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Improvements of Slater's Inequality by Means of 4-Convexity and Its Applications

Xuexiao You [1], Muhammad Adil Khan [2,*], Hidayat Ullah [2] and Tareq Saeed [3]

1 School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China; youxuexiao@126.com
2 Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan; hidayat53@uop.edu.pk
3 Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; tsalmalki@kau.edu.sa
* Correspondence: madilkhan@uop.edu.pk

Abstract: In 2021, Ullah et al., introduced a new approach for the derivation of results for Jensen's inequality. The purpose of this article, is to use the same technique and to derive improvements of Slater's inequality. The planned improvements are demonstrated in both discrete as well as in integral versions. The quoted results allow us to provide relationships for the power means. Moreover, with the help of established results, we present some estimates for the Csiszár and Kullback–Leibler divergences, Shannon entropy, and Bhattacharyya coefficient. In addition, we discuss some additional applications of the main results for the Zipf–Mandelbrot entropy.

Keywords: convex function; Slater's inequality; Jensen's inequality; power mean; information theory; Zipf–Mandelbrot entropy

MSC: 26A51; 26D15; 68P30

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1. Introduction

Functions are the most important and fundamental concepts in almost all areas of science, especially in mathematics. The functions are used as key research objects in mathematics for modeling and solving many real world phenomena. There are numerous important classes of functions, one of the most interesting classes of functions is the class of convex functions [1–5]. This class of functions has several interesting properties and due to such properties and its behavior with solving problems, it become a focus point for the researchers [6–8]. This class of functions has been applied in many fields, including engineering [9], statistics [10], optimization [11] economics [12], information theory [13] and epidemiology [14], etc. Due to the huge importance of this class, it has been generalized, improved, and expanded in diverse directions while utilizing its behavior and properties [15]. In an elegant manner, convex function can be defined as:

Definition 1. *A real valued function Ψ is said to be convex on $[a, b]$, if the inequality*

$$\Psi(\xi_1\gamma + (1-\xi_1)\zeta) \leq \xi_1\Psi(\gamma) + (1-\xi_1)\Psi(\zeta) \tag{1}$$

is valid, for all γ, $\zeta \in [a, b]$ and $\xi_1 \in [0, 1]$.

If for the aforesaid conditions, the inequality (1) is valid in the reverse direction, then the function Ψ is said to be concave.

As a result of considerable applicability of the convex functions class, many important generalizations of this class have been investigated such like $P-$convex, $s-$convex, coordinate convex and quasi convex functions and many more. Among these generalizations of convex functions, one of them is the class of 4-convex functions. To give the definition of 4-convex function, first we present divided difference:

Consider the arbitrary function $\Psi : [a,b] \to \mathbb{R}$ and let $\zeta_0, \zeta_1, \dots, \zeta_m$ be any distinct points from $[a,b]$, then the m^{th} ordered divided difference of Ψ at the selected points is defined recursively as:

$$[\zeta_i]\Psi = \Psi(\zeta_i), \qquad i = 1, 2, \dots, m,$$

$$[\zeta_0, \zeta_1, \dots, \zeta_m]\Psi = \frac{[\zeta_1, \dots, \zeta_m]\Psi - [\zeta_0, \dots, \zeta_{m-1}]\Psi}{\zeta_m - \zeta_0}$$

and the 4th ordered divided difference is given by:

$$[\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4]\Psi = \frac{[\zeta_1, \zeta_2, \zeta_3, \zeta_4]\Psi - [\zeta_0, \zeta_1, \zeta_2, \zeta_3]\Psi}{\zeta_4 - \zeta_0}.$$

Now, we give the definition of 4-convex function.

Definition 2 ([2])**.** *A function $\Psi : [a,b] \to \mathbb{R}$ is said to be 4-convex, if the relation*

$$[\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4]\Psi \geq 0 \tag{2}$$

is valid for all distinct points $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [a,b]$.

If the relation (2) is valid in the reverse sense with the mentioned conditions, then the function Ψ is said to be 4-concave.

The following theorem provides a criteria for a function to be 4-convex.

Theorem 1 ([2])**.** *Let $\Psi : [a,b] \to \mathbb{R}$ be any function such that Ψ'''' exists. Then Ψ is a 4-convex if and only if $\Psi'''' \geq 0$ on $[a,b]$.*

Due to the massive properties and consequences of convex functions, a lot of problems have been solved and modeled in diverse fields of science with the help of this class of functions [16–20]. It has been ascertained that, the convex functions played a very meaningful performance in the field of mathematical inequalities [2,21–24]. There are many consequential inequalities that have been established via convex functions, such as majorization [25], Favard's [26], Hermaite–Hadamard inequalities [27] and many more [28–32]. Besides these inequalities, one of the most attractive inequalities for the class of convex functions is the Jensen inequality [10]. This inequality is also of the great interest in the sense that many classical inequalities can be deduced from it [10,33]. The formal form of the Jensen inequality is verbalized in the next theorem:

Theorem 2. *Assume that $\gamma_i \geq 0$ and $\zeta_i \in [a,b]$ for each $i \in \{1, 2, \cdots, m\}$ such that $\sum\limits_{i=1}^{m} \gamma_i > 0$. Further, suppose that the real valued function Ψ is convex on $[a,b]$, then*

$$\Psi\left(\frac{\sum\limits_{i=1}^{m} \gamma_i \zeta_i}{\sum\limits_{i=1}^{m} \gamma_i}\right) \leq \frac{\sum\limits_{i=1}^{m} \gamma_i \Psi(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i}. \tag{3}$$

The inequality (3) flips for the function Ψ to be concave on $[a,b]$.

The integral variant of the Jensen inequality is stated in the following theorem.

Theorem 3. *Let $g_1, g_2 : [a,b] \to [c,d]$ be any integrable functions such that $g_1(y) \geq 0$, for $y \in [a,b]$ with $\int_a^b g_1(y)dy > 0$. Also, assume that Ψ is a convex function on $[c,d]$ and $\Psi \circ g_2$ is an integrable, then*

$$\Psi\left(\frac{\int_a^b g_1(y)g_2(y)dy}{\int_a^b g_1(y)dy}\right) \leq \frac{\int_a^b g_1(y)\Psi(g_2(y))dy}{\int_a^b g_1(y)dy}. \tag{4}$$

For the concave function Ψ, the inequality (4) holds in reverse direction.

