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Oscillation and Nonoscillatory Criteria of Higher Order Dynamic Equations on Time Scales

Ya-Ru Zhu 1, Zhong-Xuan Mao 1,* 1, Jing-Feng Tian 1, Ya-Gang Zhang 1 and Xin-Ni Lin 2

1 Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China; zhuyr1982@ncepu.edu.cn (Y.-R.Z.); tianjf@ncepu.edu.cn (J.-F.T); yagangzhang@ncepu.edu.cn (Y.-G.Z.)
2 College of Foreign Languages, Cultures and International Exchanges, Zhejiang University, Ningbo 315100, China; llxxnn@zju.edu.cn
* Correspondence: maozhongxuan@ncepu.edu.cn

Abstract: In this paper, we consider two universal higher order dynamic equations with several delay functions. We will establish two oscillatory criteria of the first equation and a sufficient and necessary condition for the second equation with a nonoscillatory solution by employing fixed point theorem.

Keywords: higher order dynamic equations; oscillation; nonoscillation; Riccati technique; fixed point theorem

1. Introduction

In 1988, Hilger [1] established a completely new theory in academia called time scales, which unites discrete and continuous representations and drew a lot of attention. The field of time scales has a tremendous amount of theoretical value, such as differential equations and difference equations can both be consolidated within the theoretical framework of dynamics equations on time scales. In this manner, it can avoid repeating studies in difference equations and differential equations while also exploring more similarities and differences between them. We refer the readers to [2–5] for more information about dynamic equations on time scales. Meanwhile, the study of dynamic equations had a wide range of applications in physics [6,7], chemistry [6], biology [6,8], engineering [6] and finance [9].

As we all know, most differential equations and difference equations with nonconstant coefficients, especially some nonlinear equations, have no analytical solution. Some researchers focus on the oscillatory and asymptotic behavior of the solutions to these equations, where a solution is oscillatory if it is neither finally positive nor eventually negative, and it is nonoscillatory if it is not oscillatory.

Since generic dynamic equations on time scales cannot be investigated directly due to technological constraints, researchers [10–25] usually confine their research to dynamic equations with specific structures under specified assumptions. Next, we list some equations that have been studied by scholars. However, it is worth pointing out that even while studying the same equation, various conditions can generate different results.

The authors [10–19,23,24] mainly focused on the second order dynamic equation on time scales. Furthermore, the paper [15] provided by Agwo and Khodier in 2017 gave an uniform form of the above equations as follows:

\[(a(t)g(x^\Delta(t)))^\Delta + p(t)f(y(\tau(t))) = 0, \quad t \in [t_0, \infty)_T.\] (1)

As research advances, more attention is being paid to the oscillatory and asymptotic behaviour of higher order dynamic nonlinear equations [26–28]. For example, in 2012, Sun, Yu and Xi [26] discussed the oscillatory and asymptotic behavior of the following higher order dynamic equation:
\[
\left( a_n(t) \left( (a_{n-1}(t) \cdots (a_1(t)x^\Delta(t))^\Delta \cdots )^\Delta \right) \right)^\Delta + \delta p(t) f(x(\tau(t))) = 0, \quad t \in [t_0, \infty)_T. \tag{2}
\]

The two papers [27,28] concerned the same equation and proposed different oscillatory criteria.

\[
\left( a_n(t) \left( (a_{n-1}(t) \cdots (a_1(t)x^\Delta(t))^\Delta \cdots )^\Delta \right) \right)^\Delta + f(t, x(\tau(t))) = 0, \quad t \in [t_0, \infty)_T. \tag{3}
\]

Very recently, Chatzarakis, Grace and Jadlovská [25] provided a sharp oscillation for the following half-linear second-order differential equations with several delay terms:

\[
(a(t)|x'|^{\alpha-1}x')' + \sum_{i=1}^{m} q_i(t)|x'|^{\alpha-1}x'(\tau_i(t)) = 0, \quad t \geq t_0 > 0. \tag{4}
\]

On the contrary, scholars also concerned the nonoscillatory behavior of higher order dynamic equations [22,29–33]. For example, in 2009, Zhang, Dong and Li et al. [29] presented some sufficient conditions for the existence of positive solutions for

\[
(x(t) + p(t)x(\tau(t)))^{\Delta^x} + \Phi(t, x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_m(t))) = 0, \quad t \in [t_0, \infty)_T. \tag{5}
\]

In 2010, some necessary and sufficient conditions for the following equation [30] were provided:

\[
\left( r(t) \left( (x(t) - p(t)x(\delta(t)))^{\Delta^w} \right) \right)^\Delta + \Phi(t, x(\tau(t))) = 0, \quad t \in [t_0, \infty)_T. \tag{6}
\]

Zhou [22] considered the following equation:

\[
(x(t) + p(t)x(t - \delta))^{\Delta^x} + \Phi(t, x(\tau(t))) = 0(t), \quad t \in [t_0, \infty)_T. \tag{7}
\]

Inspired by these works, we will establish oscillatory criteria for

\[
\left( \lambda_n(s) \Psi_n(\lambda_{n-1}(s) \cdots (\lambda_1(s) \Psi_1(y^\Delta(s)))^\Delta \cdots )^\Delta \right)^\Delta + \sum_{k=1}^{m} \phi_k(s)\Psi_k(y(\tau_k(s))) = 0, \quad s \in [s_0, \infty)_T. \tag{8}
\]

and nonoscillatory criteria for

\[
\left( \lambda_n(s) \Psi_n(\lambda_{n-1}(s) \cdots (\lambda_1(s) \Psi_1((y(s) + p(s)y(\delta(s)))^\Delta \cdots )^\Delta \right)^\Delta + \Phi(s, y(\tau_1(s)), y(\tau_2(s)), \ldots, y(\tau_m(s))) = 0, \quad s \in [s_0, \infty)_T. \tag{9}
\]

Comparing the previous equations with (8) and (9) yields that many existing research equations are covered by our research objects. Specifically, Equation (8) can be transformed into (2) by setting

\[
\Psi_n(s) = s^\gamma, \quad \Psi_k(s) = s, (k = 1, 2, \ldots, n - 1), \quad m = 1.
\]

Equations (1) and (4) are special cases of (8) when \( n = 1, m = 1 \) and \( n = 1, m = 1, \Psi_1(s) = \psi_1(s) = \sigma_t(s) = s^{\alpha-1}s, T = \mathbb{R}, \) respectively. In the same way, it is easy to check that Equation (9) is a generalized form of Equations (5)–(7). It is also worth noting that Equation (9) can change into (8) by taking \( p(s) \equiv 0 \) and

\[
\Phi(s, y(\tau_1(s)), y(\tau_2(s)), \ldots, y(\tau_m(s))) = \sum_{k=1}^{m} \phi_k(s)\phi_k(y(\tau_k(s))).
\]
That is, we investigate equations in a broader form in this paper, which means that the oscillatory behavior of some equations that cannot be judged by current oscillatory criteria may be solved, such as, Examples 1 and 2 in Section 4.

This paper is organized as follows: Some notations and hypotheses are given in Section 4. In Section 3.1, we first provided some lemmas, and then two oscillatory criteria of the Equation (8) are established via lemmas. In Section 3.2, some useful lemmas are proposed, and we will constitute sufficient and necessary conditions for the existence of positive solutions to Equation (9). Some examples are furnished to show our results in Section 4. In the end, we summarize this paper in Section 5.

