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Bifurcation Analysis of a Synthetic Drug Transmission Model with Two Time Delays

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Abstract: Synthetic drugs are taking the place of traditional drugs and have made headlines giving rise to serious social issues in many countries. In this work, a synthetic drug transmission model incorporating psychological addicts with two time delays is being developed. Local stability and exhibition of Hopf bifurcation are established analytically and numerically by taking the combinations of the two time delays as bifurcation parameters. The exhibition of Hopf bifurcation shows that it is burdensome to eradicate the synthetic drugs transmission in the population.

Keywords: synthetic drugs transmission; time delays; Hopf bifurcation; local stability; period solutions

MSC: 34C23

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1. Introduction

In recent years, synthetic drugs which consist of a variety of psychoactive substances such as cocaine and marijuana compounds, are more and more popular due to the fact that they mainly appear in public places of entertainment frequented by young people. Synthetic drugs can bring about serious deleterious effects on a user's Central Nervous System (CNS) and make the users excited or inhibited [1]. Therefore, synthetic drugs are more addictive compared with traditional drugs. On the other hand, the manufacturing method of synthetic drugs is relatively simple and they are also easy to obtain. Accordingly, this leads to a sharp rise in the number of synthetic drug users around the globe. In China, for example, synthetic drug abuse had ranked first by the end of 2017 [2]. It is much worse that infectious diseases especially the spread of AIDS can be caused by synthetic drug abuse. In order to maintain social order, it is extremely urgent to control the spread of synthetic drug abuse.

A mathematical modelling approach has been utilized to solve social issues extensively since heroin addiction was considered an infectious disease [3]. Liu et al. [4,5] studied a heroin epidemic model with bilinear incidence rate. Ma et al. [6–8] discussed dynamics of a heroin model with nonlinear incidence rate. Yang et al. [9,10] considered an age-structured multi-group heroin epidemic model. There have been also some works about giving up smoking models [11–16], and drinking abuse models [17–20]. Motivated by the aforementioned works, some synthetic drug transmission models have been formulated by scholars. In [21], Das et al. proposed a fractional order synthetic drugs transmission model and decided stability of the model and formulated the optimal control of the model. In [22], Saha and Samanta proposed a synthetic drugs transmission model considering general rate. They proved local and global stability of the model and presented sensitivity analysis. Taking into account the relapse phenomenon in synthetic drug abuse, Liu et al. [23] formulated a delayed synthetic drugs transmission model with relapse and analyzed stability of the model. Based on the work by Ma et al. [24] and in consideration of the effect of psychology and time delay, Zhang et al. [25] established the following synthetic drugs transmission model with time delay:

$$\begin{cases} \frac{dS(t)}{dt} &= A - dS(t) - \beta_1 S(t)P(t) - \beta_2 S(t)H(t), \\ \frac{dP(t)}{dt} &= \beta_1 S(t)P(t) + \beta_2 S(t)H(t) - \pi P(t) - (d+\gamma)P(t), \\ \frac{dH(t)}{dt} &= \pi P(t) + \theta T(t-\tau) - \sigma H(t) - dH(t), \\ \frac{dT(t)}{dt} &= \gamma P(t) + \sigma H(t) - \theta T(t-\tau) - dT(t), \end{cases} \tag{1}$$

where $S(t)$ denotes the number of the susceptible population at time t, $P(t)$ is the number of the psychological addicts at time t, $H(t)$ is the number of the physiological addicts at time t and $T(t)$ is the number of the drug-users in treatment at time t. A is the constant rate of entering the susceptible population; β_1 is the contact rate between the susceptible population and the psychological addicts; β_2 is the contact rate between the susceptible population and the physiological addicts; d is the natural mortality of all the populations; π is the escalation rate from the psychological addicts to the physiological addicts; γ is the treatment rate of the psychological addicts; σ is the treatment rate of the physiological addicts; θ is the relapse rate of the drug-users in treatment. The symbol τ is the relapse time period of the drug-users in treatment. Zhang et al. analyzed the effect of the time delay due to the relapse time period of the drug-users in treatment on the model (1).

Clearly, Zhang et al. considered that a drug-user in treatment usually needs a certain interval to become a physiological addict again. Likewise, we believe that both the psychological addicts and the physiological addicts need a period to accept treatment and come off drugs. In fact, the dynamical model with multiple time delays has been somewhat fruitful. Kundu and Maitra [26] formulated a three species predator-prey model with three delays and obtained the critical value of each time delay where the Hopf-bifurcation happened. Ren et al. [27] proposed a computer virus model with two time delays and found that a Hopf bifurcation may occur depending on the time delays. Xu et al. [28] investigated the influence of multiple time delays on bifurcation of a fractional-order neural network model through taking two different delays as bifurcation parameters. Motivated by the work above, we investigate the following synthetic drug transmission model with two time delays:

$$\begin{cases} \frac{dS(t)}{dt} &= A - dS(t) - \beta_1 S(t)P(t) - \beta_2 S(t)H(t), \\ \frac{dP(t)}{dt} &= \beta_1 S(t)P(t) + \beta_2 S(t)H(t) - \pi P(t) - dP(t) - \gamma P(t-\tau_2), \\ \frac{dH(t)}{dt} &= \pi P(t) + \theta T(t-\tau_1) - \sigma H(t-\tau_2) - dH(t), \\ \frac{dT(t)}{dt} &= \gamma P(t-\tau_2) + \sigma H(t-\tau_2) - \theta T(t-\tau_1) - dT(t), \end{cases} \tag{2}$$

where τ_1 is the time delay due to the relapse time period of the drug-users in treatment and τ_2 is the time delay due to the period that both the psychological addicts and the physiological addicts need to accept treatment and come off drugs.

The outline of this work is as follows. In the next Section, a series of sufficient criteria are derived by choosing four different combinations of the two time delays as bifurcation parameters. Moreover, direction and stability of the Hopf bifurcation are explored under the case when $\tau_1 \in (0, \tau_{10})$ and $\tau_2 > 0$ in Section 3. Numerical simulations are demonstrated to examine the validity of our theoretical findings in Section 4. Section 5 ends our work.

2. Positivity and Boundedness of the Solutions

Considering $\mathbb{R}^4_+ = \{(z_1, z_2, z_3, z_4) | z_j \geq 0, j = 1, 2, 3, 4\}$ and $\tau = \max\{\tau_1, \tau_2\}$. The initial conditions for the model (2) are

$$S(\vartheta) = \xi_1(\vartheta), P(\vartheta) = \xi_2(\vartheta), H(\vartheta) = \xi_3(\vartheta), T(\vartheta) = \xi_4(\vartheta), \tag{3}$$

where $\xi_j(\vartheta) \geq 0$, $\xi(0) > 0$, $j = 1, 2, 3, 4$; $\vartheta \in [-\tau, 0]$ and $(\xi_1, \xi_2, \xi_3, \xi_4) \in C([-\tau, 0], \mathbb{R}^4_+)$, where $C([-\tau, 0], \mathbb{R}^4_+)$ is the Banach Space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^4_+$.

It can be observed that all the solutions of the model (2) with the above initial conditions (3) are defined on $\mathbb{R}^4_+$ and remain positive $\forall t \geq 0$. We prove this by utilizing provided methods of Bodnar [28] and Yang et al. [29] . For this purpose we present the following result.

Theorem 1. *All the solution of model (2) with the positive initial condition (3) are positive for all $t > 0$.*

Proof. It is easy to verify for system (2) that by choosing that $S(t) = 0$ implies that $S'(t) = A > 0$ for all $t \geq 0$. Hence, $S(t) > 0$, for all $t \geq 0$.

Now, we let $\tau = \max\{\tau_1, \tau_2\}$. Suppose that there exists $t_1 \in [0, \tau]$ such that $P(t_1) = 0$ and $P'(t_1) < 0$, and $P(t) > 0$ for $t \in [0, t_1]$, and $H(t_1) > 0$, $T(t_1) > 0$, and $H(t) > 0$, $T(t) > 0$ for all $t \in [0, t_1]$, then we have

$$P'(t_1) = \beta_2 S(t_1) H(t_1) - \gamma P(t_1 - \tau_2),$$

Note that $t_1 - \tau_2 \in [-\tau_2, 0]$ therefore $P'(t_1) < 0$ not always holds (in this case for any initial condition). Therefore, we have a contradiction with $P'(t_1) < 0$. Therefore, $P(t) > 0$ for all $t \in [0, \tau]$.

Similarly, we assume that there exists $t_2 \in [0, \tau]$ such that $H(t_2) = 0$ and $H'(t_2) < 0$, and $H(t) > 0$ for $t \in [0, t_2]$, and $T(t_2) > 0$, and $T(t) > 0$ for all $t \in [0, t_2]$, then we have

$$H'(t) = \pi P(t_2) + \theta T(t_2 - \tau_1) - \sigma H(t - \tau_2).$$

Then, $t_2 - \tau_2 \in [-\tau_2, 0]$ therefore, $H'(t_2) < 0$ does not always hold, which is a contradiction. Therefore, $H(t) > 0$ for all $t \in [0, \tau]$. Using the same method we obtain $T(t) > 0$ for all $t \in [0, \tau]$. Therefore, the solution is positive for $t \in [0, \tau]$. By induction, we can show that the solution is positive for $t \in [n\tau, (n+1)\tau]$. Therefore, we deduce that the solution of the system (2) is positive under the given initial conditions (3) for all $t \geq 0$. □

Denote $N(t) = S(t) + P(t) + H(t) + T(t)$, then in view of the equations of the model (2), we obtain

$$\frac{d}{dt} N(t) = A - dN(t). \tag{4}$$

Solving Equation (4), yields

$$N(t) = \frac{A}{d} + (N(0) - \frac{A}{d}) e^{-dt}. \tag{5}$$

Accordingly, for $N(0) < \frac{A}{d}$, then we can know that $N(t) < \frac{A}{d}$ and $\lim_{t \to \infty} N(t) = \frac{A}{d}$. Conclusively, the set

$$\Delta = \{(S, P, H, T) \in \mathbb{R}^4_+ : S + P + H + T = \frac{A}{d}, S > 0, P > 0, H > 0, T > 0\}$$

is a bounded feasible region as well as positively invariant under the model (2).

3. Exhibition of the Hopf bifurcation

In this section, we shall explore the impact of the time delay τ_1 and τ_2 according to analysis of the distribution of the roots of associated characteristic equations, and using a similar process about delayed systems in [30–33].

