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Weighted Hermite–Hadamard-Type Inequalities by Identities Related to Generalizations of Steffensen's Inequality

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Abstract: In this paper, we obtain some new weighted Hermite–Hadamard-type inequalities for $(n+2)-$convex functions by utilizing generalizations of Steffensen's inequality via Taylor's formula.

Keywords: weighted Hermite–Hadamard inequality; Steffensen's inequality; Taylor's formula; n-convex functions

MSC: 26D15; 26A51

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Citation: Pečarić, J.; Perušić Pribanić, A.; Smoljak Kalamir, K. Weighted Hermite-Hadamard-Type Inequalities by Identities Related to Generalizations of Steffensen's Inequality. *Mathematics* **2022**, *10*, 1505. https://doi.org/10.3390/math10091505

Academic Editor: Jaan Janno

Received: 30 March 2022
Accepted: 29 April 2022
Published: 1 May 2022

1. Introduction

The Hermite–Hadamard inequality is one of the most important mathematical inequalities. It was discovered independently first by Hermite [1] and later by Hadamard [2]. The classical Hermite–Hadamard inequality provides an estimate from below and above the mean value of convex function $f\colon [a,b] \to \mathbb{R}$. More precisely, we have the following.

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

To illustrate the importance of the Hermite–Hadamard inequality, let us mention that the Hermite–Hadamard inequality can be considered as the necessary and sufficient condition for convexity of a function. Furthermore, the Hermite–Hadamard inequality has an important role in numerical analysis, mathematical analysis and functional analysis. Various generalizations, extensions and applications of the Hermite-Hadamard inequality have appeared in the literature (see [3–8]).

In this paper, we consider the weighted Hermite–Hadamard inequality for convex functions given in following theorem (see [8–10]).

Theorem 1. *Let $p\colon [a,b] \to \mathbb{R}$ be a non-negative function. If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then we have the following:*

$$f(m) \le \frac{1}{P(b)}\int_a^b p(x)f(x)dx \le \frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)$$

or

$$P(b)f(m) \le \int_a^b p(x)f(x)dx \le P(b)\left[\frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)\right], \tag{1}$$

where the following is the case.

$$P(t) = \int_a^t p(x)\,dx \quad and \quad m = \frac{1}{P(b)}\int_a^b p(x)x\,dx.$$

In 1918, Steffensen proved the following inequality (see [11]).

Theorem 2 ([11])**.** *Suppose that f is non-increasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then, we have the following.*

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt. \tag{2}$$

The inequalities are reversed for f non-decreasing.

Many papers have been devoted to generalizations and refinements of Steffensen's inequality and its connection to other well-known inequalities such as Gauss–Steffensen's, Hölder's, Jenssen-=Steffensen's and other inequalities. A complete overview of the results related to Steffensen's inequality can be found in monographs [12,13].

By using the Mitrinović [14] result in which the inequalities in (2) follow from identities:

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ = \int_a^{a+\lambda}[f(t)-f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)dt\end{aligned}$$

and

$$\begin{aligned}\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\ = \int_a^{b-\lambda}[f(t)-f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda)-f(t)][1-g(t)]dt\end{aligned}$$

and using Taylor's formulae in points a and b

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}dt$$

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!}(x-b)^i - \frac{1}{(n-1)!}\int_x^b f^{(n)}(t)(x-t)^{n-1}dt$$

in paper [15], the authors proved the following identities related to generalizations of Steffensen's inequality.

Theorem 3 ([15])**.** *Let $f\colon [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$ and let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$. Let $\lambda = \int_a^b g(t)dt$ and let the function G_1 be defined by the following.*

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)dt, & x \in [a+\lambda, b]. \end{cases}$$

Then, we have the following:

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx \\ = -\frac{1}{(n-2)!}\int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt\end{aligned} \tag{3}$$

and the following is obtained.

