_{r=n+1}^{m} \sum_{1\le i<j\le p} \left(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2 \right) + \widetilde{\tau}(T_x N_1^p). \tag{23}$$

Similarly, we have

$$\tau(T_x N_2^q) = \sum_{r=n+1}^{m} \sum_{p+1\le a<b\le n} \left(h_{aa}^r h_{bb}^r - (h_{ab}^r)^2 \right) + \widetilde{\tau}(T_x N_2^q). \tag{24}$$

A holomorphic submanifold is one in which J preserves every tangent space of M^n, that is, $J(T_x M) \subseteq T_x M$, for each $x \in M^n$. Similarly, for each $x \in M^n$, the totally real submanifold is defined as follows: J translates any tangent space of Mn into normal space, that is, $J(T_x M) \subseteq T^\perp M$. Aside from the holomorphic and totally real submanifolds, the CR-submanifold, slant submanifold, semi-slant submanifold, pointwise semi-slant submanifold, and pointwise slant submanifold are other important classes of submanifolds of a Kaehler manifold determined by the behavior of the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold. In [11–13] contains a comprehensive taxonomy of these submanifolds. We refer to [11,28] for several examples of a pointwise semi-slant submanifold in a Kaehler manifold, as well as related difficulties. Let us represent the dimensions of the complex distribution $\mathcal{D}^T$ and pointwise slant distribution $\mathcal{D}^\vartheta$ of pointwise semi-slant submanifold in a Kaehler manifold $\widetilde{M}^{2m}$ with p and q, respectively, using the Definition 3.1 [11]. Then the following observations apply.

Remark 1. *M^n is invariant and pointwise slant submanifold for $p = 0$ and $q = 0$, respectively.*

Remark 2. *If the slant function $\vartheta : M^n \to R$ is globally constant on M^n and $\vartheta = \frac{\pi}{2}$, then M^n is a called CR-submanifold.*

Remark 3. *M^n is defined as a proper pointwise semi-slant submanifold if the slant function neither $\vartheta = 0$ nor $\vartheta = \frac{\pi}{2}$.*

We will follows the definition of the warped product manifold of [3,29]. According them, the following remarks are consequences of Lemma 2.1 [3];

Remark 4. *A WPM $M^n = N_1 \times_f N_2$ is said to be trivial or simply a Riemannian product manifold if the WF f is a constant function along N_1.*

Remark 5. *If $M^n = N_1 \times_f N_2$ is a WPM, then N_1 is totally geodesic and N_2 is totally umbilical submanifold of M^n, respectively.*

From [30] (Equation (3.3)), the following relation is acquired.

$$\sum_{\alpha=1}^{p} \sum_{\beta=1}^{q} K(e_\alpha \wedge e_\beta) = \frac{q\Delta f}{f}. \tag{25}$$

Further, $\nabla \ln f$ is the gradient of $\ln f$ which is defined as:

$$g(\nabla \ln f, X) = X(\ln f). \tag{26}$$

3. Non-Trivial WPPSS

In this section, some basic facts and some key results recall which will be used in the proof of our main results. First, we remember that if the two factors of the warped product submanifold are holomorphic and pointwise slant submanifolds, then it is called a WPPSS of almost Hermitian manifolds. Therefore, in such a case, there are two types of WPPSSs of a Kaehler manifold such that

$$(i)\ N_\vartheta \times_f N_T, \quad and \quad (ii)\ N_T \times_f N_\vartheta, \text{ we choose } \dim N_T = p \ \&\ \dim N_\vartheta = q.$$

For the first case, let us recall Theorem 4.1 in [11] which showed that a proper WPPSS $M^n = N_\vartheta \times_f N_T$ in a Kaehler manifold $\widetilde{M}^{2m}$ does not exist such that N_ϑ is a proper pointwise slant submanifold and N_T is a holomorphic submanifold of $\widetilde{M}^{2m}$.

On the other hand, proceeding to the second case, let us recall Theorem 5.1 in [11] that many non-trivial WPPSS's of the form $N_T \times_f N_\vartheta$ with examples are studied. Now, for N_T and N_ϑ are holomorphic and pointwise slant submanifolds of $\widetilde{M}^{2m}$. The following lemma and theorems will be useful in the sequel.

