mathematics

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# Nonlinear Differential Equations with Distributed Delay: Some New Oscillatory Solutions

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**Abstract:** The oscillation of a class of fourth-order nonlinear damped delay differential equations with distributed deviating arguments is the subject of this research. We propose a new explanation of the fourth-order equation oscillation in terms of the oscillation of a similar well-studied second-order linear differential equation without damping. The extended Riccati transformation, integral averaging approach, and comparison principles are used to provide some additional oscillatory criteria. An example demonstrates the efficacy of the acquired criteria.

**Keywords:** oscillation; fourth-order; damping term; Riccati transformation; comparison theorem; distributed deviating arguments

**MSC:** 34C10; 34K11



**Citation:** Almarri, B.; Ali, A.H.; Lopes, A.M.; Bazighifani, O. Nonlinear Differential Equations with Distributed Delay: Some New Oscillatory Solutions. *Mathematics* **2022**, *10*, 995. https://doi.org/10.3390/math10060995

Academic Editor: Ioannis G. Stratis

Received: 9 March 2022
Accepted: 17 March 2022
Published: 19 March 2022



## 1. Introduction

In our current study, we take into consideration the following fourth-order nonlinear damped delay differential equations with distributed deviating arguments:

$$\left(x_2(t)\left(x_1(t)\left(u''(t)\right)^{\alpha}\right)'\right)' + p(t)\left(u''(\delta(t))\right)^{\alpha} + \int_c^d q(t,\varrho)f(t,u(g(t,\varrho)))d\varrho = 0, \quad (1)$$

where $\alpha \geq 1$ is a ratio of odd non-negative natural numbers and $c < d$. We consider the below assertions all through this article:

$$\begin{cases} x_1, x_2, p, \delta \in C\left(I, [0,\infty)\right) \text{ and } x_1, x_2 > 0, \text{ where } I = [t_0, +\infty); \\ q, g \in C[I \times [c,d], [0,\infty)), \delta(t) \leq t, \lim_{t\to+\infty} \delta(t) = \infty, \; g(t,\varrho) \text{ is a non-decreasing} \\ \text{function for } \varrho \in [c,d] \text{ satisfying } g(t,\varrho) \leq t \text{ and } \lim_{t\to+\infty} g(t,\varrho) = \infty; \\ f \in C(\mathbb{R},\mathbb{R}), \text{ there is a constant } k_1 > 0 \text{ such that } f(t, u(t))/u^{\beta} \geq k_1. \end{cases}$$

We define the operators,

$$L^{[0]}u = u,\; L^{[1]}u = u',\; L^{[2]}u = x_1\left(\left(L^{[0]}u\right)''\right)^{\alpha},\; L^{[3]}u = x_2\left(L^{[2]}u\right)' \quad \text{as well as} \quad L^{[4]}u = (L^{[3]}u)'.$$

The meaning of having a solution to Equation (1) is the function $u(t)$ in $C^2[T_u,\infty)$, for which $L^{[2]}u$, $L^{[4]}u$ is in $C^1[T_u,\infty)$, and Equation (1) holds on $[T_u,\infty)$, such that $T_u \geq t_0$. We only take into consideration the solutions $u(t)$ when $\sup\{|u(t)| : t \geq T\} > 0$ for every $T \geq T_u$. On one hand, such a solution to Equation (1) is termed oscillatory when this solution is not eventually negative and, at the same time, not eventually positive on the interval $[T_u,\infty)$. On the other hand, the same solution is termed non-oscillatory if it is eventually negative or eventually positive. Finally, when every solution is oscillating, the equation is said to be oscillatory.

We define

$$\begin{aligned}
A_1(t_1,t) &= \int_{t_1}^{t} x_1^{-1/\alpha}(s)ds, \\
A_2(t_1,t) &= \int_{t_1}^{t} x_2^{-1}(s)ds, \\
A_3(t_1,t) &= \int_{t_1}^{t} \Big( (x_1(s))^{-1} A_2(t_1,s) \Big)^{1/\alpha} ds, \\
A_4(t_1,t) &= \int_{t_1}^{t} \int_{t_1}^{u} \Big( (x_1(s))^{-1} A_2(t_1,s) \Big)^{1/\alpha} ds\, du,
\end{aligned}$$

for $t_0 \leq t_1 \leq t < \infty$ and assume that

$$A_1(t_1,t) \to \infty, \quad A_2(t_1,t) \to \infty \quad \text{as} \quad t \to \infty. \tag{2}$$

Fourth-order differential equations are often used in mathematical models of a wide range of physical, chemical, and biological processes [1–4]. Problems with elasticity, structural deformation, and soil settling are examples of applications of this type of equation. In addition, in mechanical and engineering fields, questions about the presence of oscillatory and non-oscillatory solutions are mostly arising, and the solutions require the presence of the same mentioned equation [5]. Many researchers have intensively studied the topic of oscillation of fourth or higher order differential equations in depth, and many strategies for establishing oscillatory criteria for fourth or higher order differential equations have been developed. Several works, see [6–18], contain extremely interesting results linked to oscillatory features of solutions of neutral differential equations and damped delay differential equations with or without distributed deviating arguments.