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \\ &= \frac{1}{(n-2)!} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt.\end{aligned} \tag{4}$$

Theorem 4 ([15]). *Let f: $[a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g: $[a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)dt$ and let the function G_2 be defined by the following.*

$$G_2(x) = \begin{cases} \int_a^x g(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))dt, & x \in [b-\lambda, b]. \end{cases}$$

Then, we have the following:

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx \\ &= -\frac{1}{(n-2)!} \int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt\end{aligned} \tag{5}$$

and the following is obtained.

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx \\ &= \frac{1}{(n-2)!} \int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt.\end{aligned} \tag{6}$$

Since, in this paper, we will deal with $n-$convex functions, let us recall the definition of the $n-$convex function. For more details on convex functions, we refer the interested reader to [6,8].

Let f be a real-valued function defined on the segment $[a,b]$. The *divided difference* of order n of the function f at distinct points $x_0, ..., x_n \in [a,b]$ is defined recursively (see [8]) by the following.

$$f[x_i] = f(x_i), \quad (i = 0, \dots, n)$$

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \dots, x_n]$ is independent of the order of the points $x_0, \dots, x_n$. The definition may be extended to include the case in which some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define the following.

$$f[\underbrace{x, \dots, x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

Definition 1 ([8]). *A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the $n-th$ order divided difference of f satisfies the following.*

$$f[x_0, ..., x_n] \geq 0.$$

Note that $1-$convex functions are non-decreasing functions and $2-$convex functions are convex functions. An $n-$convex function need not to be $n-$times differentiable; however, if $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$. The following property also holds: if f is an $(n+2)-$convex function, then there exists the $n-$th derivative $f^{(n)}$, which is a convex function.

The aim of this paper is to use identities related to generalizations of Steffensen's inequality, obtained by using Taylor's formula, to prove new weighted Hermite–Hadamard-type inequalities for $(n+2)-$convex functions.

2. Main Results

In this section, applying identities given in Theorems 3 and 4 and the properties of $n-$convex functions, we derive new weighted Hermite–Hadamard-type inequalities.

Theorem 5. *Let f: $[a,b] \to \mathbb{R}$ be $(n+2)-$convex on $[a,b]$ and $f^{(n-1)}$ absolutely continuous for $n \geq 2$. Let g: $[a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Let function G_1 be defined by the following.*

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)dt, & x \in [a+\lambda, b]. \end{cases} \tag{7}$$

Then, we have the following:

$$\begin{aligned} & P_1(b) \cdot f^{(n)}(m_1) \leq \\ & (n-2)!\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx\right] \\ & \leq P_1(b) \cdot \left[\frac{b-m_1}{b-a} f^{(n)}(a) + \frac{m_1-a}{b-a} f^{(n)}(b)\right], \end{aligned} \tag{8}$$

where the following is the case:

$$P_1(b) = \frac{1}{(n-1)\cdot n}\left(\int_a^b g(x)(x-a)^n dx - \frac{\lambda^{n+1}}{n+1}\right) \tag{9}$$

and the following is obtained.

$$m_1 = a + \frac{1}{(n-1)\cdot n \cdot (n+1) \cdot P_1(b)}\left(\int_a^b g(x)(x-a)^{n+1} dx - \frac{\lambda^{n+2}}{n+2}\right). \tag{10}$$

Proof. Since $f^{(n-1)}$ is absolutely continuous, function f satisfies the conditions of Theorem 3. Therefore, identity (3) holds.

From condition $0 \leq g \leq 1$, function G_1 defined by (7) is non-negative. Hence, for every $n \geq 2$, we have the following.

$$\int_t^b G_1(x)(x-t)^{n-2} dx \geq 0, \quad t \in [a,b].$$

Define

$$p(t) = \int_t^b G_1(x)(x-t)^{n-2} dx.$$

Since the function f is $(n+2)-$convex, function $f^{(n)}$ is convex. Furthermore, function p is non-negative, so we can apply Theorem 1 and obtain the following inequality:

$$\begin{aligned} P_1(b) \cdot f^{(n)}(m_1) & \leq \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx\right) f^{(n)}(t) dt \\ & \leq P_1(b) \cdot \left[\frac{b-m_1}{b-a} f^{(n)}(a) + \frac{m_1-a}{b-a} f^{(n)}(b)\right], \end{aligned} \tag{11}$$