Lemma 1 ([11])**.** *Let $M^n = N_T \times_f N_\vartheta$ be a WPPSS of a Kaehler manifold $\widetilde{M}^{2m}$. Then*

$$g(h(\mathbb{X}_1, \mathbb{Z}_1), FT\mathbb{Z}_1) = -(\mathbb{X}_1 \ln f) \cos^2\vartheta \|\mathbb{Z}_1\|^2, \tag{27}$$

$$g(h(\mathbb{Z}_1, J\mathbb{X}_1), F\mathbb{Z}_1) = (\mathbb{X}_1 \ln f) \|\mathbb{Z}_1\|^2, \tag{28}$$

$$g(h(\mathbb{X}_1, \mathbb{Y}_1), F\mathbb{Z}_1) = 0, \tag{29}$$

for any $\mathbb{X}_1, \mathbb{Y}_1 \in \mathfrak{X}(TN_T)$ and $\mathbb{Z}_1 \in \mathfrak{X}(TN_\vartheta)$.

Theorem 10 ([3])**.** *Let $\Psi : M^n = N_T \times_f N_\vartheta \longrightarrow \widetilde{M}^{2m}$ be isometrically immersed from a WPPSS $N_T \times_f N_\vartheta$ into a Kaehler manifold $\widetilde{M}^{2m}$. Then N_T is always a minimal submanifold of $\widetilde{M}^{2m}$.*

Theorem 11 ([31])**.** *Let Ψ be $\mathcal{D}^\vartheta$-minimal isometric immersion of a WPPSS $N_T \times_f N_\vartheta$ into a Kaehler manifold $\widetilde{M}^{2m}$, then Ψ is a N_ϑ-totally geodesic.*

The above notion was extended into the complex space forms and also to describe brief method to demonstrate the triviality for both inequality and equality results in [31], which holds on a compact Riemannian submanifold whose boundary is empty.

Theorem 12 ([3])**.** *On a compact orientable WPPSS $M^n = N_T \times_f N_\vartheta$ in a complex space form $\widetilde{M}^{2m}(c)$, the following inequality holds:*

$$||h||^2 \geq \frac{pqc}{2}, \tag{30}$$

where p and q are dimensions of N_T and N_ϑ, respectively. Then M^n is simply a Riemannian product manifold.

For the equality case of inequality (30), the following result was demonstrated.

Theorem 13 ([3])**.** *Let $M^n = N_T \times_f N_\vartheta$ is compact orientable WPPSS in a complex space form $\widetilde{M}^{2m}(c)$. Then M^n is simply a Riemannian product if and only if it is satisfied*

$$||h_\nu||^2 = \frac{pqc}{4}, \tag{31}$$

where h_ν is a component of h in $\mathfrak{X}(\nu)$.

Now, we demonstrate some interesting results.

4. Proof of Theorem 1

Proof. Using the Gauss Equation (17), we get

$$n^2\|\mathcal{H}\|^2 = ||h||^2 + 2\tau(T_xM^n) - 2\tilde{\tau}(T_xM^n). \tag{32}$$

We assume that $\{e_1, \dots, e_p, e_{p+1}, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ are orthonormal frames of $\mathfrak{X}(T_xM^n)$ and $\mathfrak{X}(T^\perp M^n)$, such that $\{e_1, \dots, e_p\}$ and $\{e_{p+1}, \dots, e_n\}$ are the frames of $\mathfrak{X}(TN_T)$ and $\mathfrak{X}(TN_\vartheta)$. From (21), we have

$$\begin{aligned}\tau(T_xM^n) &= \sum_{1\le\alpha<\beta\le n} K_{\alpha\beta} \\ &= \sum_{\alpha=1}^{p}\sum_{A=p+1}^{n} K_{\alpha A} + \sum_{1\le i<j\le p} K_{ij} + \sum_{p+1\le a<b\le n} K_{ab}\end{aligned} \tag{33}$$

Using (25) and (21), we derive the following relation

$$\tau(T_xM^n) = \frac{q\Delta f}{f} + \tau(T_xN_T^p) + \tau(T_xN_\vartheta^q). \tag{34}$$