In fact, for the following equation, Bazighifan et al. [19] have developed some oscillation criteria

$$\big(r(t)\big(N_x'''(t)\big)^{\beta}\big)' + \int_a^b q(t,\varrho)x^{\beta}(g(t,\varrho))d\varrho = 0.$$

Moreover, Dzurina et al. [20] introduced some oscillation findings of the below fourth-order equation

$$\big(r_3(t)\big(r_2(t)(r_1(t)y'(t))\big)'\big)' + p(t)y'(t) + q(t)y(\tau(t)) = 0.$$

More specifically, there are no requirements for the oscillation of Equation (1) in the previous studies.

By the motivations above, our contribution would be giving certain adequate conditions that ensure that every solution to Equation (1) oscillates, utilizing proper Riccati-type transformation, integral averaging condition, and comparison technique, when the following second-order equation

$$(x_2(t)z'(t))' + \frac{p(t)}{x_1(\delta(t))}z(t) = 0, \tag{3}$$

is oscillatory or non-oscillatory.

## 2. Basic Lemmas

We state in the current section several Lemmas along with their proofs, which are mostly needed in the rest of this study.

**Lemma 1** ([8]). *Assume that Equation (3) is non-oscillatory. If Equation (1) has a non-oscillatory solution $u(t)$ on $I$, $t_1 \geq t_0$, then there is a $t_2 \in I$ in a way that $u(t)L^{[2]}u(t) > 0$ or $u(t)L^{[2]}u(t) < 0$ for $t \geq t_2$.*

**Lemma 2.** *If the Equation (1) has a non-oscillatory solution $u(t)$ that satisfies $u(t)L^{[2]}u(t) > 0$ in Lemma 1 for $t \geq t_1 \geq t_0$, then*

$$L^{[2]}u(t) > A_2(t_1,t)\, L^{[3]}u(t), \quad t \geq t_1, \tag{4}$$

$$L^{[1]}u(t) > A_3(t_1,t)\, \left(L^{[3]}u(t)\right)^{1/\alpha}, \quad t \geq t_1, \tag{5}$$

*and*

$$u(t) > A_4(t_1,t)\, \left(L^{[3]}u(t)\right)^{1/\alpha}, \quad t \geq t_1. \tag{6}$$

**Proof.** We suppose that there is a $t_1 \geq t_0$ in a way that $u(t) > 0$ and $u(g(t,\varrho)) > 0$ for $t \geq t_1$. From Equation (1), we have

$$L^{[4]}u(t) = -\Big(\frac{p(t)}{x_1(\delta(t))}\Big) L^{[2]}u(\delta(t)) - k_1 \int_c^d q(t,\varrho) u^{\beta}(g(t,\varrho)) d\varrho \leq 0,$$

and $L^{[3]}u(t)$ is non increasing on $I$, we obtain

$$L^{[2]}u(t) \geq \int_{t_1}^{t} \big(L^{[2]}u(s)\big)'\, ds = \int_{t_1}^{t} (x_2(s))^{-1} L^{[3]}u(s)\, ds \geq A_2(t_1,t)\, L^{[3]}u(t),$$

which implies that

$$u''(t) \geq \left(L^{[3]}u(t)\right)^{1/\alpha} \left((x_1(t))^{-1} A_2(t_1,t)\right)^{1/\alpha}.$$

Now, twice integrating above from $t_1$ to $t$ and using $L^{[3]}u(t) \leq 0$, we find

$$u'(t) \geq \left(L^{[3]}u(t)\right)^{1/\alpha} \int_{t_1}^{t} \left((x_1(s))^{-1} A_2(t_1,s)\right)^{1/\alpha} ds$$

and

$$u(t) \geq \left(L^{[3]}u(t)\right)^{1/\alpha} \int_{t_1}^{t} \int_{t_1}^{u} \left((x_1(s))^{-1} A_2(t_1,s)\right)^{1/\alpha} ds\, du \quad \text{for } t \leq t_1.$$

□

**Lemma 3** ([10]). *Let $\xi \in C^1(I,\mathbb{R}^+)$, $\xi(t) \leq t$, $\xi'(t) \geq 0$ and $G(t) \in C(I,\mathbb{R}^+)$ for $t \geq t_0$. Assume that $y(t)$ is a bounded solution of a second-order delay differential equation:*

$$\big(x_2(t)\, y'(t)\big)' - \Theta(t)\, y(\xi(t)) = 0. \tag{7}$$

*If*

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} \Theta(s)\, A_2(\xi(t),\, \xi(s))\, ds > 1 \tag{8}$$

