Brownian Motion Effects on the Stabilization of Stochastic Solutions to Fractional Diffusion Equations with Polynomials

Wael W. Mohammed, Mohammed Alshammari, Clemente Cesarano, and Sultan Albadrani

Abstract: A class of stochastic fractional diffusion equations with polynomials is considered in this article. This equation is used in numerous applications, such as ecology, bioengineering, biology, and mechanical and chemical engineering. As a result, it is critical to obtain exact solutions to this equation. To obtain these solutions, the tanh-coth method is utilized. Furthermore, we clarify the impact of noise on solution stabilization by simulating our solutions.

MSC: 35R60; 60H15; 60H10

1. Introduction

Fractional derivatives have recently received much attention, owing to their potential applications in a variety of fields, such as chemistry, biochemistry [1], finance [2,3], biology [4], physics [5,6], and hydrology [7,8]. Because fractional-order derivatives allow the memory and hereditary properties of different substances to be recognized, these fractional-order equations are more suitable than integer-order equations [9].

On the other hand, in geophysics, climatic dynamics, chemistry, biology, physics, and other fields, the importance of including random effects in modeling, forecasting, simulating, and analyzing complex processes has recently been widely recognized. Equations that take into account random fluctuations relying on time are known as stochastic differential equations.

Here, we are interested in the stochastic reaction fractional-diffusion equation (SRFDE) perturbed by multiplicative noise in the following type:

$$du = \left[D_{x}^{\alpha}u + au - bu^{n+1}\right]dt + \sigma ud\beta,$$

where $a, b$ are positive real numbers, $n$ is a positive integer, $\sigma$ is the noise intensity, and $\beta(t)$ is the standard Brownian motion.

We note that if $a = n = 1, \sigma = 0$ and $b = a$, then Equation (1) becomes the well-known Fisher equation [10], which is used as the temporal and spatial propagation model in an infinite medium of a virile gene. Moreover, it is utilized in chemical kinetics [11], logistics population growth [12], nuclear reactor theory [13], autocatalytic chemical reaction [14], flame propagation [15], and neurophysiology [16].
When \( n = q - 1, \alpha = 1 \) and \( \sigma = 0 \), then it gives the Newell-Whitehead equation (NWE), which describes the appearance of the stripe pattern in two-dimensional systems. The NWE equation has numerous applications in ecology, bioengineering, biology, and mechanical and chemical engineering. For more information, see Nagumo, Arimoto, and Yoshizawa [17], Kastenberg and Chambré [18], FitzHugh [19], and the references therein.

Moreover, if \( a = b = 1 \) and \( n = 2 \), then Equation (1) reduces to Allen-Cahn equation, which represents a natural physical phenomenon [20]. It has been widely applied to the study of a variety of physical problems, such as the motion by mean curvature flows [21], image segmentation [22], antifreeze proteins [23], structure formation [24], and crystal growth [25].

Recently, Equation (1), with \( \alpha = 1 \) and \( \sigma = 0 \) (i.e., with the integer order and without noise) was studied analytically by [26–31]. While in the stochastic case, this equation with \( \alpha = 1 \) was investigated by [32–36].

Our main objective here was to obtain the analytical fractional stochastic solutions to Equation (1) via the tanh-coth method. This method is used to find the solutions for many equations, such as the stochastic (2+1)-dimensional breaking soliton equation [37], stochastic fractional-space Allen-Cahn equation [38], stochastic Ginzburg-Landau equation [39], etc. The achieved solutions would be extremely useful in explaining definite interesting physical phenomena to physicists. Furthermore, we studied the influence of multiplicative noise on the obtained solutions of Equation (1) by bringing some graphical representations via the MATLAB package.

The article is organized as follows. In Section 2, we define and declare the features of CD. In Section 3, the wave equation for the SRFDE (1) is presented. In Section 4, we utilize the tanh-coth method to obtain the exact stochastic fractional solutions of the SRFDE (1). In Section 5, we present graphs that display the effects of multiplicative noise on the solutions of SRFDE (1). Finally, we present our conclusions.