According to the computation by Zhang et al. [25], we conclude that if the basic reproductive number $\Re_0 > 1$ then the model (2) is provided with a unique synthetic drug addiction equilibrium point $E_*(S_*, P_*, H_*, T_*)$, where

$$\begin{aligned} S_* &= \frac{(\pi+d+\gamma)P_*}{\beta_1 P_* + \beta_2 H_*}, P_* = \frac{d[(\Re_0 - 1) + U]}{\beta_1 + \beta_2 V}, \\ H_* &= \frac{\pi P_* + \theta T_*}{\sigma + d}, T_* = \frac{[d(\gamma+\pi) + d\gamma]P_*}{d(\theta+\sigma+d)}, \end{aligned}$$

and

$$\begin{aligned} U &= \frac{A\beta_2\theta[\gamma(\sigma+d)+\pi\sigma]}{d^2(\sigma+d)(\theta+\sigma+d)(\pi+d+\gamma)}, \\ V &= \frac{\pi}{\sigma+d} + \frac{\theta[\gamma(\sigma+d)+\pi\sigma]}{d(\sigma+d)(\theta+\sigma+d)}, \\ \Re_0 &= \frac{A[\beta_1(\sigma+d)+\beta_2\pi]}{d(\sigma+d)(\pi+\gamma+d)}. \end{aligned}$$

The linearized section of the model (2) around the synthetic drug addiction equilibrium point $E_*(S_*, P_*, H_*, T_*)$ is

$$\begin{cases} \frac{dS(t)}{dt} = x_{11}S(t) + x_{12}P(t) + x_{13}H(t), \\ \frac{dP(t)}{dt} = x_{21}S(t) + x_{22}P(t) + x_{23}H(t) + z_{22}P(t-\tau_2), \\ \frac{dH(t)}{dt} = x_{32}P(t) + x_{33}H(t) + z_{33}H(t-\tau_2) + y_{34}T(t-\tau_1), \\ \frac{dT(t)}{dt} = x_{44}T(t) + z_{42}P(t-\tau_2) + z_{43}H(t-\tau_2) + y_{44}T(t-\tau_1), \end{cases} \tag{6}$$

with

$$\begin{aligned} x_{11} &= -(d+\beta_1 P_* + \beta_2 H_*), x_{12} = -\beta_1 S_*, x_{13} = -\beta_2 S_*, \\ x_{21} &= \beta_1 P_* + \beta_2 H_*, x_{22} = \beta_1 S_* - (\pi+d), x_{23} = \beta_2 S_*, z_{22} = -\gamma, \\ x_{32} &= \pi, x_{33} = -d, z_{33} = -\sigma, y_{34} = \theta, \\ x_{44} &= -d, y_{44} = -\theta, z_{42} = \gamma, z_{43} = \sigma. \end{aligned}$$

Then, we can obtain the corresponding characteristic equation about the synthetic drug addiction equilibrium point $E_*(S_*, P_*, H_*, T_*)$ as follows

$$\begin{aligned} \lambda^4 &+ X_{03}\lambda^3 + X_{02}\lambda^2 + X_{01}\lambda + X_{00} \\ &+ (Y_{03}\lambda^3 + Y_{02}\lambda^2 + Y_{01}\lambda + Y_{00})e^{-\lambda\tau_1} \\ &+ (Z_{03}\lambda^3 + Z_{02}\lambda^2 + Z_{01}\lambda + Z_{00})e^{-\lambda\tau_2} \\ &+ (A_{02}\lambda^2 + A_{01}\lambda + A_{00})e^{-\lambda(\tau_1+\tau_2)} \\ &+ (B_{02}\lambda^2 + B_{01}\lambda + B_{00})e^{-2\lambda\tau_2} \\ &+ (C_{01}\lambda + C_{00})e^{-\lambda(\tau_1+2\tau_2)} = 0, \end{aligned} \tag{7}$$

where

$$
\begin{aligned}
X_{00} &= x_{11}x_{33}x_{44}(x_{22}+z_{22}),\\
X_{01} &= -(x_{22}+z_{22})(x_{11}x_{33}+x_{11}x_{44}+x_{33}x_{44})-x_{11}x_{33}x_{44},\\
X_{02} &= x_{11}x_{33}+x_{11}x_{44}+x_{33}x_{44}+(x_{22}+z_{22})(x_{11}+x_{33}+x_{44}),\\
X_{03} &= -(x_{11}+x_{22}+x_{33}+x_{44}+z_{22}),\\
Y_{00} &= x_{11}x_{22}x_{33}y_{44},\\
Y_{01} &= -y_{44}(x_{11}x_{22}+x_{11}x_{33}+x_{22}x_{33}),\\
Y_{02} &= y_{44}(x_{11}+x_{22}+x_{33}), Y_{03}=-y_{44},\\
Z_{00} &= x_{11}x_{22}x_{44}z_{33},\\
Z_{01} &= -z_{33}(x_{11}x_{22}+x_{11}x_{44}+x_{22}x_{44}),\\
Z_{02} &= z_{33}(x_{11}+x_{22}+x_{44}), Z_{03}=-z_{33},\\
A_{00} &= x_{11}x_{22}(y_{34}z_{43}+y_{44}z_{33})-x_{21}y_{34}(x_{12}z_{43}+x_{13}z_{42})\\
&\quad +x_{11}(x_{33}y_{44}z_{22}+x_{23}y_{34}z_{42}),\\
A_{01} &= x_{23}y_{34}z_{42}-y_{44}z_{22}(x_{11}+x_{33})-(x_{11}+x_{22})(y_{34}z_{43}+y_{44}z_{33}),\\
A_{02} &= y_{34}z_{43}+y_{44}(z_{33}-z_{22}),\\
B_{00} &= x_{11}x_{44}z_{22}z_{33}, B_{01}=-z_{22}z_{33}(x_{11}+x_{44}), B_{02}=z_{22}z_{33},\\
C_{00} &= x_{11}z_{22}(y_{34}z_{43}+y_{44}z_{33}), C_{01}=-z_{22}(y_{34}z_{43}+y_{44}z_{33}).
\end{aligned}
$$

Case 1. $\tau_1=\tau_2=0$, Equation (7) equals

$$\lambda^4+X_{13}\lambda^3+X_{12}\lambda^2+X_{11}\lambda+X_{10}=0, \tag{8}$$

with

$$
\begin{aligned}
X_{10} &= X_{00}+Y_{00}+Z_{00}+A_{00}+B_{00}+C_{00},\\
X_{11} &= X_{01}+Y_{01}+Z_{01}+A_{01}+B_{01}+C_{01},\\
X_{12} &= X_{02}+Y_{02}+Z_{02}+A_{02}+B_{02},\\
X_{13} &= X_{03}+Y_{03}+Z_{03}.
\end{aligned}
$$

Following the work by Ma et al. [24] and the Routh-Hurwitz theorem, it can be seen that if $X_{10}>0$, $X_{13}>0$, $X_{12}X_{13}>X_{11}$ and $X_{11}X_{12}X_{13}>X_{10}X_{13}^2+X_{11}^2$, the model (2) is locally asymptotically stable.

Case 2. $\tau_1>0$ and $\tau_2=0$, Equation (7) becomes

$$\lambda^4+X_{23}\lambda^3+X_{22}\lambda^2+X_{21}\lambda+X_{20}+(Y_{23}\lambda^3+Y_{22}\lambda^2+Y_{21}\lambda+Y_{20})e^{-\lambda\tau_1}=0, \tag{9}$$

with

$$
\begin{aligned}
X_{20} &= X_{00}+Z_{00}+B_{00}, X_{21}=X_{01}+Z_{01}+B_{01},\\
X_{22} &= X_{02}+Z_{02}+B_{02}, X_{23}=X_{03}+Z_{03},\\
Y_{20} &= Y_{00}+A_{00}+C_{00}, Y_{21}=Y_{01}+A_{01}+C_{01},\\
Y_{22} &= Y_{02}+A_{02}, Y_{23}=Y_{03}.
\end{aligned}
$$

Let $\lambda=i\varsigma_1(\varsigma_1>0)$ be a root of Equation (9), then

$$
\begin{cases}
(Y_{21}\varsigma_1-Y_{23}\varsigma_1^3)\sin(\tau_1\varsigma_1)+(Y_{20}-Y_{22}\varsigma_1^2)\cos(\tau_1\varsigma_1)=X_{22}\varsigma_1^2-\varsigma_1^4-X_{20},\\
(Y_{21}\varsigma_1-Y_{23}\varsigma_1^3)\cos(\tau_1\varsigma_1)-(Y_{20}-Y_{22}\varsigma_1^2)\sin(\tau_1\varsigma_1)=X_{23}\varsigma_1^3-X_{21}\varsigma_1.
\end{cases} \tag{10}
$$

It follows from Equation (10) that

$$\varsigma_1^8+D_{23}\varsigma_1^6+D_{22}\varsigma_1^4+D_{21}\varsigma_1^2+D_{20}=0, \tag{11}$$

with

$$\begin{aligned} D_{20} &= X_{20}^2 - Y_{20}^2, \\ D_{21} &= X_{21}^2 + 2Y_{20}Y_{22} - Y_{21}^2, \\ D_{22} &= X_{22}^2 + 2X_{20} - 2X_{21}X_{23} - Y_{22}^2 + 2Y_{21}Y_{23}, \\ D_{23} &= X_{23}^2 - 2X_{22} - Y_{23}^2. \end{aligned}$$

Denote $\varsigma_1 = \vartheta_1$, then

$$\vartheta_1^4 + D_{23}\vartheta_1^3 + D_{22}\vartheta_1^2 + D_{21}\vartheta_1 + D_{20} = 0. \tag{12}$$

Distribution of the roots of Equation (12) has been discussed by Li and Wei [34]. Next, we suppose that Equation (12) has at least one positive root ϑ_{10} such that $\varsigma_{10} = \sqrt{\vartheta_{10}}$ ensuring that Equation (9) has a pair of purely imaginary roots $\pm i\varsigma_{10}$. For ς_{10}, from Equation (10), we have

$$\tau_{10} = \frac{1}{\varsigma_{10}} \times \arccos\left[\frac{E_{21}(\varsigma_{10})}{E_{22}(\varsigma_{10})}\right], \tag{13}$$

where

$$\begin{aligned} E_{21}(\varsigma_{10}) &= (Y_{22} - X_{23}Y_{23})\varsigma_{10}^6 + (X_{23}Y_{21} + X_{21}Y_{23} - Y_{20} - X_{22}Y_{22})\varsigma_{10}^4 \\ &\quad + (X_{22}Y_{20} - X_{21}Y_{21} + X_{20}Y_{22})\varsigma_{10}^2 - X_{20}Y_{20}, \\ E_{22}(\varsigma_{10}) &= Y_{23}^2\varsigma_{10}^6 + (Y_{22}^2 - 2Y_{21}Y_{23})\varsigma_{10}^4 + (Y_{21}^2 - 2Y_{20}Y_{22})\varsigma_{10}^2 + Y_{20}^2. \end{aligned}$$

By Equation (9), one has

$$\begin{aligned} \left[\frac{d\lambda}{d\tau}\right]^{-1} &= -\frac{4\lambda^3 + 3X_{23}\lambda^2 + 2X_{22}\lambda + X_{21}}{\lambda(\lambda^4 + X_{23}\lambda^3 + X_{22}\lambda^2 + X_{21}\lambda + X_{20})} \\ &\quad + \frac{3Y_{23}\lambda^2 + 2Y_{22}\lambda + Y_{21}}{\lambda(Y_{23}\lambda^3 + Y_{22}\lambda^2 + Y_{21}\lambda + Y_{20})} - \frac{\tau}{\lambda} \end{aligned} \tag{14}$$

Further,

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda = i\varsigma_{10}} = \frac{f'(\vartheta_{10})}{E_{22}(\varsigma_{10})}, \tag{15}$$

where $f(\vartheta) = \vartheta_1^4 + D_{23}\vartheta_1^3 + D_{22}\vartheta_1^2 + D_{21}\vartheta_1 + D_{20}$ and $\vartheta_{10} = \varsigma_{10}^2$. It is apparent that if $f'(\vartheta_{10}) \neq 0$ holds, then the sufficient conditions for the appearance of a Hopf bifurcation at τ_{10} are satisfied. In conclusion, we have the following results in accordance with the Hopf bifurcation theorem in [35].