*or*

$$\limsup_{t\to\infty} \int_{\xi(t)}^{t} \Big( \big(x_2(t)\big)^{-1} \int_u^t \Theta(s)\, ds \Big) du > 1, \tag{9}$$

*where $x_2(t)$ is as in Equation (1); thus, the solutions of Equation (7) are oscillatory.*

## 3. Oscillation—Comparison Principle Method

In this section, we shall establish some oscillation criteria for Equation (1). For convenience, we denote

$$Q(t) = \Big(\frac{p(t)}{x_1(\delta(t))}\Big) A_2(t_1, \delta(t)), \quad \psi(t) = \exp\Big(\int_{t_1}^t Q(s) ds\Big),$$
$$\widetilde{q}(t,\varrho) = \int_c^d q(t,\varrho)\, d\varrho, \quad \Theta^*(t) = k_1\, \widetilde{q}(t,\varrho) \Big(A_4(t_1, g(t,d))\Big)^{\beta}.$$

**Theorem 1.** *Assume that $\alpha \geq \beta$ and the conditions in Equation (2) hold, and Equation (3) is non-oscillatory. Suppose there exists a $\xi \in C^1(I, \mathbb{R})$ such that*

$$g(t,\varrho) \leq \xi(t) \leq \delta(t) \leq t, \quad \xi'(t) \geq 0 \quad \text{for } t \geq t_1,$$

*and Equations (8) or (9) holds with*

$$\Theta(t) = \ell_*\, k_1 \widetilde{q}(t,\varrho) g^{\beta}(t,d) \big(A_1(\xi(t), g(t,d))\big)^{\beta} - \frac{p(t)}{x_1(\delta(t))} \geq 0, \quad t \geq t_1,$$

*for constant $\ell_* > 0$. Moreover, suppose that every solution of the first-order delay equation*

$$z'(t) + \psi^{1-\frac{\beta}{\alpha}}(g(t,d))\, \Theta^*(t)\, z^{\frac{\beta}{\alpha}}(g(t,d)) = 0. \tag{10}$$

*Then, every solution of Equation (1) is oscillatory.*

**Proof.** Let Equation (1) have a non-oscillatory solution $u(t)$. Assume there exists a $t \geq t_1$ such that $u(t) > 0$ and $u(g(t,\varrho)) > 0$ for some $t \geq t_0$. From Lemma 1, $u(t)$ has the conditions either $L^{[2]}u(t) > 0$ or $L^{[2]}u(t) < 0$ for $t \geq t_1$.

Assume that $u(t)$ has the condition $L^{[2]}u(t) > 0$ for $t \geq t_1$, then one can easily see that $L^{[3]}u(t) > 0$ for $t \geq t_1$. We can choose $t_2 \geq t_1$ such that $g(t,\varrho) \geq t_1$ for $t \geq t_2$, $g(t,\varrho) \to \infty$ as $t \to \infty$, and we have Equation (6),

$$u(g(t,d)) > A_4(t_1, g(t,d))\, \big(L^{[3]}u(g(t,d))\big)^{1/\alpha}, \quad t \geq t_2. \tag{11}$$

By substituting Equations (4) and (11) into Equation (1) and when $L^{[3]}u(t)$ is decreasing,

$$\Big(L^{[3]}u(t)\Big)' + \Big(\frac{p(t)}{x_1(\delta(t))}\Big) L^{[3]}u(t) A_2(t_1, \delta(t))$$
$$+ k_1\, \widetilde{q}(t,\varrho) \Big(A_4(t_1, g(t,d))\Big)^{\beta} \Big(L^{[3]}u(g(t,d))\Big)^{\beta/\alpha} \leq 0. \tag{12}$$

Taking $\phi = L^{[3]}u$, we have

$$\phi'(t) + Q(t)\phi(t) + \Theta^*(t)\phi^{\frac{\beta}{\alpha}}(g(t,d)) \leq 0 \tag{13}$$

or

$$\Big(\psi(t)\, \phi(t)\Big)' + \psi(t)\Theta^*(t)\phi^{\frac{\beta}{\alpha}}(g(t,d)) \leq 0, \quad \text{for } t \geq t_2. \tag{14}$$

Next, setting $z = \psi\,\phi > 0$ and $\psi(g(t,d)) \leq \phi(t)$, we have

$$z'(t) + \psi^{1-\frac{\beta}{\alpha}}(g(t,d))\Theta^*(t)z^{\frac{\beta}{\alpha}}(g(t,d)) \leq 0. \tag{15}$$

This means Equation (15) is positive for this inequality. Furthermore, by ([21], Corollary 2.3.5), it can be seen that Equation (1) has a positive solution, a contradiction.