From (22)–(24), one obtains:

$$\begin{aligned}\tau(T_xM^n) =& \frac{q\Delta f}{f} + \tilde{\tau}(T_xN_T^p) + \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta} \\ &- \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n_1} (h^r_{\alpha\beta})^2 + \tilde{\tau}(T_xN_\vartheta^q) \\ &+ \sum_{r=n+1}^{m}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} - \sum_{r=n+1}^{m}\sum_{p+1\le a<b\le n} (h^r_{ab})^2.\end{aligned} \tag{35}$$

Equations (32) and (35) give

$$\begin{aligned}\left(\sum_{i=1}^{n} h^{n+1}_{ii}\right)^2 =& \sum_{r=n+1}^{m}\sum_{i=1}^{n}(h_{ii})^2 + \frac{2q\Delta f}{f} + 2\tilde{\tau}(T_xN_T^p) + 2\tilde{\tau}(T_xN_\vartheta^q) \\ &+ 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n_1} h^r_{\alpha\alpha}h^r_{\beta\beta} - 2\tilde{\tau}(T_xM^n) \\ &- 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le p} (h^r_{\alpha\beta})^2 + 2\sum_{r=n+1}^{m}\sum_{p+1\le a<b\le n} h^r_{aa}h^r_{bb} \\ &- 2\sum_{r=n+1}^{m}\sum_{p+1\le a<b\le n} (h^r_{ab})^2.\end{aligned} \tag{36}$$

Exercising the computations, we derive

$$\begin{aligned}\left(\sum_{i=1}^{n} h^{n+1}_{ii}\right)^2 =& \sum_{r=n+1}^{m}\sum_{i=1}^{p}(h_{ii})^2 + \sum_{r=n+1}^{m}\sum_{j=p+1}^{n}(h_{jj})^2 + 2\sum_{r=n+1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij})^2 \\ &+ \frac{2q\Delta f}{f} + 2\tilde{\tau}(T_xN_T^p) + 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le p} h^r_{\alpha\alpha}h^r_{\beta\beta} \\ &- 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le p} (h^r_{\alpha\beta})^2 + 2\tilde{\tau}(T_xN_\vartheta^q) - 2\tilde{\tau}(T_xM^n)\end{aligned} \tag{37}$$

$$+2\sum_{r=n+1}^{m}\sum_{p+1\leq a<b\leq n}h^r_{aa}h^r_{bb}-2\sum_{r=n+1}^{m}\sum_{p+1\leq a<b\leq n}(h^r_{ab})^2.$$

As from Theorem 10 that M^n is a N_T-minimal warped product submanifold, we have

$$2\sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq p}h^r_{\alpha\alpha}h^r_{\beta\beta}+\sum_{r=n+1}^{m}\sum_{i=1}^{n}(h_{ii})^2=0. \tag{38}$$

From Theorem 11, we find that

$$2\sum_{r=n+1}^{m}\sum_{p+1\leq a<b\leq n}h^r_{aa}h^r_{bb}+\sum_{r=n+1}^{m}\sum_{j=p+1}^{n}(h_{jj})^2=\Big(\sum_{A=1}^{n}h_{AA}\Big)^2. \tag{39}$$

On substituting (38) and (39) in (37), we get

$$\begin{aligned}2\widetilde{\tau}(T_xM^n)=&\frac{2q\Delta f}{f}+2\widetilde{\tau}(T_xN^p_T)+2\widetilde{\tau}(T_xN^q_\vartheta)\\&-2\sum_{r=n+1}^{m}\Bigg\{\sum_{1\leq\alpha<\beta\leq p}(h^r_{\alpha\beta})^2+\sum_{p+1\leq a<b\leq n}(h^r_{ab})^2-\sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij})^2\Bigg\}\end{aligned} \tag{40}$$

Thus, from binomial properties, we arrive at

$$\sum_{1\leq\alpha<\beta\leq p}(h^r_{\alpha\beta})^2+\sum_{p+1\leq a<b\leq n}(h^r_{ab})^2-\sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij})^2=\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}\left(h^r_{\alpha\beta}\right)^2. \tag{41}$$

If we substituting $X=W=e_i$ and $Y=Z=e_j$ in (18), we get

$$\begin{aligned}\widetilde{R}(e_i,e_j,e_j,e_i)=&\Big\{g(e_i,e_i)g(e_j,e_j)-g(e_i,e_j)g(e_i,e_j)+g(e_i,Je_j)g(Je_j,e_i)\\&-g(e_i,Je_i)g(e_j,Je_j)+2g^2(Je_j,e_i)\Big\}.\end{aligned} \tag{42}$$