The Jensen inequality has many applications in the several fields of science for example, in information theory [10], economics [12], and statistics [34], etc. This inequality has also been acquired for several other generalized classes of convex functions. Moreover, the aforesaid inequality has also been refined [13], generalized [35] and improved [33] in many ways by consuming its behavior and properties. In 1981, Slater presented a companion inequality to the celebrated Jensen inequality, which is formally verbalized below:

Theorem 4 ([36])**.** *Assume that $\gamma_i \geq 0$ and $\zeta_i \in (a,b)$ for each $i \in \{1,2,\cdots,m\}$ such that $\sum\limits_{i=1}^{m} \gamma_i > 0$. Also, let $\Psi : (a,b) \to \mathbb{R}$ be an increasing convex function and $\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i) \neq 0$. Then*

$$\frac{\sum\limits_{i=1}^{m} \gamma_i \Psi(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i} \leq \Psi\left(\frac{\sum\limits_{i=1}^{m} \gamma_i \zeta_i \Psi'_+(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i)}\right). \tag{5}$$

In 1985, Pečarić relaxed the monotonicity condition of the function Ψ by assuming that: $\frac{\sum\limits_{i=1}^{m} \gamma_i \zeta_i \Psi'_+(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i)} \in (a,b)$ and obtained the following generalization of Slater's inequality.

Theorem 5 ([37])**.** *Assume that $\gamma_i \geq 0$ and $\zeta_i \in (a,b)$ for each $i \in \{1,2,\cdots,m\}$ such that $\sum\limits_{i=1}^{m} \gamma_i > 0$. Also, let the real valued function $\Psi : (a,b) \to \mathbb{R}$ be convex, $\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i) \neq 0$ and $\frac{\sum\limits_{i=1}^{m} \gamma_i \zeta_i \Psi'_+(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i)} \in (a,b)$. Then*

$$\frac{\sum\limits_{i=1}^{m} \gamma_i \Psi(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i} \leq \Psi\left(\frac{\sum\limits_{i=1}^{m} \gamma_i \zeta_i \Psi'_+(\zeta_i)}{\sum\limits_{i=1}^{m} \gamma_i \Psi'_+(\zeta_i)}\right). \tag{6}$$

By exploiting the behavior of Slater's inequality and the properties of the convex functions, various types of generalizations, extensions, refinements, and improvements using different methods and approaches have been established. In addition, this inequality has also been acquired for other generalized classes of convex functions. In 2006, Bakula et al. [38] considered the classes of m and $(\alpha, m)-$convex functions and acquired several significant variants of Slater's inequality. Furthermore, they also obatined variants of Jensen's inequality and more other cognate results for aforementioned classes of convex functions. Bakula et al. [39] established a couple of general inequalities of the Jensen-Steffensen type for the class of convex functions and then used these generalized inequalities to acquire some variants of the Slater as well as Jensen-Steffensen inequalities as special cases. Adil Khan and Pečarić [40] achieved a reversion and an improvement of Slater's inequality and also obtained some other related inequalities while taking differentiable functions. In 2012, Dragomir [41] considered convex functions defined on general linear spaces and acquired some Slater's type inequalities. In addition, applications of conserved inequalities for $f-$divergence measures and norm inequalities are also provided. Delavar and Dragomir [42] obtained some fundamental inequalities for the class of $\eta-$convex functions and also inequalities related to the class of differentiable $\eta-$convex functions. Furthermore, Jensen's and Slater's type and other related inequalities have also been derived for this class of convex functions.

The main theme of this article is to establish some improvements of the Slater inequality via 4-convexity. The whole article is organized in the following way:

- In Section 2, we shall establish the improvements of the Slater inequality.
- In Section 3, we shall give several relations for the power means as consequences of the main results.
- In Section 4, we shall present some applications of the obtained results in information theory.
- In Section 5, we shall achieve some bounds for the Zipf–Mandelbrot entropy as applications of the acquired results.

2. Improvements of Slater's Inequality

In this section, we are going to establish improvements of the Slater inequality. The required improvements shall be made possible by using the definition of convex function and the renowned Jensen inequality for convex functions.

Now, we commence this section by stating a lemma that establishes an identity while taking a twice differentiable function.

Lemma 1. *Presume that $y_i \in (a,b)$, $q_i \geq 0$ for each $i \in \{1,2,\cdots,m\}$ with $\sum_{i=1}^{m} q_i > 0$ and $\Psi : (a,b) \to \mathbb{R}$ is a function such that Ψ'' exists. In addition, let $\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \in (a,b)$ and $\sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0$. Then*

$$\Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \Psi(y_i) = \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)^2$$
$$\times \int_0^1 t\Psi''\left(ty_i + (1-t)\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)\right)dt. \tag{7}$$

Proof. Without loss of generality, let $y_i \neq \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}$ for each $i \in \{1, 2, \cdots, m\}$. Utilizing the integration by parts rule, we have

$$\begin{aligned}
&\frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \int_0^1 t \Psi'' \left(t y_i + (1-t) \left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right) \right) dt \\
&= \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \\
&\quad \times \left[\frac{t}{\left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)} \Psi' \left(t y_i + (1-t) \left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right) \right) \Bigg|_0^1 \right. \\
&\quad \left. - \frac{1}{\left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)} \int_0^1 \Psi' \left(t y_i + (1-t) \left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right) \right) dt \right] \\
&= \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \left[\frac{1}{\left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)} \Psi'(y_i) \right. \\
&\quad \left. - \frac{1}{\left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2} \Psi \left(t y_i + (1-t) \left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right) \right) \Bigg|_0^1 \right] \\
&= -\frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi(y_i) + \Psi \left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right).
\end{aligned}$$

Clearly, which is equivalent to (7). □

In the next theorem, we obtain an improvement for the Slater inequality by using the definition of convex function.