Next, assume $u(t)$ has the condition $L^{[2]}u(t) < 0$, for $t \geq t_1$, then one can easily see that $L^{[1]}u(t) \geq 0$ , $L^{[3]}u(t) > 0$ for $t \geq t_3(\geq t_2)$. Using the monotonicity of $u'(t)$ and mean value property of differentiation, there exists a $\theta \in (0,1)$ such that

$$u(t) \geq \theta\, t\, u'(t), \quad \text{for } t \geq t_3. \tag{16}$$

Set $w(t) = L^{[1]}u(t)$, then $w'(t) = u''(t) < 0$. Using Equation (16) in Equation (1), we obtain

$$\big(x_2(t)\big(x_1(t)\big[w'(t)\big]^{\alpha}\big)'\big)' + p(t)(w'(\delta(t)))^{\alpha} + k_1\,(t\theta)^{\beta}\,\widetilde{q}(t,\varrho)w^{\beta}(g(t,d)) \leq 0,$$

and so $\big(x_1(t)\big[w'(t)\big]^{\alpha}\big) < 0$, we have $\big(x_1(t)\big[w'(t)\big]^{\alpha}\big)' > 0$ for $t \geq t_3$. Now, for $v \geq u \geq t_3$, we obtain

$$\begin{aligned} w(u) > w(u) - w(v) &= -\int_u^v -x_1^{-1/\alpha}(\tau)(x_1(\tau)(w'(\tau))^{\alpha})^{1/\alpha}d\tau \\ &\geq x_1^{1/\alpha}(v)(-w'(v)))\left(\int_u^v x_1^{-1/\alpha}(\tau)d\tau\right) \\ &= x_1^{1/\alpha}(v)(-w'(v))A_1(u,v). \end{aligned}$$

Taking $u = \xi(t)$ and $v = g(t,d)$, we obtain

$$w(g(t,d)) > A_1(g(t,d),\xi(t))\big(x_1^{1/\alpha}(\xi(t))(-w'(\xi(t)))\big) = A_1(g(t,d),\xi(t))\; y(\xi(t)),$$

where $y(t) = x_1^{1/\alpha}(\xi(t))(-w'(\xi(t)))\, > 0$ for $t \geq t_3$. From Equation (1), we have that $y(t)$ is decreasing and $g(t,d) \leq \xi(t) \leq \delta(t) \leq t$; thus, we obtain

$$(x_2(t)z'(t))' + \frac{p(t)}{x_1(\delta(t))}z(\delta(t)) \geq k_1\,(\theta g(t,d))^{\beta}\,\widetilde{q}(t,\varrho)A_1(g(t,d),\xi(t))z^{\frac{\beta}{\alpha}-1}(\xi(t))z(\xi(t)).$$

Since $z$ is decreasing and $\alpha \geq \beta$, there exists a constant $\ell$ such that $z^{\frac{\beta}{\alpha}-1}(t) \geq \ell$ for $t \geq t_3$. Thus, we obtain

$$(x_2(t)z'(t))' \geq \left(\ell\, k_1\,(\theta\, g(t,d))^{\beta}\,\widetilde{q}(t,\varrho)A_1(g(t,d),\xi(t)) - \frac{p(t)}{x_1(\delta(t))}\right)z(\xi(t)).$$

Proceeding the rest of the proof in Lemma (3), we arrive at the required conclusion, and so it is omitted. □

## 4. Oscillation—Riccati Method

This section deals with some oscillation criteria for Equation (1) using the Ricatti Method.

**Theorem 2.** *Assume $\alpha \geq \beta$ and the conditions in Equation (2) hold, Equation (3) is non-oscillatory. Suppose there exists $\eta$, $\xi \in C^1(I,\mathbb{R})$ such that $g(t,\varrho) \leq \xi(t) \leq \delta(t) \leq t$, $\xi'(t) \geq 0$ and $\eta > 0$ for $t \geq t_1$ with*

$$\limsup_{t\to\infty} \int_{t_5}^{t}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds = \infty \text{ for all } t_1 \in I, \tag{17}$$

*where, for* $t \geq t_1$,

$$A(t) = \frac{\eta'(t)}{\eta(t)} - \frac{p(t)}{x_1(\delta(t))} A_2(t_1, \delta(t)) \tag{18}$$

*and*

$$B(t) = \frac{\beta \ell_2^{\beta-\alpha} g'(t,d)}{\eta(t)} \Big( A_4(t_1, g(t,d)) \Big)^{\beta-1} \Big( A_3(t_1, g(t,d)) \Big)^{1/\alpha}, \tag{19}$$

*also Equations (8) or (9) hold with* $\Theta(t)$ *as in Theorem 1. Then every solution of Equation (1) is oscillatory.*

**Proof.** Suppose that Equation (1) has a non-oscillatory solution $u(t)$. Assume that, there exists a $t \geq t_1$ such that $u(t) > 0$ and $u(g(t,\varrho)) > 0$ for some $t \geq t_0$. From Lemma 1, $u(t)$ has the conditions either $L^{[2]}u(t) > 0$ or $L^{[2]}u(t) < 0$ for $t \geq t_1$. If condition $L^{[2]}u(t) < 0$ holds, the proof follows from Theorem 1.