Taking summing up over the basis vector fields of TM^n. For $1\leq i\neq j\leq n$, we obtain

$$2\widetilde{\tau}(TM^n)=n(n-1)+3\sum_{1\leq i\neq j\leq n}g^2(Pe_i,e_j). \tag{43}$$

Next, we assume that M^n is a warped product of holomorphic and proper pointwise slant submanifolds in a CPS $\mathbb{C}P^{2m}(4)$. Thus, we set the following frame of orthonormal vector fields as:

$$e_1,e_2=Je_1,\cdots,e_{2d_1-1},e_{2d_1}=Je_{2d_1-1},$$

$$e_{2d_1+1},e_{2d_1+2}=\sec\vartheta Te_{2d_1+1},\cdots,e_{2d_1+2d_2-1}e_{2d_1+2d_2}=\sec\vartheta Te_{d_1-1}.$$

Using the orthonormal frame, we have

$$\begin{aligned}g^2(Je_i,e_{i+1})=&1\ \ for\ i\in\{1,\cdots,p-1\},\\=&\cos^2\vartheta\ \ for\ \ i\in\{p+1,\cdots,p+q-1\}.\end{aligned}$$

Thus, it is easily seen that

$$\sum_{i,j=1}^{n}g^2(Pe_i,e_j)=p+q\cos^2\vartheta. \tag{44}$$

From (43) and (44), it follows that

$$\widetilde{\tau}(TM^n) = \frac{1}{2}\Big\{n(n-1) + 3\big(p + q\cos^2\vartheta\big)\Big\}. \tag{45}$$

Similarly, for TN_T^p, we derive

$$\widetilde{\tau}(TN_T^p) = \frac{1}{2}p(p-1) + \frac{3}{2}p. \tag{46}$$

Now using fact that $||T||^2 = q\cos^2\vartheta$, for pointwise slant bundle TN_ϑ [3], one derives

$$\widetilde{\tau}(TN_\vartheta^q) = \frac{1}{2}q(q-1) + \frac{3}{2}q\cos^2\vartheta. \tag{47}$$

Therefore, combining Equations (40), (41), (45)–(47), we get the essential result (1). Thus, the proof is completed. □

Proof of Theorem 2

Proof. Let $X = e_\alpha(1 \le \alpha \le p)$ and $Z = e^*_\beta(1 \le \beta \le q)$, be the orthonormal basis. Then from the definition of the bilinear form h, we have

$$||h(e_\alpha, e^*_\beta)||^2 = \sum_{r=1}^{m}\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q} g\big(h(e_\alpha, e^*_\beta), e_r\big)^2 + \sum_{\alpha=1}^{p}\sum_{\beta=1}^{q}||h_\nu(e_\alpha, e^*_\beta)||^2.$$

The term in the right hand side is a $F\mathcal{D}^\vartheta$-component and the second term is a ν-component. Using the adapted orthonormal frame for vector fields of N_T^p and N_ϑ^q for pointwise semi-slant submanifold [3], and lemma 1, we obtain:

$$||h(e_\alpha, e^*_\beta)||^2 = q\big(\csc^2\vartheta + \cot^2\vartheta\big)||\nabla\ln f||^2 + ||h_\nu||^2,$$

which implies that

$$||h(e_\alpha, e^*_\beta)||^2 = q\big(1 + 2\cot^2\vartheta\big)||\nabla\ln f||^2 + ||h_\nu||^2. \tag{48}$$

From (1) and (48), we get the essential result (2). This completes the proof of the theorem. □

5. Application of Theorem 1 to Demonstrate Theorem 4

Proof. Equation (2) for the equality case is the following

$$pq = q\Delta\chi + \|h_\nu\|^2 + 2q\cot^2\vartheta||\nabla\chi||^2. \tag{49}$$

Taking integration on M^n over the volume element $\mathrm{d}V$ with nonempty boundary, we get

$$\int_{M^n} pq\mathrm{d}V = \int_{M^n}\|h_\nu\|^2\mathrm{d}V + q\int_{M^n}(\Delta\chi)\mathrm{d}V + 2q\cot^2\vartheta\int_{M^n}||\nabla\chi||^2\mathrm{d}V. \tag{50}$$