Theorem 6. *Let all the hypotheses of Lemma 1 are valid. Moreover, if* Ψ *is* 4*-convex, then*

$$\Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \Psi(y_i)$$
$$\leq \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)^2 \left(\frac{2\Psi''(y_i) + \Psi''\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)}{6}\right). \tag{8}$$

For the $4-$*concave function* Ψ*, the inequality (8) reverses its direction.*

Proof. As a results of the fact that, the function Ψ is 4-convex, therefore utilizing the convex function definition on the right side of (7), we acquire

$$\Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \Psi(y_i) \leq \frac{1}{\sum_{i=1}^{m} q_i}\sum_{i=1}^{m} q_i \left(y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)^2$$
$$\times \left[\Psi''(y_i)\int_0^1 t^2 dt + \Psi''\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right)\int_0^1 (t-t^2)dt\right]. \tag{9}$$

Now, evaluating the integrals in (9), we receive (8). □

The integral form of (8) is stated in the coming theorem.

Theorem 7. *Let* $g_1, g_2 : (a,b) \to (c,d)$ *be any integrable functions such that* $g_1 \geq 0$ *with* $\int_a^b g_1(y)dy > 0$ *and* $\Psi : (c,d) \to \mathbb{R}$ *be a twice differentiable such that* $\int_a^b g_1(y)\Psi'(g_2(y))dy \neq 0$ *and* $\frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy} \in (c,d)$*. If* $\Psi \circ g_2$ *and* $\Psi' \circ g_2$ *are integrable and* Ψ *is* 4*-convex, then*

$$\Psi\left(\frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}\right) - \frac{1}{\int_a^b g_1(y)dy}\int_a^b g_1(y)\Psi(g_2(y))dy$$
$$\leq \frac{1}{\int_a^b g_1(y)dy}\int_a^b g_1(y)\left(g_2(y) - \frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}\right)^2$$
$$\times \left(\frac{2\Psi''(g_2(y)) + \Psi''\left(\frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}\right)}{6}\right)dy. \tag{10}$$

The relation (10) is true in reverse direction for the $4-$*concave function* Ψ*.*

In the following theorem, we receive another improvement for the Slater inequality.

Theorem 8. *Assume that all the postulates of Theorem 6 are true, then*

$$\Psi\left(\frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi(y_i)$$
$$\geq \frac{1}{2\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right)^2 \Psi''\left(\frac{2y_i + \frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}}{3}\right). \tag{11}$$

The inequality (11) is valid in contrary direction, if the function Ψ is $4-$concave.

Proof. From (7), we have

$$\Psi\left(\frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi(y_i) = \frac{1}{2\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right)^2$$
$$\times \frac{\int_0^1 t\Psi''\left(ty_i + (1-t)\left(\frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right)\right)dt}{\int_0^1 t\,dt}. \tag{12}$$

From (12), we obtain the following inequality with the help of Jensen's inequality

$$\Psi\left(\frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{1}{\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi(y_i) \geq \frac{1}{2\sum\limits_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left(y_i - \frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right)^2$$
$$\times \Psi''\left(\frac{\int_0^1 t\left(ty_i + (1-t)\left(\frac{\sum\limits_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum\limits_{i=1}^{m} q_i \Psi'(y_i)}\right)\right)dt}{\int_0^1 t\,dt}\right). \tag{13}$$

Inequality (11) can easily be obtained by checking the integral on the right side of (13). □

The analogous form of the inequality (11) is given in the below theorem.

Theorem 9. *Assume that, all the hypotheses of Theorem 7 are true, then*

$$\Psi\left(\frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}\right) - \frac{1}{\int_a^b g_1(y)} \int_a^b g_1(y)\Psi(g_2(y))dy$$
$$\geq \frac{1}{2\int_a^b g_1(y)} \int_a^b g_1(y)\left(g_2(y) - \frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}\right)^2$$
$$\times \Psi''\left(\frac{2g_2(y) + \frac{\int_a^b g_1(y)g_2(y)\Psi'(g_2(y))dy}{\int_a^b g_1(y)\Psi'(g_2(y))dy}}{3}\right)dy. \tag{14}$$

If the function Ψ is $4-$concave, then (14) is true in opposite sense.

3. Applications for the Power Means

In the current section, some of the consequences of the established results will be discussed in the form of inequalities for the notable power means. Here, we put some particular 4-convex functions in the main results for the obtaining of intended relations of the power means. Now, we initiate this with the definition of power mean.

Definition 3. *Let $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$ and $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ be arbitrary positive $m-$tuples and r be any real number. Then the power mean of order r is defined by:*

$$M_r(\boldsymbol{m}_1, \boldsymbol{m}_2) = \begin{cases} \left(\sum\limits_{i=1}^{m} \gamma_i \zeta_i^r \right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{i}^{m} \zeta_i^{\gamma_i} \right)^{\frac{1}{\sum_{i=1}^{m} \gamma_i}}, & r = 0. \end{cases}$$

In the below corollary, we present some inequalities for the power means as a consequence of Theorem 6.

Corollary 1. *Presume that $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$, $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ are any positive $m-$tuples and r, t are arbitrary non zero real numbers such that $t < r$, then the following statements are true:*
(i) If $r > 0$ such that $3r \leq t$ or $r \leq t \leq 2r$ or $t < 0$, then

$$M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) - \left(\frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum\limits_{i=1}^{m} \gamma_i}{\sum\limits_{i=1}^{m} \gamma_i \zeta_i^{t-r}} \right)^{\frac{t}{r}} \leq \frac{1}{\sum\limits_{i=1}^{m} \gamma_i} \sum_{i=1}^{m} \gamma_i \left(\zeta_i^r - \frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum\limits_{i=1}^{m} \gamma_i}{\sum\limits_{i=1}^{m} \gamma_i \zeta_i^{t-r}} \right)^2$$
$$\times \left(\frac{2\zeta_i^{t-2r} + \left(\frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum_{i=1}^{m} \gamma_i}{\sum_{i=1}^{m} \gamma_i \zeta_i^{t-r}} \right)^{\frac{t}{r}-2}}{6} \right). \tag{15}$$

(ii) If $r < 0$ such that $3r \geq t$ or $2r \leq t \leq r$ or $t > 0$, then (15) holds.
(iii) If $r > 0$ such that $2r < t < 3r$ or $r < 0$ with $2r > t > 3r$, then (15) holds in the opposite direction.