Next, if condition $L^{[2]}u(t) > 0$ holds, define

$$\omega(t) = \eta(t) \frac{L^{[3]}u(t)}{u^{\beta}(g(t,d))}, \quad t \in I, \tag{20}$$

then $\omega(t) > 0$ for $t \geq t_1$. From Equation (6) and $L^{[4]}u(t) < 0$, we have

$$\omega(t) = \eta(t) \frac{L^{[3]}u(t)}{u^{\beta}(g(t,d))} \leq \eta(t) \frac{L^{[3]}u(g(t,d))}{u^{\beta}(g(t,d))} \leq \eta(t)(A_4(t_1, g(t,d)))^{-\alpha} u^{\alpha-\beta}(g(t,d)), \tag{21}$$

for $t \geq t_1$. From Equation (5) and definition $L^{[2]}u(t)$, we find

$$u'(g(t,d)) = L^{[1]}u(g(t,d)) \geq A_3(t_1, g(t,d))(L^{[3]}u(\delta(t)))^{1/\alpha} \geq A_3(t_1, g(t,d))(L^{[3]}u(g(t,d)))^{1/\alpha}.$$

Then,

$$\begin{aligned} \frac{u'(g(t,d))}{u(g(t,d))} &\geq \left( \frac{A_3(t_1, g(t,d))}{\eta(\delta(t))} \right)^{1/\alpha} \frac{\eta^{1/\alpha}(\delta(t))(L^{[3]}u(t))^{1/\alpha}}{u^{\beta/\alpha}(g(\delta(t),d))} u^{\beta/\alpha-1}(g(\delta(t),d)) \\ &= \left( \frac{A_3(t_1, g(t,d))}{\eta(t)} \right)^{1/\alpha} \omega^{1/\alpha}(t) u^{\beta/\alpha-1}(g(\delta(t),d)). \end{aligned} \tag{22}$$

Furthermore, since there exists a constant $\ell_1$ and $t_2 \geq t_1$ such that for $L^{[3]}u(t) \leq L^{[3]}u(t_2) = \ell_1$. Therefore,

$$\begin{aligned} L^{[2]}u(t) &= L^{[2]}u(t_2) + \int_{t_2}^{t} (L^{[2]}u(s))' ds \leq L^{[2]}u(t_2) + \ell_1 \int_{t_2}^{t} \frac{ds}{x_2(s)} \\ &= L^{[2]}u(t_2) + \ell_1 A_2(t_2, t) = \left[ \frac{L^{[2]}u(t_2)}{A_2(t_2, t)} + \ell_1 \right] A_2(t_2, t) \\ &\leq \left[ \frac{L^{[2]}u(t_2)}{A_2(t_2, t_3)} + \ell_1 \right] A_2(t_2, t) = \ell_1^* A_2(t_2, t), \end{aligned} \tag{23}$$

holds for all $t \geq t_2$, where $\ell_1^* = \ell_1 + \frac{L^{[2]}u(t_1)}{A_2(t_2,t_3)}$, which implies that

$$u'(t) = u'(t_3) + \int_{t_3}^{t} u''(s)ds \leq u'(t_3) + \int_{t_3}^{t} \left( \frac{\ell_1^* A_2(t_2,s)}{x_1(s)} \right)^{1/\alpha} ds$$
$$= u(t_3) + \left(\ell_1^*\right)^{1/\alpha} A_3(t_3,t) = \ell_2\, A_3(t_3,t),$$

holds for all $t \geq t_3 (\geq t_2)$, where $\ell_2 = \frac{u(t_2)}{A_3(t_3,t_4)} + (\ell_1^*)^{1/\alpha}$. Then,

$$u(t) = u(t_4) + \int_{t_4}^{t} u'(s)ds \leq u(t_4) + \int_{t_4}^{t} \Big(\ell_2\, A_3(t_3,s)\Big) ds$$
$$= u(t_4) + \ell_2\, A_4(t_4,t) = \ell_2^*\, A_4(t_4,t), \tag{24}$$

holds for all $t \geq t_4 (\geq t_3)$, where $\ell_2^* = \frac{u(t_4)}{A_4(t_4,t_1)} + \ell_2$. Further,

$$u^{\beta/\alpha - 1}(g(t,d)) \geq \left(\ell_2^*\right)^{\beta/\alpha - 1} \left(A_4(t_4, g(t,d))\right)^{\beta/\alpha - 1}, \quad t \geq t_4. \tag{25}$$