2. Notations and Hypotheses

For convenience, notations and hypotheses are listed in this section. First, using recursive definition, we rewrite Equation (8) as follows

\[ L^A_n(s) + \sum_{k=1}^{m} \Phi_k(s) \Phi_k(y(\tau_k(s))) = 0, \]  

(10)

where \( L_k(s)(k = 0, 1, 2, \cdots, n) \) is defined as

\[ L_k(s) = \begin{cases} y(s), & \text{if } k = 0, \\ \lambda_k(s) \Phi_k(L^A_{k-1}(s)), & \text{if } k = 1, 2, \cdots, n. \end{cases} \]

Likewise, Equation (9) can be rewritten as

\[ Q^A_n(s) + \Phi(s, y(\tau_1(s)), y(\tau_2(s)), \cdots, y(\tau_m(s))) = 0, \]  

(11)

where \( Q_k(s)(k = 0, 1, 2, \cdots, n) \) is defined as

\[ Q_k(s) = \begin{cases} y(s) + p(s)y(d(s)), & \text{if } k = 0, \\ \lambda_k(s) \Phi_k(Q^A_{k-1}(s)), & \text{if } k = 1, 2, \cdots, n. \end{cases} \]

For convenience, we denote

\[ T\{\Phi\}(s) = \int_{s}^{\infty} \Psi^{-1}_1\left( \frac{1}{\lambda_1(s_1)} \right) \int_{s_1}^{\infty} \Psi^{-1}_2\left( \frac{1}{\lambda_2(s_2)} \right) \cdots \int_{s_{n-1}}^{\infty} \Psi^{-1}_n\left( \frac{1}{\lambda_n(s_n)} \right) \int_{s_n}^{\infty} \Phi(s_{n+1}, y(\tau_1(s_{n+1})), y(\tau_2(s_{n+1})), \cdots, y(\tau_m(s_{n+1}))) \Delta s_{n+1} \cdots \Delta s_2 \Delta s_1, \]

and

\[ T^r\{\Phi(r)\}(s) = \int_{s}^{\infty} \Psi^{-1}_1\left( \frac{1}{\lambda_1(s_1)} \right) \int_{s_1}^{\infty} \Psi^{-1}_2\left( \frac{1}{\lambda_2(s_2)} \right) \cdots \int_{s_{n-1}}^{\infty} \Psi^{-1}_n\left( \frac{1}{\lambda_n(s_n)} \right) \left( \frac{1}{\lambda_n(s_n)} \right) \int_{s_n}^{\infty} \Phi(s_{n+1}, r, r, \cdots, r) \Delta s_{n+1} \cdots \Delta s_2 \Delta s_1, \]  

(12)

where \( r > 0 \) is a constant.

We state the hypotheses in the following and will not repeat in the next section.

**Hypothesis 1.** \( \mathbb{T} \) is an unbounded time scale, we write \( [s_0, \infty) \cap \mathbb{T} = [s_0, \infty)_\mathbb{T} \) and obediently assume \( s_0 \in \mathbb{T} \).

**Hypothesis 2.** Function \( \Psi_k \) are odd, continuous and increasing, have inverse functions \( \Psi^{-1}_k \) for all \( k = 1, 2, \cdots, n \), and satisfy

\[ s \Psi_k(s) > 0, \quad \Psi^{-1}_k(xy) \geq K \Psi^{-1}_k(x) \Psi^{-1}_k(y), \quad x, y > 0, K > 0, k = 1, 2, \cdots, n. \]
Hypothesis 3. Function $\lambda_k, (k = 1, 2, \ldots, n)$ are positive functions and satisfy
$$
\int_{s_0}^{\infty} \Psi_{k}^{-1} \left( \frac{1}{\lambda_k(s)} \right) \Delta s = \infty, \quad k = 1, 2, \ldots, n.
$$

Hypothesis 4. $\phi_k, \tau_k, \varphi_k, (k = 1, 2, \ldots, n)$ are increasing continuous functions with
$$
\lim_{s \to \infty} \tau_k(s) = \infty. \quad \phi_k \text{ are positive functions and satisfy } \phi_k(s) \geq L \phi(s). \quad \varphi_k(s) \text{ satisfy } \varphi_k(s) > 0, \varphi_k(s) \geq s.
$$

Hypothesis 5.
$$
\int_{t}^{\infty} \frac{1}{\Psi_n^{-1} \left( \lambda_n^{-1}(t) \right)} \frac{1}{\Psi_n^{-1} \left( \lambda_n(s) \right)} \int_{s}^{\infty} \frac{1}{\Psi_n^{-1} \left( \lambda_n(u) \right)} \Delta u \Delta s \Delta t = \infty
$$

Hypothesis 6. Exists an $N_1 > 0$ subject to
$$
\frac{\phi_k(t)}{\Psi_n^{-1} \left( \lambda_n^{-1}(\cdots \Psi_1(t)) \right)} \geq N_1 \text{ and } \tau_k(t) \geq t,
$$
for all $t \in [s_0, \infty)$ and $k = 1, 2, \ldots, n$.

Hypothesis 7. Exists an $N_2 > 0$ subject to
$$
\frac{t^2 \Psi_n^{-1} \left( \lambda_n^{-1}(\cdots \Psi_1(t)) \right)}{\Psi_n^{-1} \left( \lambda_n^{-1}(\cdots \Psi_1(t)) \right)} \geq N_2 \text{ and } \Psi_n^{-1} \left( \lambda_n^{-1}(\cdots \Psi_1(t)) \right) \leq t,
$$
for all $t \in [s_0, \infty)$.

Hypothesis 8. Function $p \neq -1, 1$. Function $\Phi(v_0, v_1, v_2, \ldots, v_m)$ is positive and increasing with respect to $v_j (j = 0, 1, 2, \ldots, m)$. Exists a positive function $h(k)$ such that
$$
\Phi(v_0, kv_1, kv_2, \ldots, kv_m) \geq h(k) \Phi(v_0, v_1, v_2, \ldots, v_m).
$$

Hypothesis 9.
$$
\Psi_k^{-1}(xy) \leq K_2 \Psi_k^{-1}(x) \Psi_k^{-1}(y), \quad x, y > 0, K_2 > 0, k = 1, 2, \ldots, n.
$$

3. Main Results

In what follows, we use (10) and (11) to refer to the equations we considered rather than (8) and (9).

3.1. Oscillatory Criteria

Before we establish oscillatory criteria in Theorems 1 and 2, we need the following four lemmas to explore some properties of Equation (10).

Lemma 1. Assume Hypotheses 1–3 hold, then the following conclusions are true.

1. If $L_p(s) > 0$ on $[S, \infty)$ and $\lim_{s \to \infty} L_p(s) \neq 0$, then $\lim_{s \to \infty} L_j(s) = \infty$ for all $j = 0, 1, 2, \ldots, p - 1$.
2. If $L_p(s) < 0$ on $[S, \infty)$ and $\lim_{s \to \infty} L_p(s) \neq 0$, then $\lim_{s \to \infty} L_j(s) = -\infty$ for all $j = 0, 1, 2, \ldots, p - 1$.

Proof. (1) Clearly, exists a $l_1 > 0$ such that $L_p(s) \geq l_1$ on $[S, \infty)$, that is,
$$
L_{p-1}^\Delta(s) \geq \Psi_p^{-1} \left( \frac{l_1}{\lambda_p(s)} \right).
$$
Integrating from $S$ to $s$, we obtain
\[L_{p-1}(s) - L_{p-1}(S) \geq \int_S^s \Psi_p^{-1}\left(\frac{l_1}{\lambda_p(s)}\right) \Delta s \geq K \Psi_p^{-1}(l_1) \int_S^s \Psi_p^{-1}\left(\frac{1}{\lambda_p(s)}\right) \Delta s,\]
therefore, \(\lim_{s \to \infty} L_{p-1}(s) = \infty\). Moreover, we have \(\lim_{s \to \infty} L_k(s) = \infty(k = 0, 1, \ldots, p - 2)\).