From (6) and setting $\omega = \chi = \ln f$ and (50), it follows that

$$\int_{M^n} p\mathrm{d}V = \frac{1}{q}\int_{M^n}\|h_\nu\|^2\mathrm{d}V + \int_{M^n}\Delta\chi\mathrm{d}V + 4\cot^2\vartheta E(\chi). \tag{51}$$

If the equality assumption in (7) is satisfied, we get the following relation from (51).

$$\int_{M^n}\Delta\chi\mathrm{d}V = 0 \quad on\ M^n,$$

which gives with $\chi = \ln f$ as

$$\Delta(\ln f) = 0. \tag{52}$$

If M^n is a connected and compact WPPSS, from (52) and Theorem 3 it implies that $\ln f = 0 \Longrightarrow f = 1$, that is, f is a constant on N_T. Hence, from Remark 4, the warped product submanifold M^n is a simply Riemannian product manifold. This completes the proof of the theorem. □

6. Classifications of the Ricci Curvature and Divergence of the Hessian Tensor

Let us define the $(0,2)$-tensor T on M with a $(1,1)$-tensor by the following equation:

$$g(T(\mathbb{Z}_1), \mathbb{Y}_1) = T(\mathbb{Z}_1, \mathbb{Y}_1).$$

for all $\mathbb{Y}_1, \mathbb{Z}_1 \in \Gamma(TM)$. Thus, we get

$$div(\omega T) = \omega div T + T(\nabla\omega, \bullet) \;\text{ and }\; \nabla(\omega T) = \omega\nabla T + d\omega \otimes T,$$

for all $\omega \in C^\infty(M)$. In particular, we have $div(\omega g) = d\omega$. In addition, the following general facts are well-documented in the literature [32].

$$(i)\;\; div\nabla^2\omega = Ric(\nabla\omega, \bullet) + d\Delta\omega \;\text{ and }\; (ii)\;\; \frac{1}{2}d\|\nabla\omega\|^2 = \nabla^2\omega(\nabla, \bullet). \tag{53}$$

6.1. Proof of Theorem 5

Proof. Applying Ricci identity (53) to the warping function $\omega = \chi = \ln f$, we get

$$div\nabla^2\chi = d\big(\Delta\chi\big) + \mathcal{R}ic(\nabla\chi, \cdot). \tag{54}$$

We have M^n as a compact warped product submanifold without boundary, and we have dV as an integration along the volume element.

$$\Delta\chi = \int_{M^n} (div\nabla^2\chi)\mathrm{dV} - \int_{M^n} \mathcal{R}ic(\nabla\chi, \cdot)\mathrm{dV}. \tag{55}$$

Using the Stokes theorem on a compact manifold M^n (see [14]) in (55) to get

$$\Delta\chi = -\int_{M^n} \mathcal{R}ic(\nabla\chi, \cdot)\mathrm{dV}. \tag{56}$$

On the other hand, from (34) we have

$$q\Delta\chi + 2q\cot^2\vartheta||\nabla\chi||^2 = pq - \|h_\nu\|^2. \tag{57}$$

Equations (56) and (57) give

$$2\cot^2\vartheta||\nabla\chi||^2 - \int_{M^n} \mathcal{R}ic(\nabla\chi, \cdot)\mathrm{dV} = p - \frac{1}{q}\|h_\nu\|^2,$$

or equivalently

$$p = -\int_{M^n} \mathcal{R}ic(\nabla\chi, \cdot)\mathrm{dV} + \frac{1}{q}\|h_\nu\|^2 + 2\cot^2\vartheta||\nabla\chi||^2. \tag{58}$$

The above equation along with (8) yields

$$2\cot^2\vartheta||\nabla\chi||^2 = 0,$$

which either $\cot^2\vartheta = 0,\ or\ ||\nabla\chi||^2 = 0$.