By using Equation (24) in Equation (21), we obtain

$$\omega(t) \leq \left(\ell_2^*\right)^{\alpha-\beta} \eta(t)\, (A_4(t_1, g(t,d)))^{-\beta}, \tag{26}$$

and hence

$$\omega^{\frac{1}{\alpha}-1}(t) \leq \left(\ell_2^*\right)^{(\alpha-\beta)(\frac{1}{\alpha}-1)} \eta^{\frac{1}{\alpha}-1}(t)\, (A_4(t_1, g(t,d)))^{-\beta(\frac{1}{\alpha}-1)}. \tag{27}$$

Now differentiating Equation (20), we obtain

$$\omega'(t) = \frac{\eta'(t)}{\eta(t)}\omega(t) + \frac{L^{[4]}u(t)}{L^{[3]}u(t)}\omega(t) - \beta g'(t,d) \frac{u'(g(t,d))}{u(g(t,d))}\omega(t). \tag{28}$$

Using Equations (1) and (4) in Equation (28), we have

$$\begin{aligned} \omega'(t) &\leq \left[\frac{\eta'(t)}{\eta(t)} - \frac{p(t)}{x_1(g(t,d))}\, A_2(t_4, g(t,d))\right]\omega(t) - k_1\eta(t)\tilde{q}(t,\varrho) - \beta g'(t)\frac{u'(g(t,d))}{u(g(t,d))}\omega(t) \\ &\leq A(t)\omega(t) - k_1\eta(t)\tilde{q}(t,\varrho) - \beta g'(t)\frac{u'(g(t,d))}{u(g(t,d))}\omega(t). \end{aligned} \tag{29}$$

By using Equations (22), (25) and (28) in Equation (29), we have

$$\begin{aligned} \omega'(t) &\leq A(t)\omega(t) - k_1\eta(t)\tilde{q}(t,\varrho) - \frac{\beta\, \ell_2^{\beta-\alpha}\, g'(t)}{\eta(t)} \Big(A_4(t_1, g(t,d))\Big)^{\beta-1} \Big(A_3(t_1, g(t,d))\Big)^{1/\alpha} \omega^2(t) \\ &= A(t)\omega(t) - k_1\eta(t)\tilde{q}(t,\varrho) + B(t)\omega^2(t) \end{aligned} \tag{30}$$

$$\begin{aligned} &= -k_1\eta(t)\tilde{q}(t,\varrho) + \left[\sqrt{B(t)}\,\omega(t) - \frac{1}{2}\frac{A(t)}{\sqrt{B(t)}}\right]^2 + \frac{1}{4}\frac{A^2(t)}{B(t)} \\ &\leq -k_1\eta(t)\tilde{q}(t,\varrho) + \frac{1}{4}\frac{A^2(t)}{B(t)}. \end{aligned} \tag{31}$$

Integrating Equation (31) from $t_5(> t_4)$ to $t$ gives

$$\int_{t_5}^{t} \left( k_1\, \eta(s)\, \tilde{q}(s,\varrho) - \frac{1}{4}\frac{A^2(s)}{B(s)} \right) ds \leq \omega(t_5), \tag{32}$$

which contradicts Equation (17). □

**Corollary 1.** *Assume $\alpha \geq \beta$ and the conditions in Equation (2) hold, Equation (3) is non-oscillatory. Suppose there exists $\eta, \xi \in C^1(I, \mathbb{R})$ such that $g(t, \varrho) \leq \xi(t) \leq \delta(t) \leq t$, $\xi'(t) \geq 0$ and $\eta > 0$ for $t \geq t_1$ such that the function $A(t) \leq 0$,*

$$\limsup_{t\to\infty} \int_{t_5}^{t} \Big(\eta(s)\,\widetilde{q}(s,\varrho)\Big) ds = \infty \text{ for all } t_1 \in I, \tag{33}$$

*where $A(t)$ is defined in Equation (18), and Equations (8) or (9) holds with $\Theta(t)$ as in Theorem 1. Then, every solution of Equation (1) is oscillatory.*

Next, we examine the oscillation results of solutions to Equation (1) by Philos-type. Let $\mathbb{D}_0 = \{(t,s) : a \leq s < t < +\infty\}$, $\mathbb{D} = \{(t,s) : a \leq s \leq t < +\infty\}$, the continuous function $H(t,s)$, $H : \mathbb{D} \to \mathbb{R}$ belongs to the class function $\mathbb{R}$:

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $(t,s) \in \mathbb{D}_0$;

(ii) $H$ has a continuous and non-positive partial derivative on $\mathbb{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)[H(t,s)]^{1/2},$$

for all $(t,s) \in \mathbb{D}_0$.