Theorem 2. *If $\Re_0 > 1$, then $E_*(S_*, P_*, H_*, T_*)$ of the model (2) is locally asymptotically stable whenever $\tau_1 \in [0, \tau_{10})$; while the model (2) exhibits a Hopf bifurcation near $E_*(S_*, P_*, H_*, T_*)$ when $\tau_1 = \tau_{10}$ and a group of periodic solutions appear around $E_*(S_*, P_*, H_*, T_*)$.*

Remark 1. *Actually, it should be pointed out that the impact of the time delay τ_1 has been analyzed in [25]. In what follows, we shall further analyze the impact of the time delay τ_2 and the combinations of the time delay τ_1 and τ_2, which has been neglected in [25].*

Case 3. $\tau_1 = 0$ and $\tau_2 > 0$, Equation (7) equals

$$\lambda^4 + X_{33}\lambda^3 + X_{32}\lambda^2 + X_{31}\lambda + X_{30} + (Z_{33}\lambda^3 + Z_{32}\lambda^2 + Z_{31}\lambda + Z_{30})e^{-\lambda\tau_2} + (B_{32}\lambda^2 + B_{31}\lambda + B_{30})e^{-2\lambda\tau_2} = 0, \tag{16}$$

with

$$\begin{aligned} X_{30} &= X_{00} + Y_{00}, X_{31} = X_{01} + Y_{01}, X_{32} = X_{02} + Y_{02}, X_{33} = X_{03} + Y_{03}, \\ Z_{30} &= Z_{00} + A_{00}, Z_{31} = Z_{01} + A_{01}, Z_{32} = Z_{02} + A_{02}, Z_{33} = Z_{03}, \\ B_{30} &= B_{00} + C_{00}, B_{31} = B_{01} + C_{01}, B_{32} = B_{02}. \end{aligned}$$

Multiplying by $e^{\lambda\tau_2}$ on left and right of Equation (16), then

$$Z_{33}\lambda^3 + Z_{32}\lambda^2 + Z_{31}\lambda + Z_{30} + (\lambda^4 + X_{33}\lambda^3 + X_{32}\lambda^2 + X_{31}\lambda + X_{30})e^{\lambda\tau_2} + (B_{32}\lambda^2 + B_{31}\lambda + B_{30})e^{-\lambda\tau_2} = 0. \tag{17}$$

Let $\lambda = i\varsigma_2(\varsigma_2 > 0)$ be a root of Equation (17), then

$$\begin{cases} W_{31}(\varsigma_2)\cos(\tau_2\varsigma_2) - W_{32}(\varsigma_2)\sin(\tau_2\varsigma_2) = W_{33}(\varsigma_2), \\ W_{34}(\varsigma_2)\sin(\tau_2\varsigma_2) + W_{35}(\varsigma_2)\cos(\tau_2\varsigma_2) = W_{36}(\varsigma_2), \end{cases} \tag{18}$$

where

$$\begin{aligned} W_{31}(\varsigma_2) &= \varsigma_2^4 - (X_{32} + B_{32})\varsigma_2^2 + X_{30} + B_{30}, \\ W_{32}(\varsigma_2) &= (X_{31} - B_{31})\varsigma_2 - X_{33}\varsigma_2^3, \\ W_{33}(\varsigma_2) &= Z_{32}\varsigma_2^2 - Z_{30}, \\ W_{34}(\varsigma_2) &= \varsigma_2^4 - (X_{32} - B_{32})\varsigma_2^2 + X_{30} - B_{30}, \\ W_{35}(\varsigma_2) &= (X_{31} + B_{31})\varsigma_2 - X_{33}\varsigma_2^3, \\ W_{36}(\varsigma_2) &= Z_{33}\varsigma_2^3 - Z_{31}\varsigma_2. \end{aligned}$$

Then, one has

$$\cos(\tau_2\varsigma_2) = \frac{E_{31}(\varsigma_2)}{E_{33}(\varsigma_2)}, \sin(\tau_2\varsigma_2) = \frac{E_{32}(\varsigma_2)}{E_{33}(\varsigma_2)},$$

with

$$\begin{aligned} E_{31}(\varsigma_2) &= (Z_{32} - X_{33}Z_{33})\varsigma_2^6 + [Z_{33}(X_{31} - B_{31}) + X_{33}Z_{31} - Z_{32}(X_{32} - B_{32}) - Z_{30}]\varsigma_2^4 \\ &\quad + [Z_{30}(X_{32} - B_{32}) - Z_{31}(X_{31} - B_{31})]\varsigma_2^2 - Z_{30}(X_{30} - B_{30}), \\ E_{32}(\varsigma_2) &= \varsigma_2^7 + [X_{33}Z_{32} - Z_{31} - Z_{33}(X_{32} + B_{32})]\varsigma_2^5 \\ &\quad + [Z_{33}(X_{30} + B_{30}) + Z_{31}(X_{32} + B_{32}) - Z_{32}(X_{31} + B_{31}) - X_{33}Z_{30}]\varsigma_2^3 \\ &\quad + [Z_{30}(X_{31} + B_{31}) - Z_{31}(X_{30} + B_{30})]\varsigma_2, \\ E_{33}(\varsigma_2) &= \varsigma_2^8 + (X_{33}^2 - 2X_{32})\varsigma_2^6 + (X_{32}^2 + 2X_{30} - B_{32}^2 - 2X_{31}X_{33})\varsigma_2^4 \\ &\quad + (2B_{30}B_{32} - 2X_{30}X_{32} + X_{31}^2 - B_{31}^2)\varsigma_2^2 + X_{30}^2 - B_{30}^2. \end{aligned}$$

Then, one can obtain the following relation about ς_2

$$E_{33}^2(\varsigma_2) - E_{31}^2(\varsigma_2) - E_{32}^2(\varsigma_2) = 0. \tag{19}$$

It can be concluded that if we know all the values of parameters in the model (2), then all the roots of Equation (19) can be obtained with the help of Matlab software package. Therefore, we suppose that Equation (19) has at least one positive root ς_{20} such that Equation (17) has a pair of purely imaginary roots $\pm i\varsigma_{20}$. For ς_{20}, we have

$$\tau_{20} = \frac{1}{\varsigma_{20}} \times \arccos\left[\frac{E_{31}(\varsigma_{20})}{E_{33}(\varsigma_{20})}\right]. \tag{20}$$

Differentiating Equation (17) with respect to τ_2,

$$[\frac{d\lambda}{d\tau_2}]^{-1} = -\frac{U_{31}(\lambda)}{U_{32}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{21}$$

where

$$
\begin{aligned}
U_{31}(\lambda) &= 3Z_{33}\lambda^2 + 2Z_{32}\lambda + Z_{31} + (2B_{32}\lambda + B_{31})e^{-\lambda\tau_2} \\
&\quad + (4\lambda^3 + 3X_{33}\lambda^2 + 2X_{32}\lambda + X_{31})e^{\lambda\tau_2}, \\
U_{32}(\lambda) &= (\lambda^5 + X_{33}\lambda^4 + X_{32}\lambda^3 + X_{31}\lambda^2 + X_{30}\lambda)e^{\lambda\tau_2} \\
&\quad - (B_{32}\lambda^3 + B_{31}\lambda^2 + B_{30}\lambda)e^{-\lambda\tau_2}.
\end{aligned}
$$

Thus,

$$
Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\lambda=i\varsigma_{20}} = \frac{\Xi_{31}\Pi_{31} + \Xi_{32}\Pi_{32}}{\Pi_{31}^2 + \Pi_{32}^2}, \tag{22}
$$

with

$$
\begin{aligned}
\Xi_{31} &= Z_{31} - 3Z_{33}\varsigma_{20}^2 + 2B_{32}\varsigma_{20}\sin(\tau_{20}\varsigma_{20}) + B_{31}\cos(\tau_{20}\varsigma_{20}) \\
&\quad + (X_{31} - 3X_{33}\varsigma_{20}^2)\cos(\tau_{20}\varsigma_{20}) - (2X_{32}\varsigma_{20} - 4\varsigma_{20}^3)\sin(\tau_{20}\varsigma_{20}), \\
\Xi_{32} &= 2Z_{32}\varsigma_{20} + 2B_{32}\varsigma_{20}\cos(\tau_{20}\varsigma_{20}) - B_{31}\sin(\tau_{20}\varsigma_{20}) \\
&\quad + (X_{31} - 3X_{33}\varsigma_{20}^2)\sin(\tau_{20}\varsigma_{20}) + (2X_{32}\varsigma_{20} - 4\varsigma_{20}^3)\cos(\tau_{20}\varsigma_{20}), \\
\Pi_{31} &= (X_{33}\varsigma_{20}^4 - X_{31}\varsigma_{20}^2)\cos(\tau_{20}\varsigma_{20}) - (\varsigma_{20}^5 - X_{32}\varsigma_{20}^3 + X_{30}\varsigma_{20})\sin(\tau_{20}\varsigma_{20}) \\
&\quad + (B_{32}\varsigma_{20}^3 - B_{30}\varsigma_{20})\sin(\tau_{20}\varsigma_{20}) + B_{31}\varsigma_{20}^2\cos(\tau_{20}\varsigma_{20}), \\
\Pi_{32} &= (X_{33}\varsigma_{20}^4 - X_{31}\varsigma_{20}^2)\sin(\tau_{20}\varsigma_{20}) + (\varsigma_{20}^5 - X_{32}\varsigma_{20}^3 + X_{30}\varsigma_{20})\cos(\tau_{20}\varsigma_{20}) \\
&\quad + (B_{32}\varsigma_{20}^3 - B_{30}\varsigma_{20})\cos(\tau_{20}\varsigma_{20}) - B_{31}\varsigma_{20}^2\sin(\tau_{20}\varsigma_{20}).
\end{aligned}
$$

Therefore, if $\Xi_{31}\Pi_{31} + \Xi_{32}\Pi_{32} \neq 0$ then $Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\lambda=i\varsigma_{20}} \neq 0$. In conclusion, we have the following theorem.

Theorem 3. *If $\Re_0 > 1$, then $E_*(S_*, P_*, H_*, T_*)$ of the model (2) is locally asymptotically stable whenever $\tau_2 \in [0, \tau_{20})$; while the model (2) exhibits a Hopf bifurcation near $E_*(S_*, P_*, H_*, T_*)$ when $\tau_2 = \tau_{20}$ and a group of periodic solutions appear around $E_*(S_*, P_*, H_*, T_*)$.*

Case 4. $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$. Let $\lambda = i\varsigma_1$ be a root of Equation (7), then

$$
\begin{cases}
W_{41}(\varsigma_1)\sin(\tau_1\varsigma_1) + W_{42}(\varsigma_1)\cos(\tau_1\varsigma_1) = W_{43}(\varsigma_1), \\
W_{41}(\varsigma_1)\cos(\tau_1\varsigma_1) - W_{42}(\varsigma_1)\sin(\tau_1\varsigma_1) = W_{44}(\varsigma_1),
\end{cases} \tag{23}
$$

where

$$
\begin{aligned}
W_{41}(\varsigma_1) &= Y_{01}\varsigma_1 - Y_{03}\varsigma_1^3 + A_{01}\varsigma_1\cos(\tau_2\varsigma_1) - (A_{00} - A_{02}\varsigma_1^2)\sin(\tau_2\varsigma_1) \\
&\quad + C_{01}\varsigma_1\cos(2\tau_2\varsigma) - C_{00}\sin(2\tau_2\varsigma), \\
W_{42}(\varsigma_1) &= Y_{00} - Y_{02}\varsigma_1^2 + A_{01}\varsigma_1\sin(\tau_2\varsigma_1) + (A_{00} - A_{02}\varsigma_1^2)\cos(\tau_2\varsigma_1) \\
&\quad + C_{01}\varsigma_1\sin(2\tau_2\varsigma) + C_{00}\cos(2\tau_2\varsigma), \\
W_{43}(\varsigma_1) &= X_{02}\varsigma_1^2 - \varsigma_1^4 - X_{00} + (Z_{03}\varsigma_1^3 - Z_{01}\varsigma_1)\sin(\tau_2\varsigma_1) + (Z_{02}\varsigma_1^2 - Z_{00})\cos(\tau_2\varsigma_1) \\
&\quad - B_{01}\varsigma_1\sin(2\tau_2\varsigma_1) + (B_{02}\varsigma_1^2 - B_{00})\cos(2\tau_2\varsigma_1), \\
W_{44}(\varsigma_1) &= X_{03}\varsigma_1^3 - X_{01}\varsigma_1 + (Z_{03}\varsigma_1^3 - Z_{01}\varsigma_1)\cos(\tau_2\varsigma_1) - (Z_{02}\varsigma_1^2 - Z_{00})\sin(\tau_2\varsigma_1) \\
&\quad - B_{01}\varsigma_1\cos(2\tau_2\varsigma_1) - (B_{02}\varsigma_1^2 - B_{00})\sin(2\tau_2\varsigma_1).
\end{aligned}
$$