Case 1: When $\cot^2\vartheta = 0$, that is $\frac{\cos^2\vartheta}{\sin^2\vartheta} = 0$, which implies that $\cos\vartheta = 0$. Then from Remark 2, we conclude that a pointwise slant submanifold N_ϑ becomes a totally real submanifold; hence, M^n becomes a CR-warped product submanifold of a complex projective m-space $CP^m(4)$. The proof of (i) from Theorem 5 is now complete.
Case 2: When $||\nabla\chi||^2 = 0$, that is, $\nabla\chi = 0$, which implies that $grad\ln f = 0$. it shows that f is a constant function on N_T. Hence, from Remark 4, we conclude that M^n is a trivial WPPSS of a CPS $CP^{2m}(4)$. This is the second part (ii) of Theorem 5. □

6.2. Proof of Corollary 2

If M^n is a Ricci flat and this means that the Ricci curvature of M^n has vanished everywhere, that is

$$\mathcal{R}ic(\nabla\chi,\bullet) = 0.$$

Using (8) in the above equation, we get the proof of the corollary.

7. Application to the Ordinary Differential Equation

Proof of Theorem 6

Proof. Let we define the following equation as

$$\left\|\nabla^2\chi + t\chi I\right\|^2 = \|\nabla^2\chi\|^2 + t^2(\chi)^2\|I\|^2 + 2t\chi g(\nabla^2\chi, I).$$

As we define $\|I\|^2 = trace(II^*) = p$ and $g(\nabla^2\chi, I^*) = tr(\nabla^2\chi I^*) = tr(\nabla^2\chi)$. Then, from (5), one obtains:

$$\left\|\nabla^2\chi + t\chi I\right\|^2 = \|\nabla^2\chi\|^2 + pt^2(\chi)^2 - 2t\chi\Delta\chi. \tag{59}$$

Let π_1 be an eigenvalue of the eigenfunction $\chi = \ln f$ such that $\Delta\chi = \pi_1\chi$. Then above equation reduce to

$$\left\|\nabla^2\chi + t\chi I\right\|^2 = \|\nabla^2\chi\|^2 + (pt^2 - 2t\pi_1)(\chi)^2. \tag{60}$$

On the other hand, we obtain

$$\Delta\Big(\frac{\omega^2}{2}\Big) = -div\Big(\nabla\Big(\frac{\omega^2}{2}\Big)\Big) = div(\omega\nabla\omega) = \omega\Delta\omega - \|\nabla\omega\|^2.$$

Setting $\omega = \chi = \ln f$ and utilizing the Stokes theorem on a compact manifold M^n, we have

$$\int_{N_T\times\{q\}} \chi^2 dV = \frac{1}{\pi_1}\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV. \tag{61}$$

It follows from (60) and (61), we find that

$$\int_{N_T\times\{q\}} \|\nabla^2\chi + t\chi I\|^2 dV = \int_{N_T\times\{q\}} \|\nabla^2\chi\|^2 dV + \int_{N_T\times\{q\}} \Big(\frac{pt^2}{\pi_1} - 2t\Big)\|\nabla\chi\|^2 dV. \tag{62}$$

Putting $t = \frac{\pi_1}{p}$ in (62) and taking integration both sides, we get

$$\int_{N_T\times\{q\}} \left\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\right\|^2 dV = \int_{N_T\times\{q\}} \|\nabla^2\chi\|^2 dV - \frac{\pi_1}{p}\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV. \tag{63}$$

Using integration on (1) and the Stokes theorem once more, we have

$$\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV = \frac{1}{q}\int_{N_T\times\{q\}} \Big(\mathbf{S} - pq\Big) dV. \quad (64)$$

From (63) and (64), we derive

$$\int_{N_T\times\{q\}} \Big\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\Big\|^2 dV = \int_{N_T\times\{q\}} \|\nabla^2\chi\|^2 dV - \frac{\pi_1}{p}\int_{N_T\times\{q\}} \Big\{\frac{\mathbf{S}}{q} - p\Big\} dV. \quad (65)$$

If Equation (10) is satisfied, then from (65), we get

$$\Big\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\Big\|^2 = 0 \implies \nabla^2\chi = -\frac{\pi_1}{p}\chi I. \quad (66)$$

Since the warping function $\chi = \ln f$ is a non-constant because of warped product manifold M^n is a non-trivial. Inlvolving the Obata's theorem [17] for a differential equation with setting constant $c = \frac{\pi_1}{p} > 0$ as $\pi_1 > 0$ in (66). We conclude that N_T is isometric to the sphere $\mathbb{S}^n(\sqrt{\frac{\pi_1}{p}})$ with constant curvature $c = \sqrt{\frac{\pi_1}{p}}$. This complete the proof of the theorem. □