(2) In the same way, there exists a \(l_2 > 0\) such that \(L_p(s) \leq -l_2\) on \([S, \infty)\), noting that \(\Psi_k\) are odd functions for all \(k = 1, 2, \ldots, n\), we have
\[L_{p-1}^\Delta(s) \leq \Psi_p^{-1}\left(-\frac{l_2}{\lambda_p(s)}\right) = -\Psi_p^{-1}\left(\frac{l_2}{\lambda_p(s)}\right).\]

Integrating from $S$ to $s$, we obtain
\[L_{p-1}(s) - L_{p-1}(S) \leq - \int_S^s \Psi_p^{-1}\left(\frac{l_2}{\lambda_p(s)}\right) \Delta s \leq -K \Psi_p^{-1}(l_2) \int_S^s \Psi_p^{-1}\left(\frac{1}{\lambda_p(s)}\right) \Delta s,\]
therefore, \(\lim_{s \to \infty} L_k(s) = -\infty(k = 0, 1, 2, \ldots, p - 1)\).

Under certain conditions, the following lemma says there are only two possibilities if Equation (10) has an eventually positive solution.

**Lemma 2.** Assume Hypotheses 1–5 hold, Equation (10) has an eventually positive solution. Then one of the following conclusions holds.

1. \(L_k(s) > 0\) for all \(s > S\) and \(k = 0, 1, 2, \ldots, n\), where \(S\) is a sufficiently large number.

2. \(\lim_{s \to \infty} y(s) = 0\).

**Proof.** Based on the fact that Equation (10) have a eventually positive solution, therefore exits \(s_1\) subject to \(\min\{\tau_1(s_1), \tau_2(s_1), \ldots, \tau_n(s_1)\} > 0\). Then
\[(\lambda_n(s) \Psi_n(L_{n-1}^\Delta(s)))^\Delta = - \sum_{k=1}^m \phi_k(s) \phi_k(y(\tau_k(s))) < 0,
\]
which means \(L_n = \lambda_n \Psi_n(L_{n-1}^\Delta)\) is strictly decreasing on \([s_1, \infty)\).

We claim that \(L_n(s) > 0\) on \([s_1, \infty)\), if not, there exists a \(s_2\) such that \(L_n(s) < 0\) on \([s_2, \infty)\). Due to \(L_n\) is strictly decreasing and \(L_n(s) < 0\), \(\lim_{s \to \infty} L_n(s) = 0\) is impossible. Based on Lemma 1, we have \(\lim_{s \to \infty} L_j(s) = -\infty\) for \(j = 0, 1, 2, \ldots, n - 1\), which contradicts the fact that \(y(s)\) is eventually positive.

Therefore, we know
\[L_{n-1}^\Delta(s) = \Psi_n^{-1}\left(\frac{L_n(s)}{\lambda_n(s)}\right) > 0,
\]
namely, \(L_{n-1}\) is strictly increasing.

In the same way, if \(L_{n-1}\) is eventually positive, then we have \(\lim_{s \to \infty} L_k(s) = \infty(k = 1, 2, \ldots, n - 2)\) by employing Lemma 1. Thus conclusion (1) holds. On the other hand, if \(L_{n-1} < 0\) for all \(s \in [s_1, \infty)\), then
\[L_{n-2}^\Delta = \Psi_{n-2}^{-1}\left(\frac{L_{n-1}(s)}{\lambda_{n-2}(s)}\right) < 0,
\]
which means \(L_{n-2}\) is strictly decreasing. Same as the cases \(L_n\), we can deduce that \(L_{n-2}(s) > 0\) on \([s_1, \infty)\). We claim that the conclusion (2) hold. If not, we have \(\lim_{s \to \infty} y(s) > 0\), there exists \(s_3\) and \(M > 0\) such that \(\min\{y(\tau_1(s)), y(\tau_2(s)), \ldots, y(\tau_n(s))\} \geq M\) on \([s_3, \infty)\). Integrating Equation (10) from \(s\) to \(\infty\), with respect to \(s\), we can obtain
\[
\lim_{s \to \infty} L_n(s) - L_n(s) = - \int_s^\infty \sum_{k=1}^m \phi_k(u) \phi_k(y(\tau_k(u))) \Delta u
\leq - \sum_{k=1}^m \phi_k(M) \int_s^\infty \sum_{k=1}^m \phi_k(u) \Delta u.
\]
namely

\[-L_n(s) = -\left(\lambda_n(s)\Psi_n(L_{n-1}^\Delta(s))\right) \leq -\sum_{k=1}^m \varphi_k(M) \int_s^\infty \sum_{k=1}^m \varphi_k(u) \Delta u.\]

Integrating on both sides and using Hypothesis 2, we have

\[L_{n-1}(t) \leq -\int_t^\infty \Psi_n^{-1}\left(\frac{1}{\lambda_n(s)} \int_s^\infty \sum_{k=1}^m \varphi_k(u) \Delta u\right) \Delta s.\]

Integrating and using Hypothesis 2 twice, we obtain

\[-L_{n-2}(v) \leq -l_2 \int_v^\infty \Psi_n^{-1}\left(\frac{1}{\lambda_{n-1}(t)} \int_t^\infty \Psi_n^{-1}\left(\frac{1}{\lambda_n(s)} \int_s^\infty \sum_{k=1}^m \varphi_k(u) \Delta u\right) \Delta s\right) \Delta t,\]

where \(l_2 = \frac{K^2}{\Psi_n^{-1}(K\Psi_n^{-1}(\sum_{k=1}^m \varphi_k(M))) > 0}.\) We can derive the contradiction from Hypothesis 5.

Lemma 3 establishes estimations of \(L_k(s)\) and \(L_k^\Delta(s)\), \((k = 0, 1, 2, \ldots, n-1)\) under the assumption that \(\lim_{s \to \infty} y(s) \neq 0\), that is, we find function \(f(s, L_n(s))\) and \(g(s, L_n(s))\) such that \(L_k(s) \geq f(s, L_n(s))\), \(L_k(s) \geq g(s, L_n(s))\) for all \(s \in [s_0, \infty)\) and \(k = 0, 1, 2, \ldots, n-1\).

**Lemma 3.** Assume Hypotheses 1–3 and the case (1) in Lemma 2 hold, then

\[L_k(s) \geq \eta_{k+1}(s)\Psi_n^{-1}(\cdots \Psi_n^{-1}(L_n(s)))\), \quad k = 0, 1, 2, \ldots, n-1,\]

and

\[L_k^\Delta(s) \geq \frac{\eta_{k+1}(s)}{\lambda_{k+1}(s)}\Psi_n^{-1}(\cdots \Psi_n^{-1}(L_n(s)))\), \quad k = 0, 1, 2, \ldots, n-1,\]

where \(\eta_k, k = 1, 2, \ldots, n+1\) are defined as

\[\eta_k(s) = \begin{cases} 1, & \text{if } k = n+1, \\ K \int_{s_1}^{s} \Psi_k^{-1}\left(\frac{\eta_{k+1}(u)}{\lambda_k(u)}\right) \Delta u, & \text{if } k = 1, 2, \ldots, n. \end{cases}\]

**Proof.** Integrating \(L_n^{-1}(s) = \Psi_n^{-1}\left(\frac{y(s)}{\lambda_n(s)}\right)\) on both sides from \(s_1\) to \(s\) and noting the fact that \(L_n\) is strictly decreasing which we have proved in Lemma 2, we yield

\begin{align*}
L_{n-1}(s) & \geq \int_{s_1}^{s} \Psi_n^{-1}\left(\frac{L_n(u)}{\lambda_n(u)}\right) \Delta u \\
& \geq \int_{s_1}^{s} \Psi_n^{-1}\left(\frac{L_n(s)}{\lambda_n(u)}\right) \Delta u \\
& \geq \frac{K}{\lambda_n(s)} \int_{s_1}^{s} \Psi_n^{-1}\left(\frac{1}{\lambda_n(u)}\right) \Delta u \\
& = \eta_n(s)\Psi_n^{-1}(L_n(s)).
\end{align*}