Based on Equation (23), we obtain

$$
\cos(\tau_1\varsigma_1) = \frac{E_{41}(\varsigma_1)}{E_{43}(\varsigma_1)}, \sin(\tau_1\varsigma_1) = \frac{E_{42}(\varsigma_1)}{E_{43}(\varsigma_1)},
$$

where

$$
\begin{aligned}
E_{41}(\varsigma_1) &= W_{41}(\varsigma_1)W_{44}(\varsigma_1) + W_{42}(\varsigma_1)W_{43}(\varsigma_1), \\
E_{42}(\varsigma_1) &= W_{41}(\varsigma_1)W_{43}(\varsigma_1) - W_{42}(\varsigma_1)W_{44}(\varsigma_1), \\
E_{43}(\varsigma_1) &= W_{41}^2(\varsigma_1) + W_{42}^2(\varsigma_1).
\end{aligned}
$$

Then, we have the following relation about ς_1

$$E_{43}^2(\varsigma_1) - E_{41}^2(\varsigma_1) - E_{42}^2(\varsigma_1) = 0. \tag{24}$$

Similarly, we suppose that Equation (24) has at least one positive root ς_{1*} such that Equation (7) has a pair of purely imaginary roots $\pm i\varsigma_{1*}$. For ς_{1*}, we have

$$\tau_{1*} = \frac{1}{\varsigma_{1*}} \times \arccos\left[\frac{E_{41}(\varsigma_{1*})}{E_{43}(\varsigma_{1*})}\right]. \tag{25}$$

Differentiating Equation (7) with respect to τ_1, we have

$$[\frac{d\lambda}{d\tau_1}]^{-1} = \frac{U_{41}(\lambda)}{U_{42}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{26}$$

where

$$\begin{aligned}
U_{41}(\lambda) \;&=\; 4\lambda^3 + 3X_{03}\lambda^2 + 2X_{02}\lambda + X_{01} + (3Y_{03}\lambda^2 + 2Y_{02}\lambda + Y_{01})e^{-\lambda\tau_1} \\
&\quad +(-\tau_2 Z_{03}\lambda^3 + (3Z_{03} - \tau_2 Z_{02})\lambda^2 + (2Z_{02} - \tau_2 Z_{01})\lambda + Z_{01} - \tau_2 Z_{00})e^{-\lambda\tau_2} \\
&\quad +(-\tau_2 A_{02}\lambda^2 + (2A_{02} - \tau_2 A_{01})\lambda + A_{01} - \tau_2 A_{00})e^{-\lambda(\tau_1+\tau_2)} \\
&\quad +(-2\tau_2 B_{02}\lambda^2 + (2B_{02} - 2\tau_2 B_{01})\lambda + B_{01} - 2\tau_2 B_{00})e^{-2\lambda\tau_2} \\
&\quad +(-2\tau_2 C_{01}\lambda + C_{01} - 2\tau_2 C_{00})e^{-\lambda(\tau_1+2\tau_2)}, \\
U_{42}(\lambda) \;&=\; (Y_{03}\lambda^4 + Y_{02}\lambda^3 + Y_{01}\lambda^2 + Y_{00}\lambda)e^{-\lambda\tau_1} \\
&\quad +(A_{02}\lambda^3 + A_{01}\lambda^2 + A_{00}\lambda)e^{-\lambda(\tau_1+\tau_2)} + (C_{01}\lambda^2 + C_{00}\lambda)e^{-\lambda(\tau_1+2\tau_2)}.
\end{aligned}$$

Further

$$Re[\frac{d\lambda}{d\tau_1}]^{-1}_{\lambda=i\varsigma_{1*}} = \frac{\Xi_{41}\Pi_{41} + \Xi_{42}\Pi_{42}}{\Pi_{41}^2 + \Pi_{42}^2}, \tag{27}$$

with

$$\begin{aligned}
\Xi_{41} \;&=\; X_{01} - 3X_{03}\varsigma_{1*}^2 + Y_{02}\varsigma_{1*}\sin(\tau_{1*}\varsigma_{1*}) + (Y_{01} - 3Y_{03}\varsigma_{1*}^2)\cos(\tau_{1*}\varsigma_{1*}) \\
&\quad +((2Z_{02} - \tau_2 Z_{01})\varsigma_{1*} + \tau_2 Z_{03}\varsigma_{1*}^3)\sin(\tau_2\varsigma_{1*}) \\
&\quad +(Z_{01} - \tau_2 Z_{00} - (3Z_{03} - \tau_2 Z_{02})\varsigma_{1*}^2)\cos(\tau_2\varsigma_{1*}) \\
&\quad +(2A_{02} - \tau_2 A_{01})\varsigma_{1*}\sin((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad +(\tau_2 A_{02}\varsigma_{1*}^2 + A_{01} - \tau_2 A_{00})\cos((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad +2(B_{02} - \tau_2 B_{01})\varsigma_{1*}\sin(2\tau_2\varsigma_{1*}) + (2\tau_2 B_{02}\varsigma_{1*}^2 + B_{01} - 2\tau_2 B_{00})\cos(2\tau_2\varsigma_{1*}) \\
&\quad -2\tau_2 C_{01}\varsigma_{1*}\sin((\tau_{1*} + 2\tau_2)\varsigma_{1*}) + (C_{01} - 2\tau_2 C_{00})\cos((\tau_{1*} + 2\tau_2)\varsigma_{1*}), \\
\Xi_{42} \;&=\; 2X_{02}\varsigma_{1*} - 4\varsigma_{1*}^3 + Y_{02}\varsigma_{1*}\cos(\tau_{1*}\varsigma_{1*}) - (Y_{01} - 3Y_{03}\varsigma_{1*}^2)\cos(\tau_{1*}\varsigma_{1*}) \\
&\quad +((2Z_{02} - \tau_2 Z_{01})\varsigma_{1*} + \tau_2 Z_{03}\varsigma_{1*}^3)\cos(\tau_2\varsigma_{1*}) \\
&\quad -(Z_{01} - \tau_2 Z_{00} - (3Z_{03} - \tau_2 Z_{02})\varsigma_{1*}^2)\sin(\tau_2\varsigma_{1*}) \\
&\quad +(2A_{02} - \tau_2 A_{01})\varsigma_{1*}\cos((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad -(\tau_2 A_{02}\varsigma_{1*}^2 + A_{01} - \tau_2 A_{00})\sin((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad +2(B_{02} - \tau_2 B_{01})\varsigma_{1*}\cos(2\tau_2\varsigma_{1*}) - (2\tau_2 B_{02}\varsigma_{1*}^2 + B_{01} - 2\tau_2 B_{00})\sin(2\tau_2\varsigma_{1*}) \\
&\quad -2\tau_2 C_{01}\varsigma_{1*}\cos((\tau_{1*} + 2\tau_2)\varsigma_{1*}) - (C_{01} - 2\tau_2 C_{00})\sin((\tau_{1*} + 2\tau_2)\varsigma_{1*}), \\
\Pi_{41} \;&=\; (Y_{00}\varsigma_{1*} - Y_{02}\varsigma_{1*}^3)\sin(\tau_{1*}\varsigma_{1*}) + (Y_{03}\varsigma_{1*}^4 - Y_{01}\varsigma_{1*}^2)\cos(\tau_{1*}\varsigma_{1*}) \\
&\quad +(A_{00}\varsigma_{1*} - A_{02}\varsigma_{1*}^3)\sin((\tau_{1*} + \tau_2)\varsigma_{1*}) - A_{01}\varsigma_{1*}^2\cos((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad +C_{00}\varsigma_{1*}\sin((\tau_{1*} + 2\tau_2)\varsigma_{1*}) - C_{01}\varsigma_{1*}^2\cos((\tau_{1*} + 2\tau_2)\varsigma_{1*}), \\
\Pi_{42} \;&=\; (Y_{00}\varsigma_{1*} - Y_{02}\varsigma_{1*}^3)\cos(\tau_{1*}\varsigma_{1*}) - (Y_{03}\varsigma_{1*}^4 - Y_{01}\varsigma_{1*}^2)\sin(\tau_{1*}\varsigma_{1*}) \\
&\quad +(A_{00}\varsigma_{1*} - A_{02}\varsigma_{1*}^3)\cos((\tau_{1*} + \tau_2)\varsigma_{1*}) + A_{01}\varsigma_{1*}^2\sin((\tau_{1*} + \tau_2)\varsigma_{1*}) \\
&\quad +C_{00}\varsigma_{1*}\cos((\tau_{1*} + 2\tau_2)\varsigma_{1*}) + C_{01}\varsigma_{1*}^2\sin((\tau_{1*} + 2\tau_2)\varsigma_{1*}).
\end{aligned}$$

Clearly, if $\Xi_{41}\Pi_{41} + \Xi_{42}\Pi_{42} \neq 0$ then $Re[\frac{d\lambda}{d\tau_1}]^{-1}_{\lambda=i\varsigma_{1*}} \neq 0$. Then, we have the following theorem.

Theorem 4. *If $\Re_0 > 1$ and $\tau_2 \in (0, \tau_{20})$, then $E_*(S_*, P_*, H_*, T_*)$ of the model (2) is locally asymptotically stable whenever $\tau_1 \in [0, \tau_{1*})$; while the model (2) exhibits a Hopf bifurcation near $E_*(S_*, P_*, H_*, T_*)$ when $\tau_1 = \tau_{1*}$ and a group of periodic solutions appear around $E_*(S_*, P_*, H_*, T_*)$.*

Case 5. $\tau_1 \in (0, \tau_{10})$ and $\tau_2 > 0$. Multiplying $e^{\lambda\tau}$ on both sides of Equation (7), one can find

$$\begin{aligned} Z_{03}\lambda^3 &+ Z_{02}\lambda^2 + Z_{01}\lambda + Z_{00} \\ &+ (B_{02}\lambda^2 + B_{01}\lambda + B_{00})e^{-\lambda\tau_2} \\ &+ (\lambda^4 + X_{03}\lambda^3 + X_{02}\lambda^2 + X_{01}\lambda + X_{00})e^{\lambda\tau_2} \\ &+ (Y_{03}\lambda^3 + Y_{02}\lambda^2 + Y_{01}\lambda + Y_{00})e^{\lambda(\tau_2-\tau_1)} \\ &+ (A_{02}\lambda^2 + A_{01}\lambda + A_{00})e^{-\lambda\tau_1} \\ &+ (C_{01}\lambda + C_{00})e^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \quad (28)$$