where $P_1(b)$ and m_1 are given by

$$P_1(b) = \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) dt$$

and

$$m_1 = \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t \, dt.$$

By calculating $P_1(b)$ and m_1, we obtain the following:

$$\begin{aligned}
P_1(b) &= \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) dt \\
&= \int_a^{a+\lambda} \left(\int_a^x (1-g(s)) ds \right) \frac{(x-a)^{n-1}}{n-1} dx + \int_{a+\lambda}^b \left(\int_x^b g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\
&= \int_a^{a+\lambda} \frac{(x-a)^n}{n-1} dx + \lambda \cdot \int_{a+\lambda}^b \frac{(x-a)^{n-1}}{n-1} dx - \int_a^b \left(\int_a^x g(s) ds \right) \frac{(x-a)^{n-1}}{n-1} dx \\
&= \frac{-\lambda^{n+1}}{(n-1)\cdot n \cdot (n+1)} + \int_a^b g(x) \frac{(x-a)^n}{(n-1)\cdot n} dx
\end{aligned}$$

and

$$\begin{aligned}
m_1 &= \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t \, dt \\
&= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(\int_a^x (x-t)^{n-2} \cdot t \, dt \right) dx \\
&= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_a^x + \int_a^x \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\
&= \frac{1}{P_1(b)} \int_a^b G_1(x) \left(\frac{a \cdot (x-a)^{n-1}}{n-1} + \frac{(x-a)^n}{(n-1)\cdot n} \right) dx \\
&= a + \frac{1}{P_1(b)} \int_a^b G_1(x) \frac{(x-a)^n}{(n-1)\cdot n} dx \\
&= a + \frac{1}{P_1(b)} \left(\frac{-\lambda^{n+2}}{(n-1)\cdot n \cdot (n+1) \cdot (n+2)} + \int_a^b g(x) \frac{(x-a)^{n+1}}{(n-1)\cdot n \cdot (n+1)} dx \right).
\end{aligned}$$

Using identity (3) for the middle part of the inequality (11), inequality (11) becomes inequality (8). Hence, the proof is completed. □

Theorem 6. *Let $f\colon [a,b] \to \mathbb{R}$ be $(n+2)-$convex on $[a,b]$ and $f^{(n-1)}$ absolutely continuous for $n \geq 2$. Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Let function G_1 be defined by (7). If the following is the case:*

$$\int_a^t G_1(x)(x-t)^{n-2} dx \leq 0, \quad t \in [a,b],$$

then we have the following:

$$\begin{aligned}
&P_2(b) \cdot f^{(n)}(m_2) \leq \\
&(n-2)! \left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \right] \\
&\leq P_2(b) \cdot \left[\frac{b-m_2}{b-a} f^{(n)}(a) + \frac{m_2-a}{b-a} f^{(n)}(b) \right],
\end{aligned} \tag{12}$$

where

$$P_2(b) = \frac{1}{(n-1)\cdot n}\left(\frac{(a-b)^{n+1}-(a+\lambda-b)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n dx\right)$$

and

$$m_2 = b + \frac{1}{(n-1)\cdot n\cdot(n+1)\cdot P_2(b)} \times \left(\frac{(a-b)^{n+2}-(a+\lambda-b)^{n+2}}{n+2} + \int_a^b g(x)(x-b)^{n+1} dx\right).$$

Proof. If we assume the following:

$$\int_a^t G_1(x)(x-t)^{n-2}dx \le 0, \quad t\in[a,b]$$

then we have the following.