Using inequality (16), we can immediately get
Assume hypotheses 1–5 hold and there exits a function δ and Lemmas 1 and 2. Then \( f \) is \( \delta \)-differentiable. Then \( f \) satisfies the following inequality by employing conclusion (13)

\[
L_{n-2}(s) \geq \int_{s_1}^{s} \Psi_{n-1}^{-1} \left( \frac{L_{n-1}(u)}{\lambda_{n-1}(u)} \right) \Delta u
\]

\[
\geq \int_{s_1}^{s} \Psi_{n-1}^{-1} \left( \frac{\eta_n(u)}{\lambda_{n-1}(u)} \right) \Delta u
\]

\[
\geq K \int_{s_1}^{s} \Psi_{n-1}^{-1} \left( \Psi_{n-1}(L_n(u)) \right) \Psi_{n-1}^{-1} \left( \frac{\eta_n(u)}{\lambda_{n-1}(u)} \right) \Delta u
\]

\[
\geq K \Psi_{n-1}^{-1} \left( \Psi_{n-1}(L_n(s)) \right) \int_{s_1}^{s} \Psi_{n-1}^{-1} \left( \frac{\eta_n(u)}{\lambda_{n-1}(u)} \right) \Delta u
\]

\[
= \eta_{n-1}(s) \Psi_{n-1}^{-1} (\Psi_{n-1}(L_n(s))).
\]

We complete the proof by employing induction. Inequality (13) is held when \( j = n \) and \( j = n - 1 \). Supposing it’s held for \( j = k + 1 \), we will show the case \( j = k \) is also true. More specifically, we know that

\[
L_{k+1}(s) \geq \eta_{k+2}(s) \Psi_{k+2}^{-1} \left( \Psi_{k+3}^{-1}(\ldots \Psi^{-1}_n(L_n(s))) \right).
\]

Consequently, integrating \( L^\lambda_k(s) = \Psi_{k+1}^{-1} \left( \frac{L_{k+1}(s)}{\lambda_{k+1}(s)} \right) \) from \( s_1 \) to \( s \) on both sides, we have

\[
L_k(s) \geq \int_{s_1}^{s} \Psi_{k+1}^{-1} \left( \frac{L_{k+1}(u)}{\lambda_{k+1}(u)} \right) \Delta u
\]

\[
\geq \int_{s_1}^{s} K \Psi_{k+1}^{-1} \left( \Psi_{k+2}^{-1}(\ldots \Psi^{-1}_n(L_n(u))) \right) \Psi_{k+1}^{-1} \left( \frac{\eta_{k+2}(u)}{\lambda_{k+1}(u)} \right) \Delta u
\]

\[
\geq K \Psi_{k+1}^{-1} \left( \Psi_{k+1}(\ldots \Psi^{-1}_n(L_n(s))) \right) \eta_{k+1}(s).
\]

Hence, we can deduce the conclusion (13). Noting that \( L^\lambda_k(s) = \Psi_{k+1}^{-1} \left( \frac{L_{k+1}(s)}{\lambda_{k+1}(s)} \right) \), \( k = 0, 1, 2, \ldots, n - 1 \), we have the following inequality by employing conclusion (13)

\[
L^\lambda_k(s) \geq \Psi_{k+1}^{-1} \left( \frac{\eta_{k+2}(s)}{\lambda_{k+1}(s)} \right) \Psi_{k+2}^{-1}(\ldots \Psi^{-1}_n(L_n(s)))
\]

\[
\geq K \Psi_{k+1}^{-1} \left( \frac{\eta_{k+2}(s)}{\lambda_{k+1}(s)} \right) \Psi_{k+1}^{-1} \left( \Psi_{k+2}^{-1}(\ldots \Psi^{-1}_n(L_n(s))) \right),
\]

thereby, we complete the proof. \( \square \)

Lemma 4 is the chain rule on time scales and it will be used in Theorem 2.

Lemma 4 ([2], Theorem 2.57). Suppose function \( f \) is continuous and function \( g : T \to \mathbb{R} \) is delta-differentiable. Then \( f(g(x)) \) is delta-differentiable with

\[
(f(g(x)))^\lambda = \left( \int_0^1 f^\lambda (g(t) + h\mu(t)g^\lambda(t)) \Delta h \right) g^\lambda(t).
\]

The following Theorem can be established only when it’s based on hypotheses 1–5 and Lemmas 1 and 2.

Theorem 1. Assume hypotheses 1–5 hold and there exits a function \( \delta(s) \) defined on \( \mathbb{R} \) satisfies

\[
\int_{s_1}^{\infty} \delta(s)e^{-A(t)}(s,s_1) \Delta s = \infty,
\]

where

\[
A(t) = \frac{\delta^\lambda(t)}{\delta^\mu(t)} \frac{L_0(t)}{s \delta^\mu(t)}.
\]
Then Equation (10) is oscillatory or tends to zero.

**Proof.** We assume Equation (10) has an eventually positive or negative solution. Without loss of generality, we regard it positive. In fact, if \(y(s)\) is an eventually negative solution, we can prove that \(-y(s)\) is also a solution of Equation (10). If \(y(s)\) tends to zero, then the proof is complete. Based on Lemma 2, we can assume \(L_k(s) > 0\), \(k = 0, 1, 2, \cdots, n\). Set

\[
\omega(s) = \delta(s)
\left(\frac{L_n(s)}{\sum_{k=1}^{m} \phi_k(s) \varphi(y(\tau_k(s)))}\right),
\]

\[
\omega(s) = \delta(s)
\left(\frac{L_n(s)}{\Phi(s)}\right)
\]

for short, namely, we denote \(\Phi(s) = \sum_{k=1}^{m} \phi_k(s) \varphi(y(\tau_k(s)))\). Differentiating \(\omega\) with respect to \(s\) and using the delta quotient rule, we have

\[
\omega^\Delta(s) = \left(\frac{\delta(s)}{\Phi(s)}\right) L_n^\Delta(s) + \left(\frac{\delta(s)}{\Phi(s)}\right) \Delta L_n(s)
\]

\[
= -\delta(s) + \phi^\sigma(s) \Phi^\sigma(s) \frac{\delta^\Delta(s)}{\Phi^\sigma(s)}
\]

\[
= -\delta(s) + \omega^\sigma(s) \phi^\sigma(s) \frac{\delta^\Delta(s)}{\Phi^\sigma(s)}
\]

Now using Hypothesis 4, we can obtain

\[
\Phi^\Delta(s) = \sum_{k=1}^{m} \Phi_k(s) \varphi(y(\tau_k(s))) \Delta
\]

\[
\Sigma_{k=1}^{m} \left(\phi_k(s) \varphi(y(\tau_k(s))) + s \phi_k^\sigma(s) \varphi(y(\tau_k(s)))\right)
\]

\[
\sum_{k=1}^{m} \phi_k(s) \varphi(y(\tau_k(s)))
\]

Hence, we have

\[
\omega^\Delta(s) \leq -\delta(s) + \omega^\sigma(s) \frac{\delta^\Delta(s)}{\Phi^\sigma(s)} - \omega^\sigma(s) \frac{L \delta(s)}{s \Phi^\sigma(s)} = -\delta(s) + A(s) \omega^\sigma(s).
\]