Proof. (i) Consider the function $\Psi(y) = y^{\frac{t}{r}}$ defined on $(0, \infty)$. Then $\Psi''''(y) = \frac{t}{r}\left(\frac{t}{r} - 1\right)\left(\frac{t}{r} - 2\right)\left(\frac{t}{r} - 3\right)y^{\frac{t}{r}-4}$, obviously $\Psi'''' > 0$. Which substantiate the 4-convexity of the function Ψ on $(0, \infty)$ for the mentioned values of t and r. Therefore, utilizing (8) for $\Psi(y) = y^{\frac{t}{r}}$, $q_i = \gamma_i$ and $y_i = \zeta_i^r$, we get (15).
(ii) For the specified values of r and t, the function $\Psi(y) = y^{\frac{t}{r}}$ is convex on $(0, \infty)$. Therefore, applying (8) while choosing $\Psi(y) = y^{\frac{t}{r}}$, $q_i = \gamma_i$ and $y_i = \zeta_i^r$, we obtain (15).
(iii) For the mentioned conditions on r and t, the function $\Psi(y) = y^{\frac{t}{r}}$ is concave on $(0, \infty)$. Therefore, taking $\Psi(y) = y^{\frac{t}{r}}$, $q_i = \gamma_i$ and $y_i = \zeta_i^r$ in (8), we acquire the reverse inequality of (15). □

The following corollary is the direct consequence of Theorem 8 for the power means.

Corollary 2. *Let $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$, $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ be arbitrary positive $m-$tuples and r, t be any non zero real numbers such that $t < r$, then the following assertions are valid:*
(i) If $r > 0$ such that $3r \leq t$ or $r \leq t \leq 2r$ or $t < 0$, then

$$M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) - \left(\frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum_{i=1}^m \gamma_i}{\sum_{i=1}^m \gamma_i \zeta_i^{t-r}}\right)^{\frac{t}{r}} \geq \frac{1}{2\sum_{i=1}^m \gamma_i} \sum_{i=1}^m \gamma_i \left(\zeta_i^r - \frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum_{i=1}^m \gamma_i}{\sum_{i=1}^m \gamma_i \zeta_i^{t-r}}\right)^2 \times \left(\frac{2\zeta_i^r + \frac{M_t^t(\boldsymbol{m}_1, \boldsymbol{m}_2) \sum_{i=1}^m \gamma_i}{\sum_{i=1}^m \gamma_i \zeta_i^{t-r}}}{3}\right)^{\frac{t}{r}-2}. \tag{16}$$

(ii) If $r < 0$ such that $3r \geq t$ or $r \geq t \geq 2r$ or $t > 0$, then (16) holds.
(iii) If $r > 0$ such that $2r < t < 3r$ or $r < 0$ with $2r > t > 3r$, then (16) holds in the opposite direction.

Proof. (i) Let $\Psi(y) = y^{\frac{t}{r}}$ be a function defined on $(0, \infty)$. Then clearly, the function Ψ is 4-convex with the given conditions. Therefore, putting $\Psi(y) = y^{\frac{t}{r}}$, $q_i = \gamma_i$ and $y_i = \zeta_i^r$ in (11), we receive (16).
(ii) For the stated conditions, the function $\Psi(y) = y^{\frac{t}{r}}$ is 4-convex. Therefore, to deduce (16) follow the procedure of (i).
(iii) Obviously the function $\Psi(y) = y^{\frac{t}{r}}$ is 4-concave for the aforementioned conditions. Therefore, the reverse inequality of (16) can be obtained by adopting the method of (i). □

Another relation for the power means is deduced from Theorem 6.

Corollary 3. *Suppose that $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$, $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ are any $m-$tuples such that γ_i, $\zeta_i > 0$, for each $i \in \{1, 2, \cdots, m\}$, then*

$$\frac{M_0(\boldsymbol{m}_1, \boldsymbol{m}_2)}{M_{-1}(\boldsymbol{m}_1, \boldsymbol{m}_2)} \leq \exp\left(\frac{1}{\sum_{i=1}^m \gamma_i} \sum_{i=1}^m \gamma_i \Big(\zeta_i - M_{-1}(\boldsymbol{m}_1, \boldsymbol{m}_2)\Big)^2 \left(\frac{2\zeta_i^{-2} + M_{-1}^2(\boldsymbol{m}_1, \boldsymbol{m}_2)}{6}\right)\right). \tag{17}$$

Proof. Consider $\Psi = -\ln y$, $y > 0$. Then $\Psi''''(y) = 6y^{-4}$, clearly $\Psi''''(y) > 0$ for all $y \in (0, \infty)$. This confirms the 4-convexity of Ψ. Therefore, utilizing (8) for $\Psi(y) = -\ln y$, $q_i = \gamma_i$ and $y_i = \zeta_i$, we acquire (17). □

By taking the 4-convex function $\Psi(y) = -\ln y$ in (11), we acquire a relation for the power means which is verbalized in the next corollary.

Corollary 4. *Let all the hypotheses of Corollary 3 are true. Then*

$$\frac{M_0(\boldsymbol{m}_1, \boldsymbol{m}_2)}{M_{-1}(\boldsymbol{m}_1, \boldsymbol{m}_2)} \geq \exp\left(\frac{1}{2\sum_{i=1}^m \gamma_i} \sum_{i=1}^m \gamma_i \Big(\zeta_i - M_{-1}(\boldsymbol{m}_1, \boldsymbol{m}_2)\Big)^2 \left(\frac{2\zeta_i^{-1} + M_{-1}(\boldsymbol{m}_1, \boldsymbol{m}_2)}{3}\right)^{-2}\right). \tag{18}$$

Proof. Inequality (18) can easily be obtained by taking $\Psi(y) - \ln y$, $q_i = \gamma_i$ and $y_i = \zeta_i$, in (11). □

The below corollary is the another consequence of Theorem 6 for the power means.