$$-\int_a^t G_1(x)(x-t)^{n-2}dx \ge 0, \quad t\in[a,b].$$

Now similarly to the proof of Theorem 5 using the following non-negative function:

$$p(t) = -\int_a^t G_1(x)(x-t)^{n-2}dx$$

and identity (4), we obtain inequality (12). Similarly, we calculate the expressions for $P_2(b)$ and m_2 and obtain the following:

$$\begin{aligned}
P_2(b) &= -\int_a^b\left(\int_a^t G_1(x)(x-t)^{n-2}dx\right)dt \\
&= \int_a^{a+\lambda}\left(\int_a^x (1-g(s))ds\right)\frac{(x-b)^{n-1}}{n-1}dx + \int_{a+\lambda}^b\left(\int_x^b g(s)ds\right)\frac{(x-b)^{n-1}}{n-1}dx \\
&= \int_a^{a+\lambda}(x-a)\frac{(x-b)^{n-1}}{n-1}dx + \lambda\cdot\int_{a+\lambda}^b\frac{(x-b)^{n-1}}{n-1}dx - \int_a^b\left(\int_a^x g(s)ds\right)\frac{(x-b)^{n-1}}{n-1}dx \\
&= \frac{(a-b)^{n+1}}{(n-1)\cdot n\cdot(n+1)} - \frac{(a+\lambda-b)^{n+1}}{(n-1)\cdot n\cdot(n+1)} + \int_a^b g(x)\frac{(x-b)^n}{(n-1)\cdot n}dx
\end{aligned}$$

and

$$\begin{aligned}
m_2 &= -\frac{1}{P_2(b)}\int_a^b\left(\int_a^t G_1(x)(x-t)^{n-2}dx\right)t\,dt \\
&= -\frac{1}{P_2(b)}\int_a^b G_1(x)\left(\int_x^b (x-t)^{n-2}\cdot t\,dt\right)dx \\
&= -\frac{1}{P_2(b)}\int_a^b G_1(x)\left(t\cdot\frac{-(x-t)^{n-1}}{n-1}\Big|_x^b + \int_x^b\frac{(x-t)^{n-1}}{n-1}dt\right)dx \\
&= -\frac{1}{P_2(b)}\int_a^b G_1(x)\left(-b\cdot\frac{(x-b)^{n-1}}{n-1} - \frac{(x-b)^n}{(n-1)\cdot n}\right)dx \\
&= b + \frac{1}{P_2(b)}\int_a^b G_1(x)\frac{(x-b)^n}{(n-1)\cdot n}dx \\
&= b + \frac{1}{(n-1)\cdot n\cdot(n+1)\cdot P_1(b)} \\
&\times\left(\frac{(a-b)^{n+2}}{n+2} - \frac{(a+\lambda-b)^{n+2}}{n+2} + \int_a^b g(x)(x-b)^{n+1}dx\right).
\end{aligned}$$

Hence, the proof is completed. □

Theorem 7. *Let $f\colon [a,b] \to \mathbb{R}$ be $(n+2)-$convex on $[a,b]$ and $f^{(n-1)}$ absolutely continuous for $n \geq 2$. Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Let function G_2 be defined by the following.*

$$G_2(x) = \begin{cases} \int_a^x g(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))dt, & x \in [b-\lambda, b]. \end{cases} \tag{13}$$

Then, the following is obtained:

$$\begin{aligned} &P_3(b) \cdot f^{(n)}(m_3) \leq \\ &(n-2)!\left[\int_{b-\lambda}^b f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx - \int_a^b f(t)g(t)dt\right] \\ &\leq P_3(b) \cdot \left[\frac{b-m_3}{b-a} f^{(n)}(a) + \frac{m_3-a}{b-a} f^{(n)}(b)\right], \end{aligned} \tag{14}$$

where

$$P_3(b) = \frac{1}{(n-1)\cdot n}\left(\frac{(b-a)^{n+1} - (b-\lambda-a)^{n+1}}{n+1} - \int_a^b g(x)(x-a)^n dx\right)$$

and

$$\begin{aligned} m_3 = a + &\frac{1}{(n-1)\cdot n \cdot (n+1) \cdot P_3(b)} \\ &\times \left(\frac{(b-a)^{n+2} - (b-\lambda-a)^{n+2}}{n+2} - \int_a^b g(x)(x-a)^{n+1} dx\right). \end{aligned}$$

Proof. We follow the similar arguments as in the proof of Theorem 5. As function $f^{(n-1)}$ is absolutely continuous, the identity (5) holds. The inequality (14) follows directly from Theorem 1, substituting the non-negative function p by a non-negative function of the following:

$$p(t) = \int_t^b G_2(x)(x-t)^{n-2} dx$$

and a convex function f by a convex function $f^{(n)}$, and then using identity (5) for integral $\int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt$. Furthermore, we calculate $P_3(b)$ and m_3 as follows.