Note that

\[
\left((\omega(s)e_{-A(s)}(s, s_1))^\Delta\right) = \omega^\Delta(s)e_{-A(s)}(s, s_1) - A(s) \omega^\sigma(s) e_{-A(s)}(s, s_1)
\]

\[
= \left(\omega^\Delta(s) - A(s) \omega^\sigma(s)\right)e_{-A(s)}(s, s_1) \leq -\delta(s)e_{-A(s)}(s, s_1),
\]

Delta integrating from \(s_1\) to \(s\) and letting \(s \to \infty\), we have

\[
\lim_{s \to \infty} \omega(s)e_{-A(s)}(s, s_1) - \omega(s_1)e_{-A(s)}(s_1, s_1) + \lim_{s \to \infty} \int_{s_1}^{s} \delta(u)e_{-A(u)}(u, s_1) \Delta u \leq 0,
\]

which is a contradiction based on the condition (17), hence we complete the proof. \(\Box\)

The second oscillatory criterion is established with more hypotheses and lemmas, however, it is more precise. In fact, this theorem has more applications for it has two arbitrary functions \(\delta(s)\) and \(m(s)\).
Theorem 2. Assume Hypotheses 1–7 hold and there exit functions \( \delta(s), m(s) \) defined on \( \mathbb{R} \) satisfies

\[
\int_{s_1}^{\infty} B(s) e^{-\frac{\lambda_n(s)}{P(y(s))}} (s, s_1) \Delta s = \infty,
\]  
(18)

where

\[
B(s) = -N_1 \delta(s) \frac{m}{k=1} \phi_k(s) + \delta(s) \left( \lambda_n(s)m(s) \right)^\Delta
\]

\[
- \delta(s) \eta_1(s) K \Psi^{-1}_1 \left( \eta_2(s) \left( \int_{t}^{\infty} \sum_{k=1}^{m} \phi_k(t) \Delta t \right) \right) \int_{s}^{\infty} \sum_{k=1}^{m} \phi_k(t) \Delta t.
\]

Then Equation (10) is oscillatory or tends to zero.

Proof. It suffices to prove Equation (10) is impossible to have an eventually positive solution is impossible under the assumption that \( L_k(s) > 0 \) for all \( k = 0, 1, 2, \cdots, n \). We set

\[
\omega(s) = \delta(s) \left( \frac{L_n(s)}{P(y(s))} + \lambda_n(s)m(s) \right),
\]

where \( P(v) = \Psi_{n-1}(\cdots \Psi_1(v)) \). Then

\[
\omega^\Delta(s) = \left( \delta(s) \frac{L_n(s)}{P(y(s))} \right)^\Delta + \left( \delta(s) \lambda_n(s)m(s) \right)^\Delta
\]

\[
= \delta(s) \frac{L_n(s)}{P(y(s))} \Delta \left( \lambda_n(s)m(s) \right)^\Delta + \delta^\Delta \lambda_n^\sigma(s)m^\rho(s)
\]

\[
= \delta(s) \frac{L_n(s)}{P(y(s))} \Delta \left( \lambda_n(s)m(s) \right)^\Delta + \delta^\Delta \lambda_n^\sigma(s)m^\rho(s)
\]

\[
= \delta(s) \left( \lambda_n(s)m(s) \right)^\Delta + \delta^\Delta \lambda_n^\sigma(s)m^\rho(s)
\]

(19)

Based on Hypothesis 6, we have

\[
- \delta(s) \sum_{k=1}^{m} \phi_k(s) \phi_k(y(\tau_k(s))) \leq - \delta(s) \frac{P(y(s))}{P(y(s))} \sum_{k=1}^{m} \phi_k(s) \phi_k(y(s)) \leq - N \delta(s) \sum_{k=1}^{m} \phi_k(s).
\]  
(20)

Since \( P(t) \leq t \) on \( t \in [s_0, \infty) \), then \( p^{\Delta \Delta} \leq 0 \), i.e., \( p^{\Delta} \) is decreasing. Then we have the following inequality by mean of Lemma 4.

\[
\left( P(y(s)) \right)^\Delta = \left( \int_{0}^{1} P^\Delta (h y^\rho(s) + (1 - h)y(s)) \, dh \right) y^\Delta(s)
\]

\[
\geq \left( \int_{0}^{1} P^\Delta (h y^\rho(s) + (1 - h)y(s)) \, dh \right) y^\Delta(s)
\]

\[
= P^\Delta (y^\rho(s)) y^\Delta(s)
\]
where \( P^\Delta(y^\sigma(s)) \) means derivative \( P \) with respect to \( y \) rather than \( s \). Noting that

\[
y^\Delta(s) = \frac{\left(\Psi^{-1}_1(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))\right)^2 y(s)}{\Psi_1^{-1}(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))} \frac{y^\Delta(s)}{\Psi^{-1}_1(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))}
\]

\[
\geq \frac{\left(\Psi^{-1}_1(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))\right)^2 y(s)}{\Psi_1^{-1}(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))} \eta_1(s) K_{\Psi^{-1}_1} \left( \frac{\eta_2(s)}{\lambda_1(s)} \right) ,
\]

(21)

then inequality (19) leads to

\[
\omega^\Delta(s) = -\frac{\delta(s)}{P(y(s))} \sum_{k=1}^{m} \phi_k(s) \phi_k(y(t_k(s))) + \delta(s) \left( \lambda_n(s) m(s) \right)^\Delta + \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s)
\]

\[
- \delta(s) \left( \frac{P(y(s))}{P(y(s))} \right)^\Delta L_n^\sigma(s)
\]

\[
\leq -N\delta(s) \sum_{k=1}^{m} \phi_k(s) + \delta(s) \left( \lambda_n(s) m(s) \right)^\Delta + \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s)
\]

\[
- \delta(s) \left( \frac{P(y^\sigma(s))}{P(y^\sigma(s))} \right)^\Delta L_n^\sigma(s)
\]

\[
\leq -N\delta(s) \sum_{k=1}^{m} \phi_k(s) + \delta(s) \left( \lambda_n(s) m(s) \right)^\Delta + \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s)
\]

\[
- \delta(s) \left( \frac{P(y^\sigma(s))}{P(y^\sigma(s))} \right)^\Delta \left( \Psi^{-1}_1(\Psi^{-1}_2(\cdots \Psi^{-1}_n(L_n(s))))\right)^2 \eta_1(s) K_{\Psi^{-1}_1} \left( \frac{\eta_2(s)}{\lambda_1(s)} \right) L_n^\sigma(s)
\]

By employing Hypothesis 7, we know \( P(t) \leq t \) and \( \Psi^{-1}_1(\Psi^{-1}_2(\cdots \Psi^{-1}_n(t))) \geq t \), therefore

\[
\omega^\Delta(s) \leq -N\delta(s) \sum_{k=1}^{m} \phi_k(s) + \delta(s) \left( \lambda_n(s) m(s) \right)^\Delta + \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s)
\]

\[
- \delta(s) \left( \frac{L_n(s)}{y(s)} \right) \left( \frac{L_n(s)}{y^\sigma(s)} \right) \eta_1(s) K_{\Psi^{-1}_1} \left( \frac{\eta_2(s)}{\lambda_1(s)} \right) \left( \frac{L_n(s)}{y(s)} \right)^\sigma
\]

Finally, we find a lower bound of \( L_n(s) \). In fact, integrating Equation (10), we have

\[
L_n(s) \geq \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \phi_k(y(t_k(t))) \Delta t
\]

\[
\geq \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \phi_k(y^\sigma(t)) \Delta t
\]

\[
\geq y^\sigma(s) \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \Delta t
\]

\[
\geq y(s) \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \Delta t.
\]

Therefore, we arrive at

\[
\omega^\Delta(s) \leq -N\delta(s) \sum_{k=1}^{m} \phi_k(s) + \delta(s) \left( \lambda_n(s) m(s) \right)^\Delta + \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s)
\]

\[
- \delta(s) \eta_1(s) K_{\Psi^{-1}_1} \left( \frac{\eta_2(s)}{\lambda_1(s)} \right) \left( \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \Delta t \right) \left( \int_{s}^{t} \sum_{k=1}^{m} \phi_k(t) \Delta t \right)
\]

\[
= \frac{\delta^\Delta(s)}{\psi^\sigma(s)} \omega^\sigma(s) + B(s).
\]
Likewise, noting that
\[ \left( \omega(s) e_{-\varphi(t)}(s,s_1) \right)^{\Delta} \leq B(s) e_{-\varphi(t)}(s,s_1), \]

Delta integrating from \( s_1 \) to \( s \) and letting \( s \to \infty \), we can deduce a contradiction based on the condition (18), hence we complete the proof. \( \square \)

3.2. Nonoscillatory Criteria

Same as Section 3.1, the following lemma which explores the properties of Equation (11) is given at the first place.