8. Application of Bochner Formula as Proof of Theorem 6

If we remember the Bochner formula (see, for example, [33]), the following relationship holds for a differentiable function $\chi = \ln f$ defined on a Riemannian manifold:

$$\frac{1}{2}\Delta\|\nabla\chi\|^2 = \|\nabla^2\chi\|^2 + Ric(\nabla\chi, \nabla\chi) + g\big(\nabla\chi, \nabla(\Delta\chi)\big).$$

Integrating the above equation with the aid of Stokes theorem, we get

$$\int_{N_T\times\{q\}} \|\nabla^2\chi\|^2 dV + \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV + \int_{N_T\times\{q\}} g(\nabla\chi, \nabla(\Delta\chi)) dV = 0. \quad (67)$$

Now, using $\Delta\chi = \pi_1\chi$ and some rearrangement in (67), we derive

$$\int_{N_T\times\{q\}} \|\nabla^2\chi\|^2 dV = -\pi_1\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV. \quad (68)$$

Inserting (68) into (63), we get

$$\int_{N_T\times\{q\}} \Big\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\Big\|^2 dV = -\pi_1\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV - \frac{\pi_1}{p}\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV,$$

which implies that

$$\int_{N_T\times\{q\}} \Big\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\Big\|^2 dV = -\pi_1\Big(\frac{p+1}{p}\Big)\int_{N_T\times\{q\}} \|\nabla\chi\|^2 dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV. \quad (69)$$

From (64) and (69), we find that

$$\int_{N_T\times\{q\}} \Big\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\Big\|^2 dV = -\pi_1\Big(\frac{p+1}{pq}\Big)\int_{N_T\times\{q\}} \mathbf{S} dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV + \pi_1\Big(\frac{p+1}{1}\Big) Vol(N_T),$$

or equivalent to the following

$$\int_{N_T\times\{q\}} \left\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\right\|^2 dV = \pi_1\left(\frac{p+1}{pq}\right)\int_{N_T\times\{q\}} \left\{pq - \mathbf{S}\right\} dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV. \tag{70}$$

The following equality holds in (70) if the equality in (11) is satisfied, that is

$$\int_{N_T\times\{q\}} \left\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\right\|^2 dV = 0,$$

which means that

$$\nabla^2\chi = -\frac{\pi_1}{p}\chi I. \tag{71}$$

Therefore, for a ordinary differential Equation (71) with constant $c = \sqrt{\frac{\pi_1}{p}} > 0$ as $\pi_1 > 0$, we invoke Obata's theorem [17]. It implies that N_T is isometric to the sphere $\mathbb{S}^p(\sqrt{\frac{\pi_1}{p}})$. This completes the proof of the theorem.

8.1. Proof of Theorem 8

In the hypothesis of the theorem, we assumed that the base manifold N_T is connected and compact and hence from (70), we have

$$\int_{N_T\times\{q\}} \left\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\right\|^2 dV = \pi_1\left(\frac{p+1}{pq}\right)\int_{N_T\times\{q\}} \left(pq - \mathbf{S}\right) dV - \int_{N_T\times\{q\}} Ric(\nabla\chi, \nabla\chi) dV. \tag{72}$$

If the statement of the theorem and Equation (12) is satisfied, then from (72), we have

$$\left\|\nabla^2\chi + \frac{\pi_1}{p}\chi I\right\|^2 = 0,$$

which implies that

$$\nabla^2\chi = -\frac{\pi_1}{p}\chi I. \tag{73}$$

As we assumed that $\pi_1 < 0$ in the hypothesis of the theorem, therefore we invoke the result [24]. Then, N_T is isometric to a warped product of the Euclidean line $\mathbb{R}$ and a complete Riemannian manifold L, that is, $\mathbb{R}\times_\phi L$, where the warping function ϕ on $\mathbb{R}$ satisfies the equation $\frac{d^2\phi}{dt^2} + \pi_1\phi = 0$. This completes the proof of the theorem.