2. Conformable Derivative

Here, we state the definition, theorem, and properties of conformable derivative (CD) [40].

**Definition 1.** Let \( \phi : (0, \infty) \to \mathbb{R} \), then the CD of \( \phi \) of order \( \alpha \in (0, 1] \) is defined as

\[
D^\alpha \phi(z) = \lim_{h \to 0} \frac{\phi(z + h z^{1-\alpha}) - \phi(z)}{h},
\]

**Theorem 1.** Let \( \phi, g : (0, \infty) \to \mathbb{R} \) be differentiable, and also \( \alpha \) differentiable functions, then the next rule holds:

\[
D^\alpha (\phi \circ g)(z) = z^{1-\alpha} g'(z) \phi'(g(z)).
\]

Let us state some properties of the CD:

1. \( D^\alpha [c_1 \phi(z) + c_2 g(z)] = c_1 D^\alpha \phi(z) + c_2 D^\alpha g(z) \), for \( c_1, c_2 \in \mathbb{R} \),
2. \( D^\alpha [C] = 0 \), \( C \) is a constant,
3. \( D^\alpha [z^k] = k z^{k-\alpha} \), \( k \in \mathbb{R} \),
4. \( D^\alpha g(z) = z^{1-\alpha} \frac{dg}{dz} \).

3. Traveling Wave Equation

To acquire the wave equation of Equation (1), we employ the wave transformation shown below:

\[
u(x, t) = Q(\mu) e^{i \nu \rho(t) - \frac{1}{2} \nu^2 \rho^2}, \quad \mu = c(\frac{1}{\alpha} x^\alpha - \lambda t),
\]

\[
\text{(2)}
\]
where \( Q \) is a real deterministic function, \( \sigma \) is the noise intensity, and \( c, \lambda \) are unknown constants. The following changes are employed:

\[
du = \left[ (−c\lambda Q′ + \frac{\sigma^2}{2} (1-n)Q)dt + \sigma Qd\beta \right] e^{[\sigma \beta(t) − \frac{1}{2} \sigma^2 t]},
\]

\[
\mathbb{D}^2 u = cQ'e^\left[\sigma \beta(t) − \frac{1}{2} \sigma^2 t\right] \text{ and } \mathbb{D}^2 u = c^2 Q''e^\left[\sigma \beta(t) − \frac{1}{2} \sigma^2 t\right],
\tag{3}
\]
where \( Q' = \frac{dQ}{d\mu} \). Equation (1) can be converted into the following ODE using (2) and (3):

\[
c^2 Q'' + c\lambda Q′ - b Q^{n+1} e^{[\sigma \beta(t) − \frac{1}{2} \sigma^2 t]} + [a + \frac{\sigma^2}{2} (1-n)] Q = 0.
\tag{4}
\]

We have, by taking expectation \( \mathbb{E}(\cdot) \):

\[
c^2 Q'' + c\lambda Q′ - b Q^{n+1} e^{[−\frac{1}{2} \sigma^2 t]} \mathbb{E}[e^{\sigma \beta(t)}] + [a + \frac{\sigma^2}{2} (1-n)] Q = 0.
\tag{5}
\]

Since \( \mathbb{E}(e^{\sigma Z}) = e^{\frac{\sigma^2}{2} t} \) for every standard Gaussian process \( Z \), then identity \( \mathbb{E}[e^{\sigma \beta(t)}] = e^{\frac{1}{2} \sigma^2 t} \) is based on the following reality, \( \sigma \beta(t) \) is distributed, such as \( \sigma n \sqrt{t}Z \). Equation (5) now becomes

\[
c^2 Q'' + c\lambda Q′ - b Q^{n+1} + \rho Q = 0,
\tag{6}
\]
where

\[
\rho = a + \frac{\sigma^2}{2} (1-n).
\]

If one defines the degree of \( Q(\mu) \) as \( D[Q(\mu)] = M \), then

\[
D[Q''] = M + 2,
\]
and

\[
D[Q^{n+1}] = (n + 1)M
\]

By balancing \( Q^{n+1} \) with \( Q'' \) in Equation (6), we have

\[
(n + 1)M = M + 2,
\]

hence,

\[
M = \frac{2}{n}.
\tag{7}
\]