$$\begin{aligned} P_3(b) &= \int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2}dx\right)dt \\ &= \int_a^{b-\lambda}\left(\int_a^x g(s)ds\right)\frac{(x-a)^{n-1}}{n-1}dx + \int_{b-\lambda}^b \left(\int_x^b (1-g(s))ds\right)\frac{(x-a)^{n-1}}{n-1}dx \\ &= \int_{b-\lambda}^b (b-x)\frac{(x-a)^{n-1}}{n-1}dx - \lambda \cdot \int_{b-\lambda}^b \frac{(x-a)^{n-1}}{n-1}dx + \int_a^b \left(\int_a^x g(s)ds\right)\frac{(x-a)^{n-1}}{n-1}dx \\ &= \frac{(b-a)^{n+1} - (b-\lambda-a)^{n+1}}{(n-1)\cdot n \cdot (n+1)} - \int_a^b g(x)\frac{(x-a)^n}{(n-1)\cdot n}dx, \end{aligned}$$

$$
\begin{aligned}
m_3 &= \frac{1}{P_3(b)} \int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2} dx \right) t\, dt \\
&= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(\int_a^x (x-t)^{n-2} \cdot t\, dt \right) dx \\
&= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_a^x + \int_a^x \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\
&= \frac{1}{P_3(b)} \int_a^b G_2(x) \left(\frac{a \cdot (x-a)^{n-1}}{n-1} + \frac{(x-a)^n}{(n-1)\cdot n} \right) dx \\
&= a + \frac{1}{P_3(b)} \int_a^b G_2(x) \frac{(x-a)^n}{(n-1)\cdot n} dx \\
&= a + \frac{1}{P_3(b)} \left(\frac{(b-a)^{n+2} - (b-\lambda-a)^{n+2}}{(n-1)\cdot n \cdot (n+1) \cdot (n+2)} - \int_a^b g(x) \frac{(x-a)^{n+1}}{(n-1)\cdot n \cdot (n+1)} dx \right).
\end{aligned}
$$

Hence, the proof is completed. □

Theorem 8. *Let f: $[a, b] \to \mathbb{R}$ be $(n+2)-$convex on $[a, b]$ and $f^{(n-1)}$ absolutely continuous for $n \geq 2$. Let g: $[a, b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Let function G_2 be defined by (13). If the following is the case:*

$$\int_a^t G_2(x)(x-t)^{n-2} dx \leq 0, \quad t \in [a, b]$$

then we obtain the following:

$$
\begin{aligned}
&P_4(b) \cdot f^{(n)}(m_4) \leq \\
&(n-2)! \left[\int_{b-\lambda}^b f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx - \int_a^b f(t)g(t)dt \right] \\
&\leq P_4(b) \cdot \left[\frac{b-m_4}{b-a} f^{(n)}(a) + \frac{m_4 - a}{b-a} f^{(n)}(b) \right],
\end{aligned}
\tag{15}
$$

where

$$P_4(b) = \frac{-1}{(n-1)\cdot n} \left(\frac{(-\lambda)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n dx \right)$$

and

$$m_4 = b - \frac{1}{(n-1)\cdot n \cdot (n+1) \cdot P_4(b)} \left(\frac{(-\lambda)^{n+2}}{n+2} + \int_a^b g(x)(x-b)^{n+1} dx \right).$$