Let $\lambda = i\varsigma_2$ be a root of Equation (7), then

$$\begin{cases} W_{51}(\varsigma_2)\sin(\tau_2\varsigma_2) + W_{52}(\varsigma_2)\cos(\tau_2\varsigma_2) = W_{53}(\varsigma_2), \\ W_{54}(\varsigma_2)\cos(\tau_2\varsigma_2) + W_{55}(\varsigma_2)\sin(\tau_2\varsigma_2) = W_{56}(\varsigma_2), \end{cases} \quad (29)$$

where

$$\begin{aligned} W_{51}(\varsigma_2) &= X_{03}\varsigma_2^3 + (B_{01} - X_{01})\varsigma_2 - (Y_{01}\varsigma_2 - Y_{03}\varsigma_2^3)\cos(\tau_1\varsigma_2) \\ &\quad + (Y_{00} - Y_{02}\varsigma_2^2)\sin(\tau_1\varsigma_2) + C_{01}\varsigma_2\cos(\tau_1\varsigma_2) - C_{00}\sin(\tau_1\varsigma_2), \\ W_{52}(\varsigma_2) &= \varsigma_2^4 - (B_{02} + X_{02})\varsigma_2^2 + B_{00} + X_{00} + (Y_{01}\varsigma_2 - Y_{03}\varsigma_2^3)\sin(\tau_1\varsigma_2) \\ &\quad + (Y_{00} - Y_{02}\varsigma_2^2)\cos(\tau_1\varsigma_2) + C_{01}\varsigma_2\sin(\tau_1\varsigma_2) + C_{00}\cos(\tau_1\varsigma_2), \\ W_{53}(\varsigma_2) &= Z_{02}\varsigma_2^2 - Z_{00} - A_{01}\varsigma_2\sin(\tau_1\varsigma_2) - (A_{00} - A_{02}\varsigma_2^2)\cos(\tau_1\varsigma_2), \\ W_{54}(\varsigma_2) &= (B_{01} + X_{01})\varsigma_2 - X_{03}\varsigma_2^3 + (Y_{01}\varsigma_2 - Y_{03}\varsigma_2^3)\cos(\tau_1\varsigma_2) \\ &\quad - (Y_{00} - Y_{02}\varsigma_2^2)\sin(\tau_1\varsigma_2) + C_{01}\varsigma_2\cos(\tau_1\varsigma_2) - C_{00}\sin(\tau_1\varsigma_2), \\ W_{55}(\varsigma_2) &= \varsigma_2^4 + (B_{02} - X_{02})\varsigma_2^2 - B_{00} + X_{00} + (Y_{01}\varsigma_2 - Y_{03}\varsigma_2^3)\sin(\tau_1\varsigma_2) \\ &\quad + (Y_{00} - Y_{02}\varsigma_2^2)\cos(\tau_1\varsigma_2) - C_{01}\varsigma_2\sin(\tau_1\varsigma_2) - C_{00}\cos(\tau_1\varsigma_2), \\ W_{56}(\varsigma_2) &= Z_{03}\varsigma_2^3 - Z_{01}\varsigma_2 - A_{01}\varsigma_2\cos(\tau_1\varsigma_2) + (A_{00} - A_{02}\varsigma_2^2)\sin(\tau_1\varsigma_2). \end{aligned}$$

Accordingly, one has

$$\cos(\tau_2\varsigma_2) = \frac{E_{51}(\varsigma_2)}{E_{53}(\varsigma_2)}, \sin(\tau_2\varsigma_2) = \frac{E_{51}(\varsigma_2)}{E_{53}(\varsigma_2)},$$

with

$$\begin{aligned} E_{51}(\varsigma_2) &= W_{51}(\varsigma_2)W_{56}(\varsigma_2) - W_{53}(\varsigma_2)W_{55}(\varsigma_2), \\ E_{52}(\varsigma_2) &= W_{53}(\varsigma_2)W_{54}(\varsigma_2) - W_{52}(\varsigma_2)W_{56}(\varsigma_2), \\ E_{53}(\varsigma_2) &= W_{51}(\varsigma_2)W_{54}(\varsigma_2) - W_{52}(\varsigma_2)W_{55}(\varsigma_2). \end{aligned}$$

Then, one has

$$E_{53}^2(\varsigma_1) - E_{51}^2(\varsigma_1) - E_{52}^2(\varsigma_1) = 0. \quad (30)$$

Next, we suppose that Equation (30) has at least one positive root ς_{2*} such that Equation (28) has a pair of purely imaginary roots $\pm i\varsigma_{2*}$. For ς_{2*}, we have

$$\tau_{2*} = \frac{1}{\varsigma_{2*}} \times \arccos\left[\frac{E_{51}(\varsigma_{2*})}{E_{53}(\varsigma_{2*})}\right]. \quad (31)$$

Differentiating Equation (28) regarding τ_2 and substituting $\lambda = i\varsigma_{2*}$, we have

$$Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\lambda=i\varsigma_{2*}} = \frac{\Xi_{51}\Pi_{51} + \Xi_{52}\Pi_{52}}{\Pi_{51}^2 + \Pi_{52}^2}, \tag{32}$$

where

$$\begin{aligned}
\Xi_{51} &= Z_{01} - 3Z_{03}\varsigma_{2*}^2 + 2B_{02}\varsigma_{2*}\sin(\tau_{2*}\varsigma_{2*}) + B_{01}\cos(\tau_{2*}\varsigma_{2*}) \\
&+(X_{01} - 3X_{03}\varsigma_{2*}^2)\cos(\tau_{2*}\varsigma_{2*}) - (2X_{02}\varsigma_{2*} - 4\varsigma_{2*}^3)\sin(\tau_{2*}\varsigma_{2*}) \\
&+(Y_{01} - \tau_1 - (3Y_{03} - \tau_1 Y_{02})\varsigma_{2*}^2)\cos((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&-((2Y_{02} - \tau_1 Y_{01})\varsigma_{2*} + \tau_1 Y_{03}\varsigma_{2*}^3)\sin((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&+(2A_{02} - \tau_1 A_{01})\varsigma_{2*}\sin(\tau_{2*}\varsigma_{2*}) + (\tau_1 A_{02}\varsigma_{2*}^2 + A_{01} - \tau_1 A_{00})\cos(\tau_{2*}\varsigma_{2*}) \\
&+(C_{01} - \tau_1 C_{00})\cos((\tau_1 + \tau_{2*})\varsigma_{2*}) - \tau_1 C_{01}\varsigma_{2*}\sin((\tau_1 + \tau_{2*})\varsigma_{2*}), \\
\Xi_{52} &= 2Z_{02}\varsigma_{2*} + 2B_{02}\varsigma_{2*}\cos(\tau_{2*}\varsigma_{2*}) - B_{01}\sin(\tau_{2*}\varsigma_{2*}) \\
&+(X_{01} - 3X_{03}\varsigma_{2*}^2)\sin(\tau_{2*}\varsigma_{2*}) + (2X_{02}\varsigma_{2*} - 4\varsigma_{2*}^3)\cos(\tau_{2*}\varsigma_{2*}) \\
&+(Y_{01} - \tau_1 - (3Y_{03} - \tau_1 Y_{02})\varsigma_{2*}^2)\sin((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&+((2Y_{02} - \tau_1 Y_{01})\varsigma_{2*} + \tau_1 Y_{03}\varsigma_{2*}^3)\cos((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&+(2A_{02} - \tau_1 A_{01})\varsigma_{2*}\cos(\tau_{2*}\varsigma_{2*}) - (\tau_1 A_{02}\varsigma_{2*}^2 + A_{01} - \tau_1 A_{00})\sin(\tau_{2*}\varsigma_{2*}) \\
&-(C_{01} - \tau_1 C_{00})\sin((\tau_1 + \tau_{2*})\varsigma_{2*}) - \tau_1 C_{01}\varsigma_{2*}\cos((\tau_1 + \tau_{2*})\varsigma_{2*}), \\
\Pi_{51} &= (X_{03}\varsigma_{2*}^4 - X_{01}\varsigma_{2*}^2)\cos(\tau_{2*}\varsigma_{2*}) * -(\varsigma_{2*}^5 - X_{02}\varsigma_{2*}^3 + X_{00}\varsigma_{2*})\sin(\tau_{2*}\varsigma_{2*}) \\
&+(Y_{03}\varsigma_{2*}^4 - Y_{01}\varsigma_{2*}^2)\cos((\tau_{2*} - \tau_1)\varsigma_{2*}) - (Y_{00}\varsigma_{2*} - Y_{02}\varsigma_{2*}^3)\sin((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&+(B_{02}\varsigma_{2*}^3 - B_{00}\varsigma_{2*})\sin(\tau_{2*}\varsigma_{2*}) + B_{01}\varsigma_{2*}^2\cos(\tau_{2*}\varsigma_{2*}) \\
&+C_{00}\varsigma_{2*}\sin((\tau_1 + \tau_{2*})\varsigma_{2*}) + C_{01}\varsigma_{2*}^2\cos((\tau_1 + \tau_{2*})\varsigma_{2*}), \\
\Pi_{52} &= (X_{03}\varsigma_{2*}^4 - X_{01}\varsigma_{2*}^2)\sin(\tau_{2*}\varsigma_{2*}) * +(\varsigma_{2*}^5 - X_{02}\varsigma_{2*}^3 + X_{00}\varsigma_{2*})\cos(\tau_{2*}\varsigma_{2*}) \\
&+(Y_{03}\varsigma_{2*}^4 - Y_{01}\varsigma_{2*}^2)\sin((\tau_{2*} - \tau_1)\varsigma_{2*}) + (Y_{00}\varsigma_{2*} - Y_{02}\varsigma_{2*}^3)\cos((\tau_{2*} - \tau_1)\varsigma_{2*}) \\
&+(B_{02}\varsigma_{2*}^3 - B_{00}\varsigma_{2*})\cos(\tau_{2*}\varsigma_{2*}) - B_{01}\varsigma_{2*}^2\sin(\tau_{2*}\varsigma_{2*}) \\
&+C_{00}\varsigma_{2*}\cos((\tau_1 + \tau_{2*})\varsigma_{2*}) - C_{01}\varsigma_{2*}^2\sin((\tau_1 + \tau_{2*})\varsigma_{2*}).
\end{aligned}$$

Then, we can see that if $\Xi_{51}\Pi_{51} + \Xi_{52}\Pi_{52} \neq 0$ then $Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\lambda=i\varsigma_{2*}} \neq 0$. Thus, we have the following theorem.