In order to obtain a closed form solution, \( M \) should be an integer. So that, we suppose

\[
Q = \psi^{2/n}.
\tag{8}
\]

Substituting (8) into Equation (6), we have

\[
2c^2 (2-n) \psi^2 + 2c^2 n \psi \psi'' + 2c\lambda n \psi \psi' - bn^2 \psi^4 + \rho n^2 \psi^2 = 0.
\tag{9}
\]

Now, by balancing \( \psi \psi'' \) and \( \psi^4 \), we have

\[
M = 1.
\tag{10}
\]

3.1. Tanh-Coth Method

The tanh-coth method was defined by Malfliet [41]. Defining the solution \( \psi \) of Equation (9), using Equation (10), as follows:

\[
\psi(\mu) = \sum_{k=0}^{M} \ell_k \chi^k = \sum_{k=0}^{\ell_k} \chi^k = \psi_0 + \ell_1 \chi,
\tag{11}
\]
where $\chi = \tanh \mu$ or $\chi = \coth \mu$. Plugging Equation (11) into Equation (9), we attain

$$
-2\ell_1 c^2 (1 - \chi^2) \chi + c \lambda \ell_1 (1 - \chi^2) - (\ell_0 + \ell_1 \chi + \ell_2 \chi^2)^2 + \rho (\ell_0 + \ell_1 \chi + \ell_2 \chi^2) = 0.
$$

Hence,

$$
[2\ell_1^2 c^2 (2 + n) - bn^2 \ell_1^4] \chi^4 + [4c^2 n^2 \ell_0 \ell_1 - 2c \lambda n \ell_1^2 - 4bn^2 \ell_1^4] \chi^3 + \rho \ell_1^2 n - 6bn^2 \ell_1^2 \ell_0 - 2c \lambda n \ell_1 \ell_0] \chi^2 + 2c(2 - n) \ell_1^2 + 2c \lambda n \ell_1 \ell_0 - bn^2 \ell_0^2 + \rho n^2 \ell_0^2 = 0.
$$

Putting each coefficient of $\chi^k (k = 0, 1, 2, 3, 4)$ equal to zero, we obtain

$$
2c^2 (2 - n) \ell_1^2 + 2c \lambda n \ell_1 \ell_0 - bn^2 \ell_0^2 + \rho n^2 \ell_0^2 = 0,
$$

$$
[ -4c^2 n + 2pn^2] \ell_1 \ell_0 + 2c \lambda n \ell_1^2 - 4bn^2 \ell_1 \ell_0 = 0,
$$

$$
-8c^2 \ell_1^2 + \rho \ell_1^2 n^2 - 6bn^2 \ell_1^2 \ell_0 - 2c \lambda n \ell_1 \ell_0 = 0,
$$

$$
4c^2 n \ell_0 \ell_1 - 2c \lambda n \ell_1^2 - 4bn^2 \ell_1^2 \ell_0 = 0.
$$

and

$$
2\ell_1^2 c^2 (2 + n) - bn^2 \ell_1^4 = 0.
$$

We solve this system, obtaining

$$
\ell_0 = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}}, \ell_1 = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}}, c = n \sqrt{\frac{\rho}{8(n + 2)}}, \lambda = -(n + 4) \sqrt{\frac{\rho}{2(n + 2)}},
$$

and

$$
\ell_0 = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}}, \ell_1 = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}}, c = n \sqrt{\frac{\rho}{8(n + 2)}}, \lambda = (n + 4) \sqrt{\frac{\rho}{2(n + 2)}}.
$$

For the first set: substituting into (11), we have

$$
\psi(\mu) = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \tanh \mu \text{ or } \psi(\mu) = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \coth \mu.
$$