Lemma 5. If Equation (11) has a bounded positive solution and Hypotheses 1–3 and 8 hold, then we have the following conclusions.

(1) \((-1)^{n-j+1} Q^j_n(s) > 0 \) for all \( j = 0, 1, 2, \cdots, n \).

(2) \((-1)^{n-j} Q^j s > 0 \) for all \( j = 1, 2, \cdots, n \).

(3) \( \lim_{s \to \infty} Q^j s = 0 \) for all \( j = 1, 2, \cdots, n \).

Proof. Based on Equation (11) and Hypothesis 8, we have
\[ Q^\Delta_n(s) = -\Phi(s, y(t_1(s)), y(t_2(s)), \cdots, y(t_m(s))) < 0. \]

We claim that \( Q^\Delta_n(s) \geq 0 \), and \( \lim_{s \to \infty} Q^\Delta_n(s) = 0 \). If not, there exits \( s_1 \) and \( l_1 > 0 \) such that \( |Q^\Delta_n(s)| \geq l_1 \) on \([s_1, \infty)\). If \( Q^\Delta_n(s) \) is eventually positive, then
\[ Q^\Delta_{n-1}(s) - Q^\Delta_{n-1}(s_1) \geq \int_{s_1}^{s} \Psi_n^{-1} \left( \frac{l_1}{\lambda_n(s)} \right) \Delta s \geq K \Psi_n^{-1}(l_1) \int_{s_1}^{s} \Psi_n^{-1} \left( \frac{l_1}{\lambda_n(s)} \right) \Delta s, \]

therefore \( \lim_{s \to \infty} Q^\Delta_n(s) = 0 \), which contradicts the fact that \( Q^\Delta_n(s) \) is bounded. If \( Q^\Delta_n(s) \) is eventually negative, it can deduce \( \lim_{s \to \infty} Q^\Delta_n(s) = -\infty \) in the same way. It also contradicts the conclusion that \( Q^\Delta_n(s) \) is bounded.

Noting that \( Q^\Delta_{n-1} = \Psi_n \left( \frac{Q^\Delta_n(s)}{\lambda_n(s)} \right) > 0 \), We can know that \( Q^\Delta_{n-1}(s) < 0 \) and \( \lim_{s \to \infty} Q^\Delta_{n-1}(s) = 0 \). Repeating the process, we can get the conclusions. \( \square \)

Next two lemmas are about fixed point theorem which can be found in [29,34].

Lemma 6 ([29]). If \( f^\Delta \) is uniformly bounded, then \( f \) is equicontinuous.

Lemma 7 (Kranoselkii’s fixed point theorem, see [34]). If \( \Omega \subset X \) is closed, convex and bounded, exist two maps \( L_1 \) and \( L_2 : \Omega \to X \) such that

(1) \( L_1 y_1 + L_2 y_2 \in \Omega \) for all \( y_1, y_2 \in \Omega \).

(2) \( L_1 \) is a contraction,

(3) \( L_2 \) is completely continuous.

Then the equation
\[ L_1 y + L_2 y = y, \]

has a solution in \( \Omega \).

The following Theorem establishes a sufficient and necessary condition for the existence of positive solution for Equation (11).

Theorem 3. If Hypotheses 1–3, 8 and 9 hold, then Equation (11) has an eventually bounded nonoscillatory solution \( y(s) \) with \( \lim_{s \to \infty} y(s) \neq 0 \) if and only if \( T^* \{ \Phi(u) \} \{ s \} < \infty \), where \( T^* \{ \Phi(u) \} \{ s \} \) is defined by (12) and \( u \geq \max_{s \in [s_0, \infty)} y(s) \).
Proof. Sufficiency. If Equation (11) has an eventually positive bounded solution, then there exist a constant \( w \) and \( S > 0 \) such that \( \min\{y(s), y(t_1(s)), y(t_2(s)), \ldots, y(t_m(s))\} > u/w \) on \([S, \infty)\). Noting
\[
Q_n^+(s) = -\Phi(s, y(t_1(s)), y(t_2(s)), \ldots, y(t_m(s)))
\]
\[
< -\Phi(s, \frac{u}{w}, \ldots, \frac{u}{w}) \leq -h(\frac{1}{w})\Phi(s, u, \ldots, u),
\]
integrating from \( s_n > S \) to \( \infty \), we have
\[
Q_n(s_n) > h(\frac{1}{w}) \int_{s_n}^{\infty} \Phi(s_{n+1}, u, u, \ldots, u) \Delta s_{n+1}.
\]

Together with the conclusions (2) and (3) in Lemma 5, inequality (22) and Hypothesis 9, we have
\[
T^*\{\Phi(u)\}(s)
\]
\[
= \int_s^{\infty} \Psi_1^{-1}(\frac{1}{\lambda_1(s_1)}) \int_{s_1}^{\infty} \Psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-1}}^{\infty} \Psi_n^{-1}(\frac{1}{\lambda_n(s_n)}) \Phi(s_n, u, \ldots, u) \Delta s_1 \Delta s_2 \cdots \Delta s_n.
\]

\[
= \int_s^{\infty} \Psi_1^{-1}(\frac{1}{\lambda_1(s_1)}) \int_{s_1}^{\infty} \Psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-1}}^{\infty} \Psi_n^{-1}(\frac{1}{\lambda_n(s_n)}) Q_n(s_n) \Delta s_1 \Delta s_2 \cdots \Delta s_n.
\]

\[
\leq \int_s^{\infty} \Psi_1^{-1}(\frac{1}{\lambda_1(s_1)}) \int_{s_1}^{\infty} \Psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-1}}^{\infty} \Psi_n^{-1}(\frac{1}{\lambda_n(s_n)}) \lim_{s_n \to \infty} Q_n(s_n) \Delta s_1 \Delta s_2 \cdots \Delta s_n.
\]

\[
\leq \int_s^{\infty} \Psi_1^{-1}(\frac{1}{\lambda_1(s_1)}) \int_{s_1}^{\infty} \Psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-2}}^{\infty} \Psi_{n-2}^{-1}(K_{\Psi_n}^{-1}(\frac{1}{h(s_{n-2})})) \int_{s_{n-2}}^{\infty} -Q_n(s_n) \Delta s_{n-1} \cdots \Delta s_2 \Delta s_1.
\]

\[
\leq \cdots
\]

\[
\leq K^*(-1)^n Q_0(s) < \infty,
\]

where
\[
k^* = K\Psi_1^{-1}(K_{\Psi_2}^{-1}(K_{\Psi_3}^{-1}(\cdots K_{\Psi_n}^{-1}(\frac{1}{h(s_n)})))).
\]

Thereby we complete the proof.