Theorem 5. *If $\Re_0 > 1$ and $\tau_1 \in (0, \tau_{10})$, then $E_*(S_*, P_*, H_*, T_*)$ of the model (2) is locally asymptotically stable whenever $\tau_2 \in [0, \tau_{2*})$; while the model (2) exhibits a Hopf bifurcation near $E_*(S_*, P_*, H_*, T_*)$ when $\tau_2 = \tau_{2*}$ and a group of periodic solutions appear around $E_*(S_*, P_*, H_*, T_*)$.*

4. Stability of the Periodic Solutions

In this section, we examine direction and stability of the Hopf bifurcation at τ_{2*} for the case $\tau_1 \in (0, \tau_{10})$ and $\tau_2 > 0$. Denote $v_1(t) = S(t) - S_*$, $v_2(t) = P(t) - P_*$, $v_3(t) = H(t) - H_*$, $v_4(t) = T(t) - T_*$, $\tau_2 = \tau_{2*} + \mu$ and $t \to (t/\tau_2)$. Suppose that $\tau_{10*} \in (0, \tau_{10}) < \tau_{2*}$ in this section. Thus, the model system (2) becomes Equation (33) in $C = C([-1, 0], R^4)$:

$$\dot{v}(t) = L_\mu(v_t) + F(\mu, v_t), \tag{33}$$

where

$$L_\mu\phi = (\tau_{2*} + \mu)\Big(L_1\phi(0) + L_2\phi(-\frac{\tau_{10*}}{\tau_{2*}}) + L_3\phi(-1)\Big), \tag{34}$$

and

$$F(\mu,\phi) = \begin{pmatrix} -\beta_1\phi_1(0)\phi_2(0) - \beta_2\phi_2(0)\phi_3(0) \\ \beta_1\phi_1(0)\phi_2(0) + \beta_2\phi_2(0)\phi_3(0) \\ 0 \\ 0 \end{pmatrix}, \tag{35}$$

with

$$L_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ x_{21} & x_{22} & x_{23} & 0 \\ 0 & x_{32} & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{34} \\ 0 & 0 & 0 & y_{44} \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & z_{22} & 0 & 0 \\ 0 & 0 & z_{33} & 0 \\ 0 & z_{42} & z_{43} & 0 \end{pmatrix}.$$

Thus, there exists η function of ω and μ for $\omega \in [-1,0]$ fulfills

$$L_\mu\phi = \int_{-1}^{0} d\eta(\omega,\mu)\phi(\omega). \tag{36}$$

In fact,

$$\eta(\omega,\mu) = (\tau_{2*}+\mu)\begin{cases} (L_1+L_2+L_3), & \omega = 0, \\ (L_2+L_3), & \omega \in [-\frac{\tau_{10*}}{\tau_{2*}}, 0), \\ L_2, & \omega \in (-1, -\frac{\tau_{10*}}{\tau_{2*}}), \\ 0, & \omega = -1, \end{cases} \tag{37}$$

For $\phi \in C([-1,0], R^4)$,

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\omega)}{d\omega}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\omega,\mu)\phi(\omega), & \theta = 0, \end{cases} \tag{38}$$

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \omega < 0, \\ F(\mu,\phi), & \omega = 0, \end{cases} \tag{39}$$

Then system (33) equals

$$\dot{v}(t) = A(\mu)v_t + R(\mu)v_t. \tag{40}$$

For $\xi \in C^1([0,1], (R^4)^*)$,

$$A^*(\xi) = \begin{cases} -\frac{d\xi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\xi(-s), & s = 0, \end{cases} \tag{41}$$

and

$$\langle \xi(s), \phi(\omega) \rangle = \bar{\xi}(0)\phi(0) - \int_{\omega=-1}^{0}\int_{\chi=0}^{\omega} \bar{\xi}(\chi-\omega)d\eta(\omega)\phi(\chi)d\chi, \tag{42}$$

an inner product form is defined in this form with $\eta(\omega) = \eta(\omega,0)$.

Denote that $\Upsilon(\omega) = (1,\Upsilon_2,\Upsilon_3,\Upsilon_4)^T e^{i\varsigma_{2*}\tau_{2*}\omega}$ is the eigenvector of $A(0)$ related with $+i\varsigma_{2*}\tau_{2*}$ and $\Upsilon^*(s) = U(1,\Upsilon_2^*,\Upsilon_3^*,\Upsilon_4^*)^T e^{i\varsigma_{2*}\tau_{2*}s}$ is the eigenvector of $A^*(0)$ related with $-i\varsigma_{2*}\tau_{2*}$, respectively. Then,

$$
\begin{aligned}
\Upsilon_2 &= \frac{x_{13}x_{21} + x_{23}(i\varsigma_{2*} - x_{11})}{x_{13}(i\varsigma_{2*} - x_{22} - z_{22}e^{-i\tau_{2*}\varsigma_{2*}}) + x_{12}x_{23}}, \\
\Upsilon_3 &= \frac{i\varsigma_{2*} - x_{11} - x_{12}\Upsilon_2}{x_{13}}, \\
\Upsilon_4 &= \frac{(z_{42}\Upsilon_2 + z_{43}\Upsilon_3)e^{-i\tau_{2*}\varsigma_{2*}}}{i\varsigma_{2*} - x_{44} - y_{44}e^{-i\tau_{10*}\varsigma_{2*}}}, \\
\Upsilon_2^* &= -\frac{i\omega_0 + l_{11} + l_{31}v_3}{l_{21}}, \\
\Upsilon_2^* &= -\frac{i\varsigma_{2*} + x_{11}}{x_{21}}, \\
\Upsilon_3^* &= -\frac{x_{13} + x_{23}\Upsilon_2}{i\varsigma_{2*} + x_{33} + (z_{33} + z_{43}\Upsilon_*)e^{i\tau_{2*}\varsigma_{2*}}}, \\
\Upsilon_4^* &= \Upsilon_*\Upsilon_3^*, \Upsilon_* = -\frac{y_{34}e^{i\tau_{10*}\varsigma_{2*}}}{i\varsigma_{2*} + x_{44} + y_{44}e^{i\tau_{10*}\varsigma_{2*}}}.
\end{aligned}
$$

In view of Equation (42), one has

$$
\begin{aligned}
\bar{U} &= [1 + \Upsilon_2\bar{\Upsilon}_2^* + \Upsilon_3\bar{\Upsilon}_3^* + \Upsilon_4\bar{\Upsilon}_4^* + (\Upsilon_3\bar{\Upsilon}_3^* + \Upsilon_4\bar{\Upsilon}_4^*)e^{-i\tau_{10*}\varsigma_{2*}} \\
&\quad + \Upsilon_2(z_{22}\bar{\Upsilon}_2^* + z_{42}\bar{\Upsilon}_4^*)e^{-i\tau_{2*}\varsigma_{2*}} + \Upsilon_3(z_{33}\bar{\Upsilon}_3^* + z_{43}\bar{\Upsilon}_4^*)e^{-i\tau_{2*}\varsigma_{2*}}]^{-1}.
\end{aligned} \tag{43}
$$

Next, we can get the coefficients as follows by means of the method proposed in [35]:

$$
\begin{aligned}
\Psi_{20} &= 2\tau_{2*}\bar{U}(\Upsilon_2^* - 1)(\beta_1\Upsilon_2 + \beta_2\Upsilon_3), \\
\Psi_{11} &= \tau_{2*}\bar{U}(\Upsilon_2^* - 1)(2\beta_1 Re\{\Upsilon_2\} + 2\beta_2 Re\{\Upsilon_3\}), \\
\Psi_{02} &= \bar{g}_{20}, \\
\Psi_{21} &= 2\tau_{2*}\bar{U}(\Upsilon_2^* - 1)[\beta_1(Q_{11}^{(1)}(0)\Upsilon_2 + \frac{1}{2}Q_{20}^{(1)}(0)\bar{\Upsilon}_2 + Q_{11}^{(2)}(0) + \frac{1}{2}Q_{20}^{(2)}(0)) \\
&\quad + \beta_2(Q_{11}^{(1)}(0)\Upsilon_3 + \frac{1}{2}Q_{20}^{(1)}(0)\bar{\Upsilon}_3 + Q_{11}^{(3)}(0) + \frac{1}{2}Q_{20}^{(3)}(0))],
\end{aligned} \tag{44}
$$

with

$$
\begin{aligned}
Q_{20}(\omega) &= \frac{i\Psi_{20}}{\varsigma_{2*}\tau_{2*}}\Upsilon(\omega) + \frac{i\bar{\Psi}_{02}}{3\varsigma_{2*}\tau_{2*}}\bar{\Upsilon}(\omega) + J_1 e^{2i\varsigma_{2*}\tau_{2*}\omega}, \\
Q_{11}(\omega) &= -\frac{i\Psi_{11}}{\varsigma_{2*}\tau_{2*}}V(\theta) + \frac{i\bar{\Psi}_{11}}{\varsigma_{2*}\tau_{2*}}\bar{\Upsilon}(\omega) + J_2.
\end{aligned}
$$

where

$$
J_1 = 2\begin{pmatrix} x_{11}^* & -x_{12} & -x_{13} & 0 \\ -x_{21} & x_{22}^* & -x_{23} & 0 \\ 0 & -x_{32} & x_{33}^* & y_{34}e^{-2i\varsigma_{2*}\tau_{10*}} \\ 0 & -z_{42}e^{-2i\varsigma_{2*}\tau_{2*}} & -z_{43}e^{-2i\varsigma_{2*}\tau_{2*}} & x_{44}^* \end{pmatrix}^{-1} \times \begin{pmatrix} -(\beta_1\Upsilon_2 + \beta_2\Upsilon_2) \\ \beta_1\Upsilon_2 + \beta_2\Upsilon_2 \\ 0 \\ 0 \end{pmatrix},
$$

$$
J_2 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ x_{21} & x_{22} + z_{22} & x_{23} & 0 \\ 0 & x_{32} & x_{33} + z_{33} & y_{34} \\ 0 & z_{42} & z_{33} + z_{33} & x_{44} + y_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} -(2\beta_1 Re\{\Upsilon_2\} + 2\beta_2 Re\{\Upsilon_3\}) \\ 2\beta_1 Re\{\Upsilon_2\} + 2\beta_2 Re\{\Upsilon_3\} \\ 0 \\ 0 \end{pmatrix},
$$

with

$$\begin{aligned} x_{11}^* &= 2i\varsigma_{2*} - x_{11}, \\ x_{22}^* &= 2i\varsigma_{2*} - x_{22} - z_{22}e^{-2i\varsigma_{2*}\tau_{2*}}, \\ x_{33}^* &= 2i\varsigma_{2*} - x_{33} - z_{33}e^{-2i\varsigma_{2*}\tau_{2*}}, \\ x_{44}^* &= 2i\varsigma_{2*} - x_{44} - y_{44}e^{-2i\varsigma_{2*}\tau_{10*}} \end{aligned} \tag{45}$$

Then,

$$\begin{aligned} &C_1(0) = \frac{i}{2\tau_{2*}\varsigma_{2*}}\left(\Psi_{11}\Psi_{20} - 2|\Psi_{11}|^2 - \frac{|\Psi_{02}|^2}{3}\right) + \frac{\Psi_{21}}{2} \\ &\Lambda_1 = -\frac{Re\{C_1(0)\}}{\text{Re}\{\lambda'(\tau_{2*})\}}, \\ &\Lambda_2 = 2Re\{C_1(0)\}, \\ &\Lambda_3 = -\frac{Im\{C_1(0)\} + \Lambda_1 Im\{\lambda'(\tau_{2*})\}}{\tau_{2*}\varsigma_{2*}}, \end{aligned} \tag{46}$$

Theorem 6. *For system (2), if $\Lambda_1 > 0$, then the Hopf bifurcation at τ_{2*} is supercritical (subcritical for $\Lambda_1 < 0$); if $\Lambda_2 < 0$, then bifurcating periodic solutions showing around $E_*(S_*, P_*H_*, T_*)$ are stable (unstable for $\Lambda_2 > 0$); if $\Lambda_3 > 0$, then bifurcating periodic solutions showing at $E_*(S_*, P_*H_*, T_*)$ increase (decrease for $\Lambda_3 < 0$).*

5. Numerical Example

In this section, we shall adopt a numerical example by extracting the same values of parameters as those in [25] to certify our obtained analytical results in previous sections. Then, the following numerical example model system is obtained:

$$\begin{cases} \frac{dS(t)}{dt} &= 2 - 0.02S(t) - 0.016S(t)P(t) - 0.028S(t)H(t), \\ \frac{dP(t)}{dt} &= 0.016S(t)P(t) + 0.028S(t)H(t) - 0.05P(t) - 0.095P(t-\tau_2), \\ \frac{dH(t)}{dt} &= 0.03P(t) + 0.5T(t-\tau_1) - 0.421H(t-\tau_2) - 0.02H(t), \\ \frac{dT(t)}{dt} &= 0.095P(t-\tau_2) + 0.421H(t-\tau_2) - 0.5T(t-\tau_1) - 0.02T(t), \end{cases} \tag{47}$$

from which one has $\Re_0 = 12.3481 > 1$ and the unique synthetic drug addiction equilibrium point $E_*(1.3196, 13.6111, 45.6355, 39.4338)$.