Substituting again into (2) with $Q = \psi^{2/n}$, we have the solutions of the SRFDE (1) as

$$
u(x, t) = e^{[c \beta(t) - \frac{1}{2} n t^2]} \left\{ \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \tanh \left[ \sqrt{\frac{n^2 \rho}{8(n + 2)}} \left( \frac{x^2}{\alpha} + \sqrt{\frac{\rho(n + 4)^2}{2(n + 2)}} \right) t \right] \right\}^{2/n},
$$

and

$$
u(x, t) = e^{[c \beta(t) - \frac{1}{2} n t^2]} \left\{ \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \coth \left[ \sqrt{\frac{n^2 \rho}{8(n + 2)}} \left( \frac{x^2}{\alpha} + \sqrt{\frac{\rho(n + 4)^2}{2(n + 2)}} \right) t \right] \right\}^{2/n},
$$

where $\rho = a + \frac{c^2}{2} (1 - n)$.

For the second set: substituting into (11), we have

$$
\psi(\mu) = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \tanh \mu \text{ or } \psi(\mu) = \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \coth \mu.
$$

Substituting again into (2) with $Q = \psi^{2/n}$, then we have the solutions of the SRFDE (1) as
\[ u(x, t) = e^{[\sigma \beta(t) - \frac{1}{2} \rho^2 t]} \left\{ \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \mp \frac{1}{2} \sqrt{\frac{\rho}{b}} \tanh \left[ \sqrt{\frac{n^2 \rho}{8(n+2)}} \left( \frac{x^a}{\alpha} - \sqrt{\frac{\rho(n+4)^2}{2(n+2)}} t \right) \right] \right\}^{2/n}, \tag{14} \]

and

\[ u(x, t) = e^{[\sigma \beta(t) - \frac{1}{2} \rho^2 t]} \left\{ \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \pm \frac{1}{2} \sqrt{\frac{\rho}{b}} \coth \left[ \sqrt{\frac{n^2 \rho}{8(n+2)}} \left( \frac{x^a}{\alpha} - \sqrt{\frac{\rho(n+4)^2}{2(n+2)}} t \right) \right] \right\}^{2/n}, \tag{15} \]

where \( \rho = a + \sigma^2 (1 - n). \)

3.2. Special Cases

Here, we present special cases for different values of \( a, b, \) and \( n. \)

3.2.1. Nonlinear Heat Equation

When we put \( a = b = 1 \) and \( n = 2 \) in Equation (1), then we have the space fractional stochastic nonlinear heat equation

\[ du = \left[ D^{\alpha} x x u + \left( u - u^3 \right) \right] dt + \sigma ud\beta. \tag{16} \]

We discuss many cases:

**Case 1:** If we put \( \sigma = 0 \) (i.e., there is no noise) and \( \alpha = 1 \) (i.e., integral order), then we obtain the identical results as in [29], which are as follows:

\[ u(x, t) = \frac{1}{4} \left( 1 - \tanh \left( \frac{x}{2 \sqrt{2}} - \frac{3}{\sqrt{6}} t \right) \right) \]

and

\[ u(x, t) = \frac{1}{4} \left( 1 - \tanh \left( \frac{x}{2 \sqrt{2}} + \frac{3}{\sqrt{6}} t \right) \right). \]

**Case 2:** If we put \( \alpha = 1, \) then we have the same solutions of (16) reported in [39], as follows:

\[ u(x, t) = \frac{1}{4} \left( 1 + \tanh \left( \frac{2a - \sigma^2}{16} \left( x + \frac{3}{2} \sqrt{2 - \sigma^2} \right) t \right) \right) \]

and

\[ u(x, t) = \frac{1}{4} \left( 1 - \tanh \left( \frac{2a - \sigma^2}{16} \left( x + \frac{3}{2} \sqrt{2 - \sigma^2} \right) t \right) \right). \]

3.2.2. Fisher’s Equation

When we put \( a = b = 1 \) in Equation (1), then we have the space fractional stochastic Fisher equation

\[ du = \left[ D_{xx}^\alpha u + b(u - u^2) \right] dt + \sigma ud\beta. \tag{17} \]

If we put \( \sigma = 0 \) and \( \alpha = 1 \) in Equations (14) and (15), then we achieve the same solutions, as reported in [30,31], as follows:

\[ u(x, t) = \frac{1}{4} [1 - \tanh(\frac{x}{2 \sqrt{6}} - \frac{5}{12} t)]^2, \]
and

\[ u(x, t) = \frac{1}{4} [1 - \coth(\frac{x}{2\sqrt{6}} - \frac{5}{12}t)]^2. \]