8.2. Proof of Theorem 9

Let us consider the following equation:

$$\left\|\nabla^2\chi - tI\right\|^2 = \|\nabla^2\chi\|^2 + t^2\|I\|^2 - 2tg(\nabla^2\chi, I), \tag{74}$$

which implies the fact that the Hessian $\nabla^2\chi$ and identity operator I, are linked by the following equation:

$$\left\|\nabla^2\chi - tI\right\|^2 = \|\nabla^2\chi\|^2 + t^2p - 2t\Delta\chi.$$

Putting $t = \frac{\pi_1}{p}$ in the above equation, and integrating along the volume element dV, we derive

$$\int_{N_T \times \{q\}} \left\| \nabla^2 \chi - \frac{\pi_1}{p} I \right\|^2 dV = \int_{N_T \times \{q\}} \left(\|\nabla^2 \chi\|^2 + \frac{\pi_1^2}{p} \right) dV. \tag{75}$$

Using (68), we obtain

$$\int_{N_T \times \{q\}} \left\| \nabla^2 \chi - \frac{\pi_1}{p} I \right\|^2 dV = -\pi_1 \int_{N_T \times \{q\}} \|\nabla \chi\|^2 dV - \int_{N_T \times \{q\}} Ric(\nabla \chi, \nabla \chi) dV + \int_{N_T \times \{q\}} \frac{\pi_1^2}{p} dV. \tag{76}$$

Using (64), we obtain

$$\int_{N_T \times \{q\}} \left\| \nabla^2 \chi - \frac{\pi_1}{n_1} I \right\|^2 dV = -\frac{\pi_1}{q} \int_{N_T \times \{q\}} \mathbf{S} dV - \int_{N_T \times \{q\}} Ric(\nabla \chi, \nabla \chi) dV + \int_{N_T \times \{q\}} \frac{\pi_1}{p} (p^2 + \pi_1) dV.$$

From the above equation, one obtains:

$$\int_{N_T \times \{q\}} \left\| \nabla^2 \chi - \frac{\pi_1}{p} I \right\|^2 dV = \int_{N_T \times \{q\}} \left(\pi_1 p + \frac{\pi_1^2}{p} - \frac{\mathbf{S}\pi_1}{q} \right) dV - \int_{N_T \times \{q\}} Ric(\nabla \chi, \nabla \chi) dV. \tag{77}$$

If Equation (13) is satisfied, then from (77), we arrive at

$$\int_{N_T \times \{q\}} \left\| \nabla^2 \chi - \frac{\pi_1}{p} I \right\|^2 dV = 0.$$

It follows from the definition of the norm

$$\nabla^2 \chi = \frac{\pi_1}{p} I,$$

which implies that

$$\nabla^2 \chi(X, X) = \frac{\pi_1}{p} g(X, X), \tag{78}$$

for any $X \in \mathfrak{X}(N_T)$. Note that if the potential function $\chi = \ln f$ is a constant then M^n is a trivial warped product submanifold that leads to a contradiction as M^n is a non-trivial. Hence (78) is a differential equation [25] with positive constant $c = \frac{\pi_1}{p} > 0$, as $\pi_1 > 0$. Therefore, N_T is isometric to the Euclidean space $\mathbb{R}^p$. This complete the proof of the theorem.

9. Conclusions

On warped product submanifolds, the current work has used an ordinary differential equation. Some characterisation theorems for the base of a WPPSS in a CPS have been researched based on the optimization of the warping function of a WPPSS in a CPS. In summary, the study of warped product submanifolds has recently gotten increased attention due to its importance in mathematics and application to other fields such as mathematical physics. Robertson-Walker spacetime is a classic cosmological model of the universe that consists of a perfect fluid whose molecules are galaxies. Theorems that relate the intrinsic and extrinsic curvatures play an important role in physics in differential geometry of submanifolds [15]. Furthermore, the concept of second order differential Equations (PDEs) has made a significant contribution to the study of issues in fluid mechanics, heat conduction in solids, diffusive transport of chemicals in porous media, and wave propagation in strings, as well as in solid mechanics. The eigenvalue challenges are attempts to find every possible real π_1 such that a nontrivial solution to second order partial differential Equations (PDEs) $\Delta\chi + \pi_1\chi = 0$ exists [34]. Similarly, eigenvalue equations in differential geometry are intriguing topics with a physical grounding. Finding isometrics on a given manifold is a prominent task in Riemannian geometry. As a result, the article features outstanding Riemannian geometry and ordinary differential equation combinations.

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