For the case when $\tau_1 > 0$ and $\tau_2 = 0$, one has $\varsigma_{10} = 1.0902$ and $\tau_{10} = 9.7367$. In line with Theorem 1, $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ is locally asymptotically stable in the interval $\tau_1 \in [0, \tau_{10} = 9.7367)$. Figure 1 shows the local asymptotical stability of the model system (47). Whereas, Figure 2 shows the exhibition of a Hopf bifurcation at $\tau_{10} = 9.7367$.

For $\tau_1 = 0$ and $\tau_2 > 0$, we have $\varsigma_{20} = 1.6264$ and $\tau_{20} = 20.8839$ based on some calculations. It can be observed that the model system (47) is locally asymptotically stable around $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ when $\tau_2 = 18.6934 < \tau_{20} = 20.8839$, which is depicted in Figure 3. Nevertheless, $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ loses its stability and the model system (47) experiences a Hopf bifurcation as the value of τ_2 crossed τ_{20}. The loss of stability dynamics of $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ for $\tau_2 = 25.9358 > \tau_{20} = 20.8839$ is shown in Figure 4.

For $\tau_1 > 0$ and $\tau_2 = 2.5 \in (0, \tau_{20})$ and supposing τ_1 as a parameter, we obtain $\varsigma_{1*} = 3.2156$ and $\tau_{1*} = 1.4096$ through some computations. In such a case, the model system (47) is locally asymptotically stable when $\tau_1 < \tau_{1*}$ but as τ_1 passes through τ_{1*} the model system (47) exhibits a Hopf bifurcation and the model system (47) loses stability. This property is depicted in Figures 5 and 6 for $\tau_1 = 1.3785$ $(< \tau_{1*})$ and $\tau_1 = 1.1.4308$ $(> \tau_{1*})$, respectively.

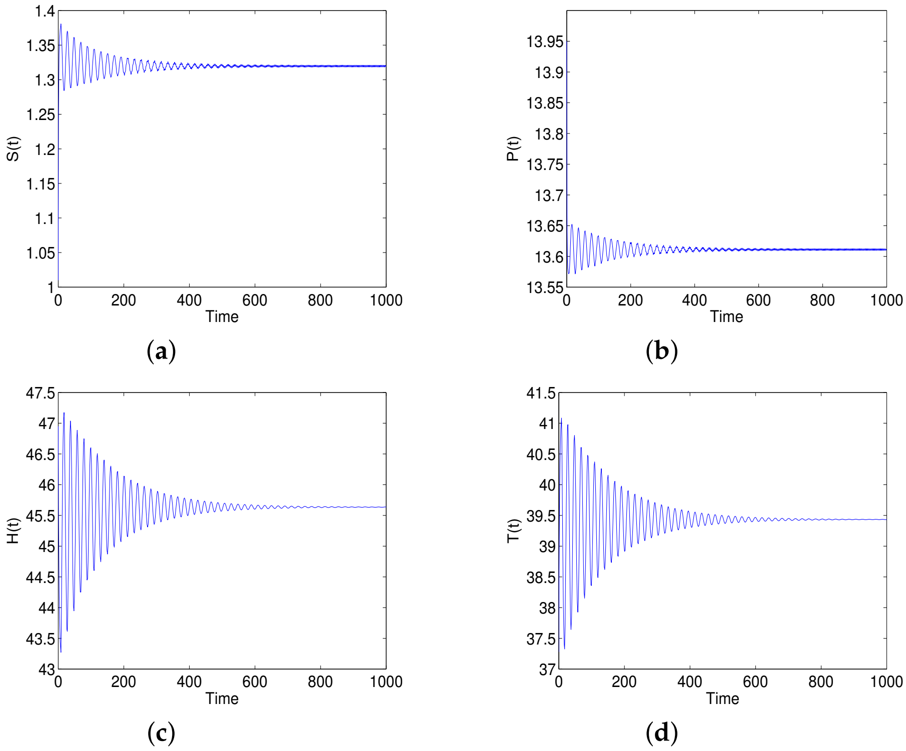

Figure 1. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_1 = 8.2247 < \tau_{10} = 9.7367$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

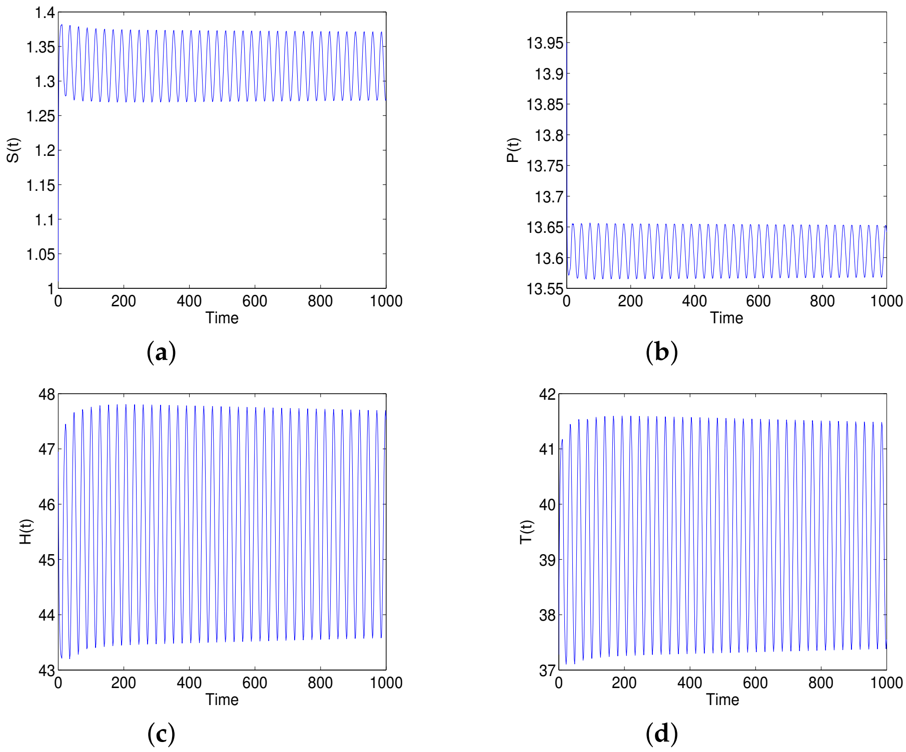

Figure 2. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_1 = 11.1421 > \tau_{10} = 9.7367$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

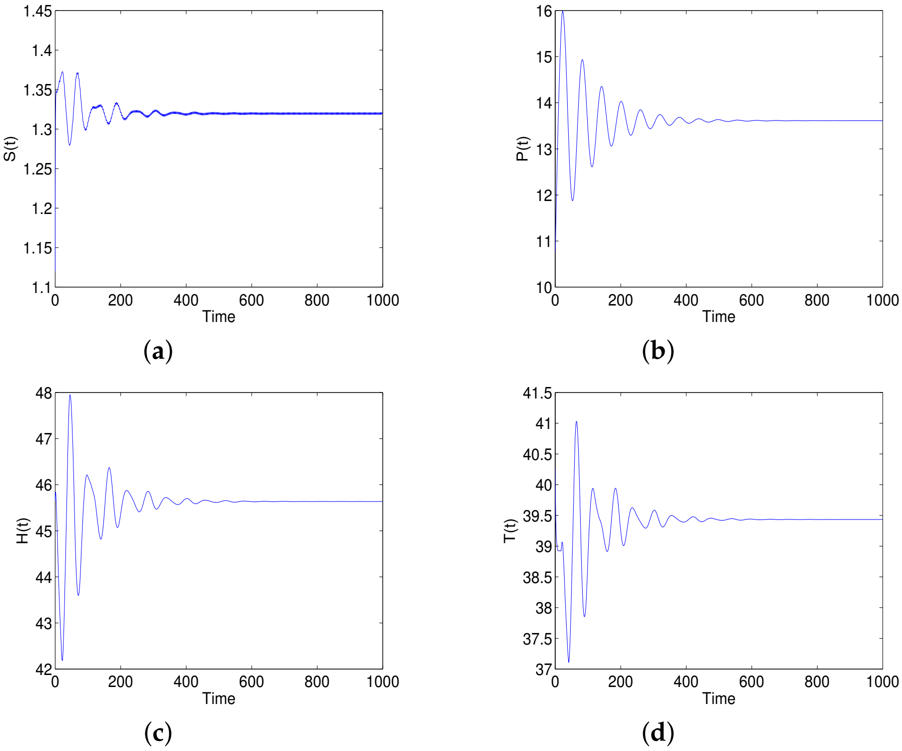

Figure 3. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_2 = 18.6934 < \tau_{20} = 20.8839$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

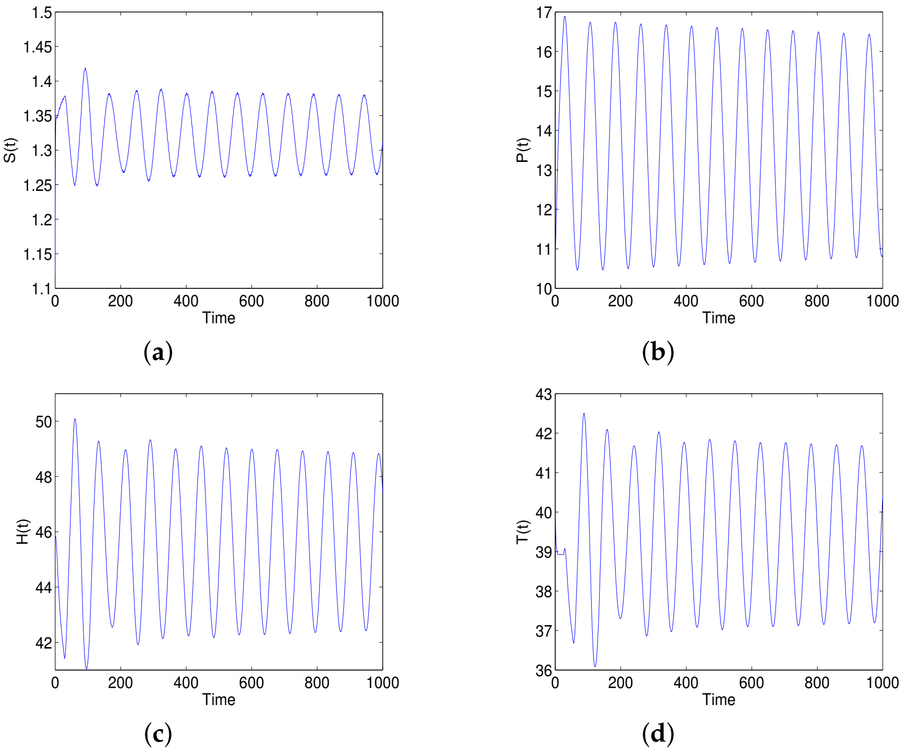

Figure 4. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_2 = 25.9358 > \tau_{20} = 20.8839$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

For $\tau_2 > 0$ and $\tau_1 = 1.5 \in (0, \tau_{10})$ and supposing τ_2 as a parameter, we get $\varsigma_{2*} = 0.7849$ and $\tau_{2*} = 8.9875$. The model system (47) is locally asymptotically stable for $\tau_2 < \tau_{2*}$ and unstable for $\tau_2 > \tau_{2*}$. Stability and instability behavior of the model system (47) is presented in Figures 7 and 8 for different values of τ_2, respectively.