3.3. Newell-Whitehead Equation

If we put \( \sigma = 0, \alpha = 1 \) and \( n = q - 1 \) in Equations (14) and (15), then we attain the following same solutions as announced in [30]:

\[ u(x, t) = \left\{ \pm \frac{1}{2} \sqrt{\frac{a}{b}} \pm \frac{1}{2} \sqrt{\frac{a}{b}} \tanh \left[ \sqrt{\frac{(q - 1)^2a}{8(q + 1)}} \left( x - \sqrt{\frac{\rho(q + 3)^2}{2(q + 1)}} t \right) \right] \right\}^{2/(q-1)}, \quad (18) \]

and

\[ u(x, t) = \left\{ \pm \frac{1}{2} \sqrt{\frac{a}{b}} \pm \frac{1}{2} \sqrt{\frac{a}{b}} \coth \left[ \sqrt{\frac{(q - 1)^2a}{8(q + 1)}} \left( x - \sqrt{\frac{\rho(q + 3)^2}{2(q + 1)}} t \right) \right] \right\}^{2/(q-1)}. \quad (19) \]

4. The Influence of Noise

In this article, we explore the effect of the noise term on the solutions of the SRFDE (1). We utilize MATLAB tools to present several graphs for different values of the noise strength in order to analyze the effects of multiplicative noise on these solutions. Below is a simulation of the solution (14) for \( x \in [0, 5] \) and \( t \in [0, 5] \):

From Figure 1: we can find that the surface is not flat and that there are some irregularities. Moreover, we note that the surface expands as the fractional order \( \alpha \) increases.

\[ \alpha = 0.5, \sigma = 0 \quad \text{and} \quad u(x, t) = \left\{ \pm \frac{1}{2} \sqrt{\frac{a}{b}} \pm \frac{1}{2} \sqrt{\frac{a}{b}} \tanh \left[ \sqrt{\frac{(q - 1)^2a}{8(q + 1)}} \left( x - \sqrt{\frac{\rho(q + 3)^2}{2(q + 1)}} t \right) \right] \right\}^{2/(q-1)}, \]

and

\[ \alpha = 1, \sigma = 0 \quad \text{and} \quad u(x, t) = \left\{ \pm \frac{1}{2} \sqrt{\frac{a}{b}} \pm \frac{1}{2} \sqrt{\frac{a}{b}} \coth \left[ \sqrt{\frac{(q - 1)^2a}{8(q + 1)}} \left( x - \sqrt{\frac{\rho(q + 3)^2}{2(q + 1)}} t \right) \right] \right\}^{2/(q-1)}. \]

Figure 1. 3D shapes of the solution (14) for \( \sigma = 0 \) and \( \alpha = 0.5, 1 \).

From Figure 2: we observe that after embedding noise and increasing its strength by \( \sigma = 1, 2 \), the surface becomes significantly flatter after minor transit patterns.
\[ \alpha = 1, \sigma = 1 \]
\[ \alpha = 0.5, \sigma = 1 \]

Figure 2. 3D shapes of the solution (14) for \( \sigma = 1, 2 \) and \( \alpha = 0.5, 1 \).

We can deduce from Figures 1 to 3, the multiplicative noise stabilizes the solutions of SRFDE (1) around zero.

Figure 3. 2D shapes of the solutions (14) with \( \alpha = 1 \).

5. Conclusions

The exact solutions to a class of stochastic reaction fractional diffusion equations with the multiplicative Brownian motion were reported in this paper. To obtain these solutions,
we used the tanh–coth method. Since the SRFDE (1) is widely used in numerous applications, such as ecology, bioengineering, biology, mechanical, and chemical engineering, these solutions can be applied to a wide range of fascinating and complex physical phenomena. Furthermore, we proved how multiplicative noise influences the solution behavior and concluded that multiplicative noise stabilizes the SRFDE (1) solutions around zero.

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