**Theorem 3.** *Assume $\alpha \geq 1$ and the conditions in Equation (2) hold, and Equation (3) is non-oscillatory. Suppose there exists $\eta, \xi \in C^1(I, \mathbb{R})$ such that $g(t, \varrho) \leq \xi(t) \leq \delta(t) \leq t$, $\xi'(t) \geq 0$, $\eta > 0$ and $H(t,s) \in \mathbb{R}$ for $t \geq t_1$ with*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_5)} \int_{t_5}^{t} \left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) H(t,s) - \frac{P^2(t,s)}{4B(s)}\right) ds = \infty \quad \text{for all} \quad t_1 \in I, \tag{34}$$

*where $P(t,s) = h(t,s) - A(s)\sqrt{H(t,s)}$ and $A(t)$, $B(t)$ are defined in Theorem 2, and Equations (8) or (9) holds with $\Theta(t)$ as in Theorem 1. Then, every solution of Equation (1) is oscillatory.*

**Proof.** Suppose that Equation (1) has a non-oscillatory solution $u(t)$. Assume that there exists a $t \geq t_1$ such that $u(t) > 0$ and $u(g(t,\varrho)) > 0$ for some $t \geq t_0$. Proceeding as in the proof of Theorem 2, we obtain the inequality from Equation (30), i.e.,

$$\omega'(t) \quad \leq \quad A(t)\omega(t) - k_1\eta(t)\widetilde{q}(t,\varrho) + B(t)\omega^2(t),$$

and so,

$$\begin{aligned}
\int_{t_5}^{t} H(t,s)\eta(s)\widetilde{q}(s,\varrho)ds \quad &\leq \quad \int_{t_5}^{t} H(t,s)[-\omega'(s) + A(s)\omega(s) - B(s)\omega^2(s)]ds \\
&= \quad -H(t,s)\Big[\omega(s)\Big]_{t_5}^{t} + \int_{t_5}^{t} \Big[\frac{\partial H(t,s)}{\partial s}\omega(s) \\
&\quad\ + H(t,s)\Big[A(s)\omega(s) - B(s)\omega^2(s)\Big]\Big] ds \\
&= \quad H(t,t_5)\omega(t_5) - \int_{t_5}^{t} \Big[\omega^2(s)B(s)H(t,s) \\
&\quad\ + \omega(s)\Big(h(t,s)\sqrt{H(t,s)} - H(t,s)A(s)\Big)\Big] ds \\
&\leq \quad H(t,t_5)\omega(t_5) + \int_{t_5}^{t} \frac{P^2(t,s)}{4B(s)} ds,
\end{aligned}$$

which contradicts Equation (34). The rest of the proof is similar to that of Theorem 2 and hence is omitted. □

## 5. Examples

Below, we present an example to show the application of the main results. This example is given to demonstrate Theorem 2.

**Example 1.** *For $t \geq 1$, consider the fourth-order differential equation*

$$\left(1/2t\left(9e^{-t}(t)\left(u''(t)\right)\right)'\right)' + 36e^{-s/2}u''(t/2) + \int_1^2 \frac{t}{3}u(\varrho, 36e^{t/3})d\varrho = 0. \quad (35)$$

*Here, $x_1 = 9e^{-t}$, $x_2 = 1/2t$, $\alpha = \beta = 1$, $p(t) = 36e^{-s/2}$, $q(t,\varrho) = t/3$ and $\delta(t) = t/2$, $g(t,\varrho) = t/3$. Now, pick $\eta(t) = 36e^{t/3}$, so we obtain*

$$\begin{aligned} A_1(t_1,t) &= \int_1^t (9e^s)^{-1}ds = 9(e^t - e), \\ A_2(t_1,t) &= \int_1^t 2s\,ds = t^2 - 1 = (t+1)(t-1), \\ A_3(t_1,t/3) &= \int_1^{t/3} (9e^s)^{-1}(s^2-1)ds = e^{t/3}(t-3)^2, \\ \widetilde{q}(s,\varrho) &= \frac{s}{3}\int_1^2 d\varrho = s/3, \end{aligned}$$

*$A^2(s) = \frac{(3t^2-5)^2}{9}$ and $B(s) = \frac{(s-3)^2}{36}$. Now,*

$$\limsup_{t\to\infty} \int_2^t \left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds = \limsup_{t\to\infty} \int_2^t \left(12k_1\,s\,e^{s/3} - \left(\frac{3s^2-5}{s-3}\right)^2\right)ds \to \infty \text{ as } t \to \infty,$$

*and all hypotheses of Theorem 2 are satisfied, so every solution of Equation (35) is oscillatory.*

## 6. Conclusions

The form in Equation (1) is clearly more generic than all of the problems covered in the literature. In this paper, we provided some oscillatory properties using the appropriate Riccati-type transformation, integral averaging condition, and comparison method, ensuring that any solution of Equation (1) oscillates under the assumption of $A_1(t_1,t) \to \infty$, $A_2(t_1,t) \to \infty$ as $t \to \infty$. Furthermore, based on the condition of $A_1(t_1,t) < \infty$, $A_2(t_1,t) < \infty$ as $t \to \infty$, it would be desirable to expand the oscillation criteria of Equation (1).