Corollary 5. *Suppose that all the conditions of Corollary 3 are valid, then*

$$\exp\left(\frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right) - M_1(\boldsymbol{m}_1,\boldsymbol{m}_2) \le \frac{1}{\sum_{i=1}^{m}\gamma_i}\sum_{i=1}^{m}\gamma_i\left(\ln\zeta_i - \frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)^2 \times\left(\frac{2\zeta_i + \exp\left(\frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)}{6}\right). \tag{19}$$

Proof. Since, the function $\Psi(y) = \exp y$ is 4-convex on $\mathbb{R}$. Therefore, utilizing (8) while picking $\Psi(y) = \exp y$, $q_i = \gamma_i$ and $y_i = \ln\zeta_i$, we get (19). □

With the help of Theorem 8, we obtain a relation for power means given in coming corollary.

Corollary 6. *Presume that, the conditions of Corollary 3 are fulfilled, then*

$$\exp\left(\frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right) - M_1(\boldsymbol{m}_1,\boldsymbol{m}_2) \ge \frac{1}{2\sum_{i=1}^{m}\gamma_i}\sum_{i=1}^{m}\gamma_i\left(\ln\zeta_i - \frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)^2 \times\exp\left(\frac{2\ln\zeta_i + \frac{M_2^2(\boldsymbol{m}_1,\boldsymbol{m}_2)}{M_1(\boldsymbol{m}_1,\boldsymbol{m}_2)}}{3}\right). \tag{20}$$

Proof. Taking $\Psi(y) = \exp y$, $q_i = \gamma_i$ and $y_i = \ln\zeta_i$ in (11), we acquire (20). □

Remark 1. *The analogous form of the above relations for the power means can easily be obtained by utilizing Theorem 7 and Theorem 9.*

4. Applications in Information Theory

In the present section, we give some applications of the main results in information theory. The proposed applications of the main results will provide different estimates for the Csiszár and Kullback–Leibler divergences, Shannon entropy, and Bhattacharyya coefficient.

We begin this section with the definition of Csiszár divergence.

Definition 4. *Let Ψ be any real valued function defined on $(0,\infty)$ and $\boldsymbol{m}_1 = (\gamma_1,\gamma_2,\cdots,\gamma_m)$, $\boldsymbol{m}_2 = (\zeta_1,\zeta_2,\cdots,\zeta_m)$, be arbitrary positive $m-$tuples. Then, the Csiszár divergence is defined as:*

$$C_\Psi(\boldsymbol{m}_1,\boldsymbol{m}_2) = \sum_{i=1}^{m}\gamma_i\Psi\left(\frac{\zeta_i}{\gamma_i}\right).$$

The following theorem is the application of Theorem 6 for the Csiszár divergence.

Theorem 10. *Assume that Ψ is any real valued function defined on $(0,\infty)$ such that Ψ'' exists and $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$, $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ are arbitrary positive $m-$tuples. If Ψ is a 4-convex function, then*

$$\Psi\left(\frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right) - \frac{C_{\Psi}(\boldsymbol{m}_1,\boldsymbol{m}_2)}{\sum_{i=1}^{m}\gamma_i} \leq \frac{1}{\sum_{i=1}^{m}\gamma_i}\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i} - \frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)^2 \times\left(\frac{2\Psi''\left(\frac{\zeta_i}{\gamma_i}\right) + \Psi''\left(\frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)}{6}\right). \tag{21}$$

Proof. Applying (8) by choosing $q_i = \gamma_i$ and $y_i = \frac{\zeta_i}{\gamma_i}$, we receive (21). □

As an application of Theorem 8, we acquire the following relation for the Csiszár divergence.

Theorem 11. *Let all the conditions of Theorem 10 be true. Then*

$$\Psi\left(\frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right) - \frac{C_{\Psi}(\boldsymbol{m}_1,\boldsymbol{m}_2)}{\sum_{i=1}^{m}\gamma_i} \geq \frac{1}{2\sum_{i=1}^{m}\gamma_i}\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i} - \frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\right)^2 \times\Psi''\left(\frac{2\frac{\zeta_i}{\gamma_i} + \frac{\sum_{i=1}^{m}\zeta_i\Psi'\left(\frac{\zeta_i}{\gamma_i}\right)}{C_{\Psi'}(\boldsymbol{m}_1,\boldsymbol{m}_2)}}{3}\right). \tag{22}$$

Proof. Utilizing $q_i = \gamma_i$ and $y_i = \frac{\zeta_i}{\gamma_i}$ in (11), we acquire (22). □

The Shannon entropy is defined as:

Definition 5. *For any positive probability distribution $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$, the Shannon entropy is defined by:*

$$SE(\boldsymbol{m}_1) = -\sum_{i=1}^{m}\gamma_i\log\gamma_i.$$

The following corollary gives an estimate for the Shannon entropy as application of Theorem 6.

Corollary 7. *Let $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$ be an arbitrary probability distribution such $\gamma_i > 0$, for each $i \in \{1, 2, \cdots, m\}$. Then*

$$\log\sum_{i=1}^{m}\gamma_i^2 - SE(\boldsymbol{m}_1) \leq \sum_{i=1}^{m}\gamma_i\left(\frac{1}{\gamma_i} - \frac{1}{\sum_{i=1}^{m}\gamma_i^2}\right)^2\left(\frac{2\gamma_i^2 + \left(\sum_{i=1}^{m}\gamma_i^2\right)^2}{6}\right). \tag{23}$$

Proof. Consider the function $\Psi(y) = -\log y$ defined on $(0,\infty)$. Then $\Psi''''(y) = 6y^{-4}$, which shows that $\Psi'''' > 0$ on $(0,\infty)$. This confirms the 4-convexity of the said function. Therefore, take $\Psi(y) = -\log y$ and $\zeta_i = 1$, for each $i \in \{1, 2, \cdots, m\}$ in (21), we get (23). □

The following corollary is the application of Theorem 8 for the Shannon entropy.