Necessity. Case 1: \(-1 < p_1 \leq p(s) \leq 0\). Supposing solution \( y(s) \) satisfies \( l \leq y(s) \leq u \) on \([s_1, \infty)\), we define
\[
\Omega = \{ y \in C_{nu}([s_0, \infty), (-\infty, \infty)) : l \leq y(s) \leq u, s \geq s_0 \}. 
\]
Clearly, \( \Omega \) is a bounded, convex, and closed subset of \( C_{rd}([s_0, \infty), (-\infty, \infty)) \) which is a Banach space.

Since \( T^\ast \{ \Phi(u) \} \)(s) < \infty, there exists \( S > s_0 \) such that
\[
T^\ast \{ \Phi(u) \}(s) \leq \frac{(1 + p_1)(u - l)}{2}, \quad s \in [S, \infty).
\]

We set
\[
(L_1y)(s) = \begin{cases} \frac{(1 + p_1)(l + u)}{2} - p(s)y(d(s)), & s \geq S, \\ (L_1y)(S), & S \geq s \geq s_0, \end{cases}
\]
and
\[
(L_2y)(s) = \begin{cases} (-1)^n T\{ \Phi \}(s), & s \geq S, \\ (L_2y)(S), & S \geq s \geq s_0. \end{cases}
\]

1. We claim \((L_1y_1)(s) + (L_2y_2)(s) \in \Omega \) for all \( y_1, y_2 \in \Omega \). Clearly, we have
\[
(L_1y_1)(s) + (L_2y_2)(s) \leq \frac{(1 + p_1)(l + u)}{2} - p_1u + T\{ \Phi \}(s) \leq \frac{(1 + p_1)(l + u)}{2} - p_1u + \int_s^\infty \psi_1^{-1}(\frac{1}{\lambda_1(s)}) \int_{s_1}^\infty \psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \int_{s_{n-1}}^{s_n} \Phi(s_{n+1}, u, \ldots, u) \Delta s_{n+1} \Delta s_n \cdots \Delta s_1 \Delta s_1 \\
\leq \frac{(1 + p_1)(l + u)}{2} - p_1u + \frac{(1 + p_1)(u - l)}{2} = u.
\]

Moreover, the following inequalities hold
\[
(L_1y_1)(s) + (L_2y_2)(s) \geq \frac{(1 + p_1)(l + u)}{2} - p_1u - T\{ \Phi \}(s) \geq \frac{(1 + p_1)(l + u)}{2} - p_1u - (1 + p_1)(u - l) = l.
\]

Hence, we have proved that \((L_1y_1)(s) + (L_2y_2)(s) \in \Omega \) for all \( y_1, y_2 \in \Omega \).

2. We claim \((L_1y)(s) \) is a contraction on \( \Omega \). It is clear that
\[
|(L_1y_1)(s) - (L_2y_2)(s)| \leq -p_1|y_1(d(s)) - y_2(d(s))| \leq -p_1 \| y_1 - y_2 \|, \quad s > S,
\]
where \( \| x \| = \sup_{s > s_0} |x(s)| \). It completes the proof.

3. We claim \((L_2y)(s) \) is completely continuous. We have that proved \( L_2\Omega \) is uniformly bounded. Note the following inequalities hold
\[
|\psi_2^{-1}(n)| = \left| \psi_1^{-1}(\frac{1}{\lambda_1(s)}) \int_s^\infty \psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-1}}^{s_n} \psi_2^{-1}(\frac{1}{\lambda_2(s_n)}) \int_{s_{n-1}}^{s_n} \Phi(s_{n+1}, u, \ldots, u) \Delta s_{n+1} \Delta s_n \cdots \Delta s_1 \Delta s_1 \right|
\leq \left| \psi_1^{-1}(\frac{1}{\lambda_1(s)}) \int_s^\infty \psi_2^{-1}(\frac{1}{\lambda_2(s_2)}) \cdots \int_{s_{n-1}}^{s_n} \psi_2^{-1}(\frac{1}{\lambda_2(s_n)}) \int_{s_{n-1}}^{s_n} \Phi(s_{n+1}, u, \ldots, u) \Delta s_{n+1} \Delta s_n \cdots \Delta s_1 \right|.
\]

Thus, we deduce that \( L_2\Omega \) is uniformly bounded based on Lemma 6.

Now using Lemma 7, there exist a solution of equation \( L_1y + L_2y = y \), which means Equation (11) has an eventually bounded positive solution.
Case 2. If \(-\infty < p(s) \leq p_2 < -1\). Then we can deduce the same conclusion in Case 1 by setting
\[
(\mathcal{L}_1 y)(s) = \begin{cases} \frac{(1+p_2)(l+u)}{2p_2} - \frac{1}{p(d^{-1}(s))} y(d^{-1}(s)), & s \geq S, \\ (\mathcal{L}_1 y)(S), & S \geq s \geq s_0, \end{cases}
\]
and \(T^*\{\Phi(u)\}(s) \leq -\frac{(p_2+1)(l-u)}{2}\).

Case 3. If \(0 \leq p(s) \leq p_3 < 1\). In the same way, we take
\[
(\mathcal{L}_1 y)(s) = \begin{cases} \frac{(1+p_3)(l+u)}{2} - p(s) y(d(s)), & s \geq S, \\ (\mathcal{L}_1 y)(S), & S \geq s \geq s_0, \end{cases}
\]
\([L_2 y](s)\) same as case 1 and \(T^*\{\Phi(u)\}(s) \leq \frac{(p_3-1)(l-u)}{2}\).

Case 4. If \(1 < p_4 \leq p(s) < \infty\). It is sufficient to let
\[
(\mathcal{L}_1 y)(s) = \begin{cases} \frac{(1+p_4)(l+u)}{2p_4} - \frac{1}{p(d^{-1}(s))} y(d^{-1}(s)), & s \geq S, \\ (\mathcal{L}_1 y)(S), & S \geq s \geq s_0, \end{cases}
\]
\([L_2 y](s)\) same as case 2 and \(T^*\{\Phi(u)\}(s) \leq \frac{(p_4-1)(l-u)}{2}\). Thereby we complete the proof. \(\Box\)

If we take \(p(s) \equiv 0\) and
\[
\Phi(s, y(\tau_1(s)), y(\tau_2(s)), \ldots, y(\tau_m(s))) = \sum_{k=1}^m \phi_k(s) \phi_k(y(\tau_k(s))),
\]
then we have the following corollary.

**Corollary 1.** If Hypotheses 1–4 and 9 hold, then Equation (10) has an eventually bounded nonoscillatory solution \(y(s)\) with \(\lim_{s \to \infty} y(s) \neq 0\) if and only if there exits \(u\) such that
\[
\int_s^\infty \Psi^{-1}_1 \left( \frac{1}{\lambda_1(s_1)} \right) \int_{s_1}^\infty \Psi^{-1}_2 \left( \frac{1}{\lambda_2(s_2)} \right) \cdots \int_{s_{n-1}}^\infty \Psi^{-1}_n \left( \frac{1}{\lambda_n(s_n)} \right) \int_{s_n}^\infty \sum_{k=1}^m \phi_k(s_n) \phi_k(u) \Delta s_n+1 \Delta s_n+2 \Delta s_2 \Delta s_1
\]
is finite, where \(u\) is satisfying \(u \geq \max_{s \in [s_0, \infty)} y(s)\).