**Author Contributions:** Formal analysis, B.A., A.H.A. and A.M.L.; Data curation, O.B. and A.M.L.; Funding acquisition, A.M.L.; Methodology, B.A., A.H.A. and O.B.; Project administration, A.M.L.; Resources, A.H.A. and O.B.; Software, A.H.A. and O.B.; Supervision, A.M.L. and O.B.; Validation, A.M.L. and O.B.; Visualization, A.H.A.; Writing—review and editing, A.H.A. and O.B. All authors have read and agreed to the published version of the manuscript.

**Funding:** Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R216), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Althobati, S.; Bazighifan, O.; Yavuz, M. Some Important Criteria for Oscillation of Non-Linear Differential Equations with Middle Term. *Mathematics* **2021**, *9*, 346. [CrossRef]
2. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear Neutral Delay Differential Equations of Fourth-Order: Oscillation of Solutions. *Entropy* **2021**, *23*, 129. [CrossRef] [PubMed]
3. Agarwal, R.P.; Grace, S.R. The oscillation of higher-order differential equations with deviating arguments. *Comput. Math. Appl.* **1999**, *38*, 185–199. [CrossRef]
4. Agarwal, R.P.; Grace, S.R.; O'Regan, D. Oscillation of certain fourth order functional differential equations. *Ukrain. Mat. Zh.* **2007**, *59*, 291–313. [CrossRef]
5. Bartušek, M.; Cecchi, M.; Došlá, Z.; Marini, M. Fourth-order differential equation with deviating argument. *Abstr. Appl. Anal.* **2012**, *2012*, 185242. [CrossRef]
6. Agarwal, R.P.; Grace, S.R.; Wong, P.J. On the bounded oscillation of certain fourth order functional differential equations. *Nonlinear Dyn. Syst. Theory* **2005**, *5*, 215–227.
7. Bartušek, M.; Došlá, Z.: Oscillatory solutions of nonlinear fourth order differential equations with a middle term. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *55*, 1–9. [CrossRef]
8. Grace, S.R.; Sun, S.; Akın, E. Oscillation Criteria for Certain Fourth-Order Nonlinear Delay Differential Equations. *Mediterr. J. Math.* **2016**, *13*, 2383–2396. [CrossRef]
9. Grace, S.R. Oscillation theorems for nth-order differential equations with deviating arguments. *J. Math. Anal. Appl.* **1984**, *101*, 268–296. [CrossRef]
10. Grace, S.R. Oscillation criteria for third-order nonlinear delay differential equations with damping. *Opuscula Math.* **2015**, *35*, 485–497. [CrossRef]
11. Grace, S.R.; Džurina, J.; Jadlovská, I.; Li, T. On the oscillation of fourth-order delay differential equations. *Adv. Differ. Equ.* **2019**, *1*, 1–15. [CrossRef]
12. Wei, M.; Jiang, C.; Li. T. Oscillation of third-order neutral differential equations with damping and distributed delay. *Adv. Differ. Equ.* **2019**, *1*, 1–11. [CrossRef]
13. Kumar, M.S.; Bazighifan, O.; Al-Shaqsi, K.; Wannalookkhee, F.; Nonlaopon, K. Symmetry and Its Role in Oscillation of Solutions of Third-Order Differential Equations. *Symmetry* **2021**, *13*, 1485. [CrossRef]
14. Hou, C.; Cheng, S.S. Asymptotic dichotomy in a class of fourth-order nonlinear delay differential equations with damping. *Abstr. Appl. Anal.* **2009**, *2009*, 7. [CrossRef]
15. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
16. Saker, S.; Sethki, A. Riccati Technique and Oscillation of Second Order Nonlinear Neutral Delay Dynamic Equations. *J. Comput. Anal. Appl.* **2021**, *2021*, 266.
17. Jadlovská, I.; Chatzarakis, G.E.; Džurina, J.; Grace, S.R. On Sharp Oscillation Criteria for General Third-Order Delay Differential Equations. *Mathematics* **2021**, *9*, 1675. [CrossRef]
18. Tiryaki, A.; Aktas, M.F. Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping. *J. Math. Anal. Appl.* **2007**, *325*, 54–68. [CrossRef]
19. Bazighifan, O.; Minhos, F.; Moaaz, O. Sufficient Conditions for Oscillation of Fourth-Order Neutral Differential Equations with Distributed Deviating Arguments. *Axioms* **2020**, *9*, 39. [CrossRef]
20. Dzurina, J.; Baculíková, B.; Jadlovská, I. New oscillation results to fourth order delay differential equations with damping. *Electron. J. Qual. Theory Differ. Equ.* **2016**, 1–15. [CrossRef]
21. Agarwal, R.P.; Grace, S.R.; O'Regan, D. Oscillation theory for second order dynamic equations. In S*eries Mathematical Analysis Applications*; Taylor and Francis: London, UK, 2003; Volume 5.