Corollary 8. *Presume that, all the hypotheses of Corollary 7 are valid, then*

$$\log\sum_{i=1}^{m}\gamma_i^2 - SE(\boldsymbol{m}_1) \geq \frac{1}{2}\sum_{i=1}^{m}\gamma_i\left(\frac{1}{\gamma_i}-\frac{1}{\sum\limits_{i=1}^{m}\gamma_i^2}\right)^2\left(\frac{\frac{2}{\gamma_i^2}+\frac{1}{\sum\limits_{i=1}^{m}\gamma_i^2}}{3}\right)^{-2}. \tag{24}$$

Proof. Since, the function $\Psi(y) = -\log y$ is 4-convex on $(0,\infty)$. Therefore, applying (22) by putting $\Psi(y) = -\log y$ and $\zeta_i = 1$, for each $i \in \{1,2,\cdots,m\}$, we get (24). □

Now, we recall the definition of Kulback–Leibler divergence.

Definition 6. *Let $\boldsymbol{m}_1 = (\gamma_1,\gamma_2,\cdots,\gamma_m)$ and $\boldsymbol{m}_2 = (\zeta_1,\zeta_2,\cdots,\zeta_m)$ be any positive $m-$tuples such that $\sum\limits_{i=1}^{m}\gamma_i = 1$ and $\sum\limits_{i=1}^{m}\zeta_i = 1$. Then Kullback–Leibler divergence is defined as:*

$$K_d(\boldsymbol{m}_1,\boldsymbol{m}_2) = \sum_{i=1}^{m}\gamma_i\log\left(\frac{\gamma_i}{\zeta_i}\right).$$

In the next corollary, we receive a bound for the Kulback–Leibler divergence as an application of Theorem 6.

Corollary 9. *Assume that $\boldsymbol{m}_1 = (\gamma_1,\gamma_2,\cdots,\gamma_m)$ and $\boldsymbol{m}_2 = (\zeta_1,\zeta_2,\cdots,\zeta_m)$ are positive $m-$tuples such that $\sum\limits_{i=1}^{m}\gamma_i = 1$ and $\sum\limits_{i=1}^{m}\zeta_i = 1$, then*

$$\begin{aligned}\log\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right) - K_d(\boldsymbol{m}_1,\boldsymbol{m}_2) \leq \sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2 \\ \times\left(\frac{2\left(\frac{\gamma_i}{\zeta_i}\right)^2+\left(\sum\limits_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^2}{6}\right).\end{aligned} \tag{25}$$

Proof. Using the 4-convex function $\Psi(y) = -\log y$ in (21), we obtain (25). □

The following corollary is the application of Theorem 8 for the Kulback–Leibler divergence.

Corollary 10. *Assume that, the hypotheses of Corollary 9 are true, then*

$$\begin{aligned}\log\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right) - K_d(\boldsymbol{m}_1,\boldsymbol{m}_2) \geq \frac{1}{2}\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2 \\ \times\left(\frac{2\frac{\zeta_i}{\gamma_i}+\left(\sum\limits_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}}{3}\right)^{-2}.\end{aligned} \tag{26}$$

Proof. Inequality (26) can easily be deduced by taking $\Psi(y) = -\log y$ in (22). □

Instantly, we give the definition of Bhattacharyya coefficient.

Definition 7. *Let $\boldsymbol{m}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_m)$ and $\boldsymbol{m}_2 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ be any $m-$tuples with the positive entries such that $\sum\limits_{i=1}^{m} \gamma_i = 1$ and $\sum\limits_{i=1}^{m} \zeta_i = 1$. Then, the Bhattacharyya coefficient is defined by:*

$$B_c(\boldsymbol{m}_1, \boldsymbol{m}_2) = \sum_{i=1}^{m} \sqrt{\gamma_i \zeta_i}.$$

The coming corollary provide a bound for the Bhattacharyya coefficient as an application of Theorem 6.

Corollary 11. *Suppose that, all the assumptions of Corollary 9 are valid, then*

$$\sqrt{\frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}}} - B_c(\boldsymbol{m}_1, \boldsymbol{m}_2) \leq \frac{1}{4} \sum_{i=1}^{m} \gamma_i \left(\frac{\zeta_i}{\gamma_i} - \frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}} \right)^2 \times \left(\frac{2\left(\frac{\gamma_i}{\zeta_i}\right)^{-\frac{3}{2}} + \left(\frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}} \right)^{-\frac{3}{2}}}{6} \right). \tag{27}$$

Proof. Let us take the function $\Psi(y) = -\sqrt{y},\ y > 0$. Then $\Psi''''(y) = \frac{15}{16} y^{-\frac{7}{2}}$, clearly Ψ'''' is positive on $(0, \infty)$. This substantiate the 4-convexity of the aforementioned function. Therefore, the desired inequality (27) can easily be acquired by taking $\Psi(y) = -\sqrt{y}$ in (21). □

The next corollary is the application of Theorem 8 for Bhattacharyya coefficient.

Corollary 12. *Let the conditions of Corollary 9 be fulfilled. Then*

$$B_c(\boldsymbol{m}_1, \boldsymbol{m}_2) - \left(\frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}} \right) \geq \frac{1}{8} \sum_{i=1}^{m} \gamma_i \left(\frac{\zeta_i}{\gamma_i} - \frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}} \right)^2 \times \left(\frac{\frac{\zeta_i}{\gamma_i} + \frac{B_c(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\sum\limits_{i=1}^{m} \gamma_i^{\frac{3}{2}} \zeta_i^{\frac{-1}{2}}}}{3} \right)^{-\frac{3}{2}}. \tag{28}$$

Proof. To obtain (28), use $\Psi(y) = -\sqrt{y}$ in (22). □

Remark 2. *The integral versions of the above aforementioned relations can also be acquired by using Theorem 7 and Theorem 9.*

5. Applications for the Zipf–Mandelbrot Entropy

The Zipf–Mandelbrot entropy is one of the important tools for solving a variety of problems in diverse areas of science [6,13]. Particular, this entropy has extensive applications in probability and statistic [10]. This section of the article concern to present some additional applications of main results for the Zipf–Mandelbrot entropy. To acquire the intended relations, first we discuss some basics.