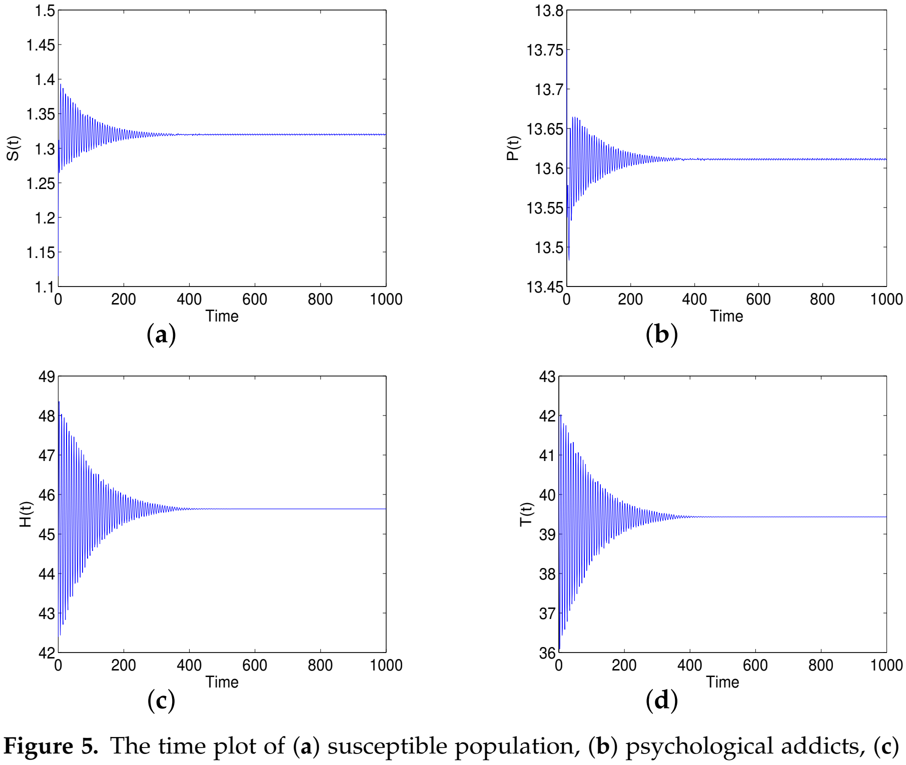

Figure 5. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_1 = 1.3785 < \tau_{1*} = 1.4096$ and $\tau_2 = 2.5 \in (0, \tau_{20})$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

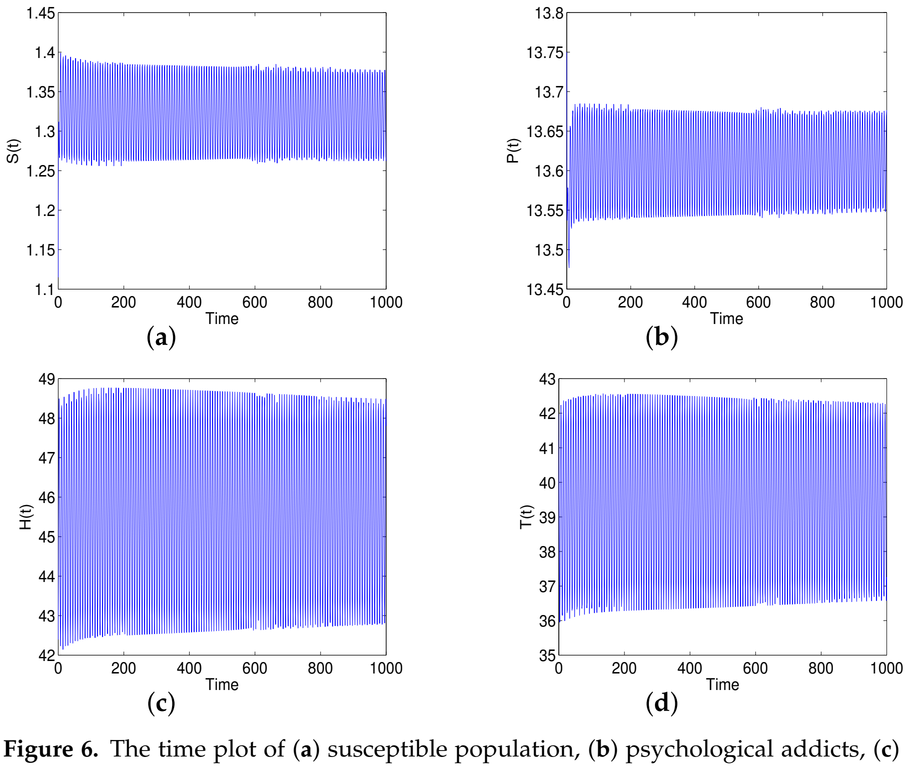

Figure 6. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_1 = 1.4308 > \tau_{1*} = 1.4096$ and $\tau_2 = 2.5 \in (0, \tau_{20})$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

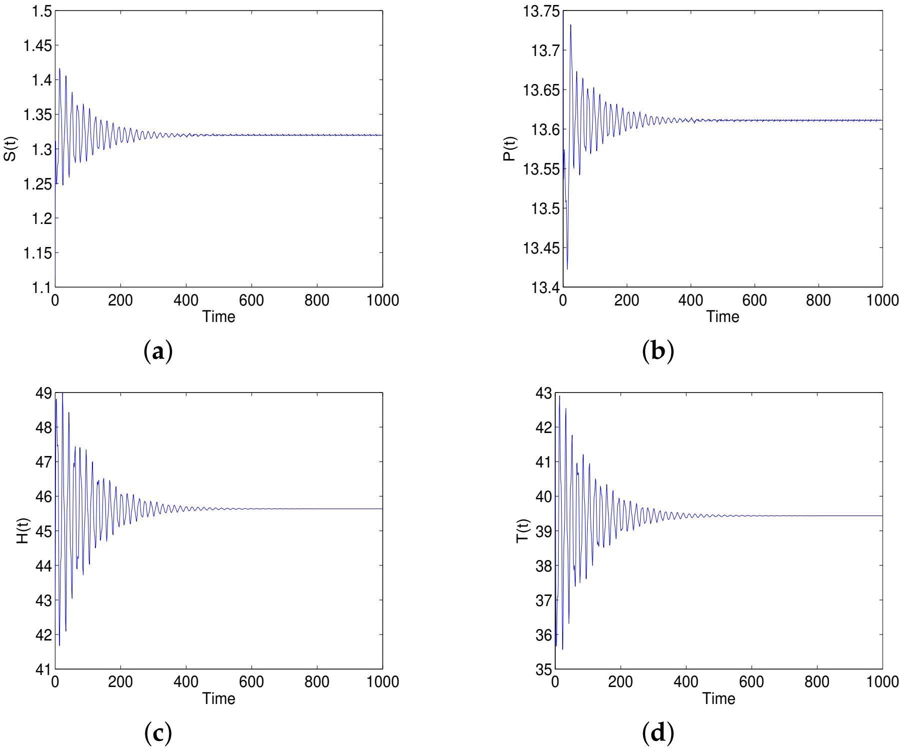

Figure 7. The time plot of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment for $\tau_2 = 8.2943 < \tau_{2*} = 8.9875$ and $\tau_1 = 1.5 \in (0, \tau_{10})$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

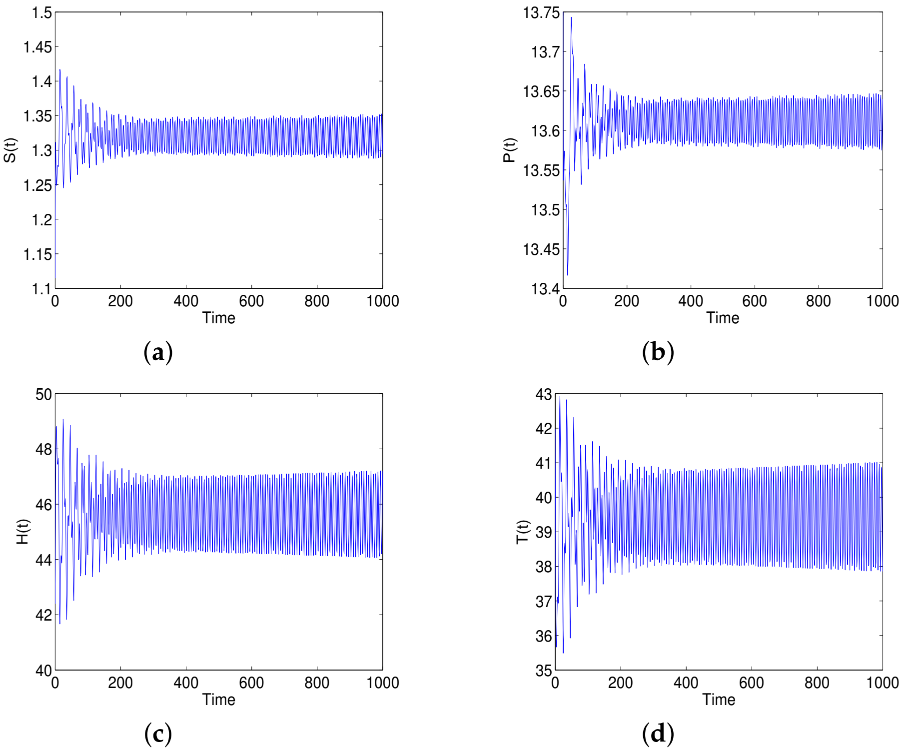

Figure 8. The time of (**a**) susceptible population, (**b**) psychological addicts, (**c**) physiological addicts and (**d**) drug-users in treatment plot for $\tau_2 = 9.3825 > \tau_{2*} = 8.9875$ and $\tau_1 = 1.5 \in (0, \tau_{10})$ with $A = 2$, $d = 0.02$, $\beta_1 = 0.016$, $\beta_2 = 0.028$, $\pi = 0.03$, $\gamma = 0.095$, $\theta = 0.5$ and $\sigma = 0.21$.

In addition, for $\tau_1 = 1.5 \in (0, \tau_{10})$ and $\tau_2 > 0$, we obtain $\lambda'(\tau_{2*}) = 0.06568892 - 0.00081555i$ and $C_0 = -4.25450964 + 13.07154877i$. Thus, we have $\Lambda_1 = 64.76753827 > 0$, $\Lambda_2 = -8.50901928 < 0$ and $\Lambda_3 = -1.84550535 < 0$. Based on the Theorem 5, we can see that the Hopf bifurcation at $\tau_{2*} = 8.9875$ is supercritical; the bifurcating periodic solutions showing around $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ are stable, and the bifurcating periodic solutions showing around $E_*(1.3196, 13.6111, 45.6355, 39.4338)$ are decreasing.

6. Conclusions

In this study, a synthetic drug transmission model with two time delays is proposed by introducing the time delay due to the period that both the psychological addicts and the physiological addicts need to accept treatment and come off drugs into the formulated model by in [25]. Through regarding the combinations of the two time delays as bifurcation parameters, sufficient criteria for local stability and exhibition of Hopf bifurcation are established. A crucial value point at which a Hopf bifurcation appears is calculated. Particularly, direction and stability of the model are explored with the aids of the normal form theory and center manifold theorem. Compared with the work in [25], we not only consider the impact of the time delay (τ_1) due to the relapse time period of the drug-users in treatment on the model system (2) but also the time delay (τ_2) due to the period that both the psychological addicts and the physiological addicts need to accept treatment and come off drugs on the model system. The results obtained in this study are supplements of the work in [25].

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