**4. Examples**

**Example 1.** Consider the equation on \(\mathbb{T} = \mathbb{N},\)
\[
L_n(s) + \sum_{k=1}^m s^{c_{2,k}} (y(\tau_k(s)))^{\gamma_{2,k}} = 0, \quad s \in [2, \infty)_\mathbb{N},
\]
where \(\Psi_k(s) = s^{\gamma_k}, \lambda_k(s) = s^{\beta_k}\) and \(\gamma_k\) are the quotient of odd positive integers satisfies
\[
\beta_k \leq \gamma_k, \quad c_{1,k}, c_{2,k} \geq 1.
\]

**Proof.** Since \(\Psi_k^{-1}(s) = s^{\frac{1}{\gamma_k}}\), Hypothesis 2 hold for \(k = 1\). Based on the fact \(\beta_k \leq \gamma_k\), Hypothesis 3 hold. Noting that
\[
\frac{s \phi_k(s)}{\phi_k(s)} = s^{\Delta_k s^{c_{2,k}}(s_{2,k})^{\gamma_k}} \geq \frac{s c_{1,k} s^{c_{2,k}} - c_{1,k}}{s^{2,k}} = c_{1,k} \phi_k(s) = s^{c_{2,k}} \geq s.
\]
Then Hypothesis 4 hold for \(L = c_{1,k}\). Hypothesis 5 is also true due to \(\int_s^\infty \phi_k(s) \Delta s = \infty\).
We take $\delta(s) = s$ in (17), then
\[
\int_{2}^{\infty} s e_{c_{1,k}}(s, s_1) \Delta s = \int_{2}^{\infty} s \prod_{k=2}^{\infty} \left(1 + \frac{c_{1,k} - 1}{k(k+1)}\right) \Delta s \\
\geq \int_{2}^{\infty} s \left(1 + \sum_{k=2}^{\infty} \frac{c_{1,k} - 1}{k(k+1)}\right) \Delta s = \int_{2}^{\infty} s \left(1 + \frac{c_{1,k} - 1}{2}\right) \Delta s = \infty.
\]

Based on Theorem 1, we know Equation (24) is oscillatory or tends to zero. \square

**Example 2.** Consider the equation on $\mathbb{T} = \mathbb{R}$,
\[
L_n(s) + \sum_{k=1}^{m} s^{c_{1,k}}(y(\tau_k(s)))^{c_{2,k}} = 0, \quad s \in [2, \infty),
\]
where $\Psi_k(s) = s^{\gamma_k}, \lambda_k(s) = s^{\beta_k}$ and $\gamma_k$ is the quotient of odd positive integers satisfies $\beta_k \leq \gamma_k(k = 1, 2, \cdots, n - 1), \beta_n - \min_{1 \leq k \leq m} \{c_{1,k} + 1\} \leq \gamma_n, \prod_{k=1}^{n} \gamma_k \leq 1, c_{2,k} \geq 1, c_{1,k} \leq -1$.

**Proof.** Clearly, Hypotheses 1–4 hold same as Example 1. Using the fact that $\beta_n - \min_{1 \leq k \leq m} \{c_{1,k} + 1\} \leq \gamma_n$, we have
\[
\int_{s}^{\infty} \Psi_n^{-1} \left(\frac{1}{\lambda_n(s)} \int_{s}^{\infty} \sum_{k=1}^{m} \phi_k(u) \Delta u\right) \Delta s
\]
\[
= \int_{s}^{\infty} \left(\frac{1}{s^{\beta_n}} \sum_{k=1}^{m} \left(-\frac{1}{1 + c_{1,k} s^{c_{1,k} + 1}}\right) \frac{1}{\gamma_k}\right) \Delta s = \infty,
\]
which means Hypothesis 5 holds. Moreover
\[
\frac{\phi_k(s)}{\Psi_n^{-1}(\Psi_{n-1}((\cdots(\Psi_{1}(s)))))} = \frac{s^{c_{2,k}}}{s^{n_{\gamma_k}}} \geq 2^1 = 2 = N_1,
\]
and
\[
\frac{s^{(s^{n_{\gamma_k} + 1})'}}{s^{n_{\gamma_k}}} \geq \prod_{k=1}^{n} \gamma_k = N_2,
\]
showing Hypotheses 6 and 7 hold.

We take $\delta(s) = \frac{1}{2}$ in (18), then $-s^{\delta(s)} - \frac{1}{s^{\delta(s)}} = \frac{1}{2}$. Based on the recursion Formula (15), we can easily check that there exist $C_1, C_2, D_1, D_2 > 0$ such that $\eta_1(s) \leq C_1 s^{D_1}$ and $\eta_2(s) \leq C_2 s^{D_2}$. Moreover, we have $\int_{0}^{\infty} \sum_{k=1}^{m} \phi_k(t) \Delta t < \infty$. Then there exist $m(s) = s^{E}$, where $E$ is sufficiently large, such that $B(s) \geq 1$. Therefore, we have
\[
\int_{2}^{\infty} B(s) e_{\frac{1}{2}}(s, 2) \Delta s = \int_{2}^{\infty} B(s) \frac{s}{2} \Delta s = \infty.
\]
Equation (25) is either oscillatory or tends to zero based on Theorem 2. \square

**Example 3.** Consider the equation on $\mathbb{T} = \mathbb{R}$,
\[
(y(s) + p(s)y(d(s)))^{[m]} + \sum_{k=1}^{m} \left(s^{c_{1,k}}(y(\tau_k(s)))^{c_{2,k}}\right) = 0, \quad s \in [1, \infty)
\]
where $c_{1,k} < -n$ and $c_{2,k} > 0$ for all $k = 1, 2, \cdots, m$.

**Proof.** In this equation
\( \lambda_k(s) \equiv 1, \quad \Psi_k(s) \equiv s, \quad \Phi(s,y(\tau_1(s)),y(\tau_2(s)),\ldots,y(\tau_n(s))) = \sum_{k=1}^{m} s^{c_{1,k}}(y(\tau_k(s)))^{r_{2,k}}. \)

It is easy to check Hypotheses 1-4 hold. We can find that

\[
T^+ \{ \Phi(r) \} (s) = \int_{s}^{\infty} \psi_1^{-1} \left( \frac{1}{\lambda_1(s_1)} \right) \int_{s_1}^{\infty} \psi_2^{-1} \left( \frac{1}{\lambda_2(s_2)} \right) \ldots \\
\int_{s_{n-1}}^{\infty} \psi_n^{-1} \left( \frac{1}{\lambda_n(s_n)} \right) \int_{s_n}^{\infty} \sum_{k=1}^{m} s^{c_{1,k}}(s_{n+1}^{r_{2,k}} \Delta s_{n+1}) \Delta s_n \ldots \Delta s_1 \Delta s_1 \\
= \int_{s}^{\infty} \int_{s_{n-1}}^{\infty} \ldots \int_{s_{n-2}}^{\infty} \int_{s_{n-1}}^{\infty} \sum_{k=1}^{m} \frac{c_{1,k}}{c_{1,k}+1} s^{c_{1,k}+1} s_{n-1} \ldots s_2 \Delta s_1 \\
= \ldots \\
= \int_{s}^{\infty} \left( -1 \right)^n \sum_{k=1}^{m} \prod_{j=1}^{n} \frac{r_{2,k}}{(c_{1,k}+j)} s_{1} \Delta s_1 \\
= (-1)^{n+1} \sum_{k=1}^{m} \prod_{j=1}^{n} \frac{r_{2,k}}{(c_{1,k}+j)} s_{1} < \infty,
\]

where use the fact that \( \lim_{t \to \infty} r_{2,k}^{c_{1,k}+l} = 0 \) for all \( l = 0, 1, 2, \ldots, n \). Based on Theorem 3, we know that Equation (26) has a nonoscillatory solution. \( \square \)

5. Conclusions

In this paper, we consider the higher order dynamic equations which have more universal functions \( \Psi_k \) and more delay functions. Some properties of the equations are presented in lemmas. Then oscillatory and nonoscillatory criteria are established in theorems via Riccati technique and fixed point theorem. In the end, we provide some examples to check our results.


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