For any $\theta \geq 0,\ s > 0, i \in \{1, 2, \cdots, m\}$, and $m \in \{1, 2, \cdots, \}$, the generalized harmonic number is defined as follows:

$$M_{m,\theta,s} = \sum_{i=1}^{m} \frac{1}{(i + \theta)^s}.$$

The expression:

$$\frac{1/(i+\theta)^s}{M_{m,\theta,s}}$$

represents the probability mass function for the Zipf–Mandelbrot law.

The following is mathematical form of the Zipf–Mandelbrot entropy:

$$Z(M,\theta,s) = \frac{s}{M_{m,\theta,s}} \sum_{i=1}^{m} \frac{\log(i+\theta)}{(i+\theta)^s} + \log M_{m,\theta,s}.$$

In the below corollary, we present an application of Theorem 6 for the Zipf–Mandelbrot entropy.

Corollary 13. *Let $\boldsymbol{m}_1 = (\zeta_1, \zeta_2, \cdots, \zeta_m)$ be any positive $m-$tuple such that $\sum_{i=1}^{m} \zeta_i = 1$. If $\theta \geq 0$ and $s > 0$, then*

$$\begin{aligned}
&\log\left(\frac{1}{M^2_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{\zeta_i (i+\theta)^{2s}}\right) + Z(M,\theta,s) + \frac{1}{M_{m,\theta,s}} \sum_{i=1}^{m} \frac{\log \zeta_i}{(i+\theta)^s} \\
&\leq \frac{1}{M_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{(i+\theta)^s} \left(M_{m,\theta,s}\zeta_i (i+\theta)^s - \left(\frac{1}{M^2_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{\zeta_i (i+\theta)^{2s}}\right)^{-1}\right)^2 \\
&\qquad \times \left(\frac{\frac{2}{\left(M_{m,\theta,s}\zeta_i (i+\theta)^s\right)^2} + \left(\frac{1}{M^2_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{\zeta_i (i+\theta)^{2s}}\right)^2}{6}\right).
\end{aligned} \tag{29}$$

Proof. To prove inequality (29), consider $\gamma_i = \frac{1}{M_{m,\theta,s}(i+\theta)^s}$, then clearly $\gamma_i > 0$ for each $i \in \{1, 2, \cdots, m\}$. Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{m} \gamma_i \log\left(\frac{\gamma_i}{\zeta_i}\right) &= \sum_{i=1}^{m} \frac{1}{M_{m,\theta,s}(i+\theta)^s} \log\left(\frac{1}{M_{m,\theta,s}(i+\theta)^s \zeta_i}\right) \\
&= \sum_{i=1}^{m} \frac{1}{M_{m,\theta,s}(i+\theta)^s} \Big(- s\log(i+\theta) - \log M_{m,\theta,s} - \log \zeta_i \Big) \\
&= -\frac{s}{M_{m,\theta,s}} \sum_{i=1}^{m} \log\left(\frac{i+\theta}{(i+\theta)^s}\right) - \log M_{m,\theta,s} - \frac{1}{M_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{(i+\theta)^s} \log \zeta_i \\
&= -Z(M,\theta,s) - \frac{1}{M_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{(i+\theta)^s} \log \zeta_i,
\end{aligned} \tag{30}$$

$$\log\left(\sum_{i=1}^{m} \frac{\gamma_i^2}{\zeta_i}\right) = \log\left(\frac{1}{M^2_{m,\theta,s}} \sum_{i=1}^{m} \frac{1}{(i+\theta)^{2s}\zeta_i}\right), \tag{31}$$

and

$$\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2\left(\frac{2\left(\frac{\gamma_i}{\zeta_i}\right)^2+\left(\sum\limits_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^2}{6}\right)$$
$$=\frac{1}{M_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{(i+\theta)^s}\left(M_{m,\theta,s}\zeta_i(i+\theta)^s-\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{(i+\theta)^{2s}\zeta_i}\right)^{-1}\right)^2$$
$$\times\left(\frac{\frac{2}{\left(M_{m,\theta,s}\zeta_i(i+\theta)^s\right)^2}+\left(\frac{1}{M^2_{m,\theta,s}}\sum\limits_{i=1}^{m}\frac{1}{(i+\theta)^{2s}\zeta_i}\right)^2}{6}\right). \tag{32}$$

Now, use (30)–(32) in (25), we acquire (29). □

The below corollary gives another bounds for the Zipf–Mandelbrot entropy.

Corollary 14. *Assume that* $\theta_1,\,\theta_2\geq 0$ *and* $s_1,\,s_2>0$, *then*

$$\log\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)+Z(M,\theta_1,s_1)-\frac{1}{M_{n,\theta_1,s_1}}\sum_{i=1}^{m}\frac{\log M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{(i+\theta_1)^{s_1}}$$
$$\leq\frac{1}{M_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{1}{(i+\theta_1)^{s_1}}\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}-\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(\frac{M_{m,\theta_2,s_2}}{M_{m,\theta_1,s_1}}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{s_1}}\right)^2+\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum\limits_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^2}{6}\right). \tag{33}$$