Proof. Under the assumption that $\int_a^t G_2(x)(x-t)^{n-2}dx \leq 0$, it is obvious that the following is the case:

$$p(t) = -\int_a^t G_2(x)(x-t)^{n-2} dx \tag{16}$$

where it is a non-negative function. Again, replacing $p(t)$ in Theorem 1 by (16) and f by $f^{(n)}$ and then using the identity (6) for

$$\int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) f^{(n)}(t) dt,$$

we obtain the required inequalities (15). Finally, a simple calculation yields the following:

$$\begin{aligned}
P_4(b) &= -\int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) dt \\
&= \int_a^{b-\lambda} \left(\int_a^x g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx + \int_{b-\lambda}^b \left(\int_x^b (1-g(s)) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\
&= -\int_{b-\lambda}^b \frac{(x-b)^n}{n-1} dx - \lambda \cdot \int_{b-\lambda}^b \frac{(x-b)^{n-1}}{n-1} dx + \int_a^b \left(\int_a^x g(s) ds \right) \frac{(x-b)^{n-1}}{n-1} dx \\
&= -\frac{(-\lambda)^{n+1}}{(n-1)\cdot n \cdot (n+1)} - \int_a^b g(x) \frac{(x-b)^n}{(n-1)\cdot n} dx
\end{aligned}$$

and

$$\begin{aligned}
m_4 &= \frac{-1}{P_4(b)} \int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) t\, dt \\
&= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(\int_x^b (x-t)^{n-2} \cdot t\, dt \right) dx \\
&= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(t \cdot \frac{-(x-t)^{n-1}}{n-1} \Big|_x^b + \int_x^b \frac{(x-t)^{n-1}}{n-1} dt \right) dx \\
&= \frac{-1}{P_4(b)} \int_a^b G_2(x) \left(-b \cdot \frac{(x-b)^{n-1}}{n-1} - \frac{(x-b)^n}{(n-1)\cdot n} \right) dx \\
&= b + \frac{1}{P_4(b)} \int_a^b G_2(x) \frac{(x-b)^n}{(n-1)\cdot n} dx \\
&= b - \frac{1}{P_4(b)} \left(\frac{(-\lambda)^{n+2}}{(n-1)\cdot n \cdot (n+1)\cdot(n+2)} + \int_a^b g(x) \frac{(x-b)^{n+1}}{(n-1)\cdot n \cdot (n+1)} dx \right).
\end{aligned}$$

□

Remark 1. *If function f is $(n+2)-$concave, the inequalities in Theorems 5–8 are reversed. This follows from the fact that for $(n+2)-$concave function, we have $-f^{(n+2)} \geq 0$. Hence, $-f^{(n)}$ is convex and we can apply inequality (1) to function $-f^{(n)}$.*

Remark 2. *The expressions $P_i(b)$ and m_i for $i = 1, \ldots, 4$ can also be achieved by the method introduced in [16]. By this method, we calculate $P_1(b)$ and m_1. Other expressions can be recaptured in a similar manner.*

The value of $P_1(b)$ can be obtained from (3) by taking $f(t) = \frac{(t-a)^n}{n!}$. Then, $f^{(n)}(t) = 1$. Thus, we have the following.

$$\begin{aligned}
P_1(b) &= -(n-2)! \left(\int_a^{a+\lambda} \frac{(x-a)^n}{n!} dt - \int_a^b \frac{(x-a)^n}{n!} g(t) dt \right) \\
&= -\frac{\lambda^{n+1}}{(n-1)\cdot n \cdot (n+1)} + \int_a^b \frac{(x-a)^n}{(n-1)\cdot n} g(t) dt.
\end{aligned}$$

Hence, we obtained expression (9).

From Theorem 1, we previously obtained the following.

$$m_1 = \frac{1}{P_1(b)} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) t\, dt.$$

To calculate m_1, we take function $f(t) = \frac{(t-a)^{n+1}}{(n+1)!}$. Then, $f^{(n)}(t) = t - a$. Hence, from the identity (3), we obtain expression (10).

3. Conclusions

In this paper, we obtained new weighted Hermite–Hadamard-type inequalities for higher order convex functions. We used previously obtained identities related to the generalizations of Steffensen's inequality. Results obtained in this paper can be considered as a starting point for some future work.

Author Contributions: Conceptualization, J.P., A.P.P. and K.S.K.; Writing — original draft, J.P., A.P.P. and K.S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Croatian Science Foundation (HRZZ 7926) "Separation of parameter influence in engineering modeling and parameter identification".

Conflicts of Interest: The authors declare no conflict of interest.

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