Proof. To get inequality (33), consider $\gamma_i=\frac{1}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}$ and $\zeta_i=\frac{1}{M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}$, then clearly both γ_i and ζ_i are positive for each $i\in\{1,2,\cdots,m\}$. Also, $\sum\limits_{i=1}^{m}\gamma_i=1$ and $\sum\limits_{i=1}^{m}\zeta_i=1$. Therefore, we have

$$\sum_{i=1}^{m}\gamma_i\log\left(\frac{\gamma_i}{\zeta_i}\right)=\sum_{i=1}^{m}\frac{1}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}\log\left(\frac{M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}\right)$$
$$=\sum_{i=1}^{m}\frac{\log M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}-\sum_{i=1}^{m}\frac{\log M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}$$
$$=\sum_{i=1}^{m}\frac{\log M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}-\frac{s_1}{M_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{\log(i+\theta_1)^{s_1}}{(i+\theta_1)^{s_1}}-\log M_{m,\theta_1,s_1}$$
$$=\sum_{i=1}^{m}\frac{\log M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}-Z(M,\theta_1,s_1), \tag{34}$$

$$\log\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)=\log\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right), \tag{35}$$

and

$$\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2\left(\frac{2\left(\frac{\gamma_i}{\zeta_i}\right)^2+\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^2}{6}\right)$$
$$=\frac{1}{M_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{1}{(i+\theta_1)^{s_1}}\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}-\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}\right)^2+\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^2}{6}\right). \tag{36}$$

Instantly, using (34)–(36) in (25), we receive (33). □

The below corollary is the application of Theorem 8.

Corollary 15. *Suppose that, all the assumptions of Corollary 13 are valid, then*

$$\log\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{\zeta_i(i+\theta)^{2s}}\right)+Z(M,\theta,s)+\frac{1}{M_{m,\theta,s}}\sum_{i=1}^{m}\frac{\log\zeta_i}{(i+\theta)^s}$$
$$\geq\frac{1}{2M_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{(i+\theta)^s}\left(M_{m,\theta,s}\zeta_i(i+\theta)^s-\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{\zeta_i(i+\theta)^{2s}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(M_{m,\theta,s}\zeta_i(i+\theta)^s\right)+\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{\zeta_i(i+\theta)^{2s}}\right)^{-1}}{3}\right)^{-2}. \tag{37}$$

Proof. Consider $\gamma_i=\frac{1}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}$ for each $i\in\{1,2,\cdots,m\}$, then we have

$$\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2\left(\frac{2\frac{\zeta_i}{\gamma_i}+\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}}{3}\right)^{-2}$$
$$=\frac{1}{M_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{(i+\theta)^s}\left(M_{m,\theta,s}\zeta_i(i+\theta)^s-\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{\zeta_i(i+\theta)^{2s}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(M_{m,\theta,s}\zeta_i(i+\theta)^s\right)+\left(\frac{1}{M^2_{m,\theta,s}}\sum_{i=1}^{m}\frac{1}{\zeta_i(i+\theta)^{2s}}\right)^{-1}}{3}\right)^{-2}. \tag{38}$$

Inequality (37) can easily be obtained by using (30), (31), and (38) in (26). □

The following corollary gives a bound for the Zipf–Mandelbrot entropy as an application of Theorem 8.

Corollary 16. *Assume that* $\theta_1,\ \theta_2 \geq 0$ *and* $s_1,\ s_2 > 0$, *then*

$$\log\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)+Z(M,\theta_1,s_1)-\frac{1}{M_{n,\theta_1,s_1}}\sum_{i=1}^{m}\frac{\log M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}{(i+\theta_1)^{s_1}}$$
$$\geq \frac{1}{2M_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{1}{(i+\theta_1)^{s_1}}\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}-\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}\right)+\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}}{3}\right). \tag{39}$$

Proof. Let us consider $\gamma_i = \frac{1}{M_{m,\theta_1,s_1}(i+\theta_1)^{s_1}}$ and $\zeta_i = \frac{1}{M_{m,\theta_2,s_2}(i+\theta_2)^{s_2}}$, then clearly both γ_i and ζ_i are positive for each $i \in \{1,2,\cdots,m\}$ such that their sums over i is unity. Therefore, we have

$$\sum_{i=1}^{m}\gamma_i\left(\frac{\zeta_i}{\gamma_i}-\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}\right)^2\left(\frac{2\frac{\zeta_i}{\gamma_i}+\left(\sum_{i=1}^{m}\frac{\gamma_i^2}{\zeta_i}\right)^{-1}}{3}\right)^{-2}=\frac{1}{M_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{1}{(i+\theta_1)^{s_1}}$$
$$\times\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}-\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}\right)^2$$
$$\times\left(\frac{2\left(\frac{M_{m,\theta_1,s_1}}{M_{m,\theta_2,s_2}}\frac{(i+\theta_1)^{s_1}}{(i+\theta_2)^{s_2}}\right)+\left(\frac{M_{m,\theta_2,s_2}}{M^2_{m,\theta_1,s_1}}\sum_{i=1}^{m}\frac{(i+\theta_2)^{s_2}}{(i+\theta_1)^{2s_1}}\right)^{-1}}{3}\right). \tag{40}$$

Now, to deduce (39), just use (34), (35), and (40) in (26). □

6. Conclusions

The convexity is the most powerful tools for solving a diverse type of problems in many areas of science such as in engineering, differential equations, analysis, information theory and statistics, etc. Due to the great importance and applicability, the convex functions have been generalized, refined and extended in many ways accordingly. One of the interesting generalized form of the class of the ordinary convexity is the 4-convexity. The class of ordinary convexity and its generalizations have played an unforgettable performance in the field of mathematical inequalities. There are a huge amount of inequalities which have been acquired with the help of convexity and its generalizations. In the present article, we established some new improvements of the Slater inequality by utilizing 4-convex functions. The proposed improvements are provided in both discrete and continuous versions. With the help of main results, we acquired some relations for the famous power means. The aforesaid relations are deduced by putting some particular 4-convex functions in main results. Furthermore, we parented applications of the established results in information theory in the form of bounds for Csiszár and Kullback–Leibler divergences, Shannon entropy and Bhattacharyya coefficient. Moreover, some additional applications of the acquired results are also discussed for the Zifp–Mandelbrot entropy. The idea and technique used in this article for obtaining the results for Slater's inequality, will motivate researchers for further work on Slater's inequality.

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