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Operational Calculus for the General Fractional Derivatives of Arbitrary Order

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Abstract: In this paper, we deal with the general fractional integrals and the general fractional derivatives of arbitrary order with the kernels from a class of functions that have an integrable singularity of power function type at the origin. In particular, we introduce the sequential fractional derivatives of this type and derive an explicit formula for their projector operator. The main contribution of this paper is a construction of an operational calculus of Mikusiński type for the general fractional derivatives of arbitrary order. In particular, we present a representation of the m-fold sequential general fractional derivatives of arbitrary order as algebraic operations in the field of convolution quotients and derive some important operational relations.

Keywords: Sonine kernel; general fractional integral; general fractional derivative of arbitrary order; fundamental theorems of fractional calculus; operational calculus; convolution series

MSC: 26A33; 26B30; 33E30; 44A10; 44A35; 44A40; 45D05; 45E10; 45J05

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1. Introduction

Starting from the early phases in development of calculus, several attempts of interpretation of derivatives and integrals as some purely algebraic symbols were undertaken. Leibniz, Euler, Lagrange, and other founders of calculus introduced and employed the algebraic rules for manipulation with the integral and differential operators. In most of the cases, these rules led to correct results. However, they were just formal algorithms without a strict mathematical background. The most prominent example of this procedure were the works by Oliver Heaviside who systematically employed an algebraic approach to investigation of a number of practical problems including differential equations of electromagnetism and theory of oscillators. Because of importance of Heaviside's approach for engineers, mathematicians started to think about its rigorous mathematical background. In the works by Bromwich, Carson, Van der Pol, Doetsch, and other mathematicians, the methods of Heaviside were justified in terms of the Laplace transform and its modifications. However, a requirement of existence of the Laplace transform led to some conditions on behavior of the functions in infinity that restricted applicability of the Heaviside operational calculus.

In the 1950s, a cardinal return to the original ideas of operational calculus was proposed in the works by Jan Mikusiński and his co-authors (see [1] and the references therein), where an operational calculus for the first order derivative has been developed and applied for a number of mathematical and real world problems. The main components of this approach are an interpretation of the Laplace convolution as a multiplication in the ring of functions continuous on the real positive semi-axis and an extension of this ring to the field of convolution quotients. Later on, the Mikusiński scheme was employed by several mathematicians for development of operational calculi for some special differential operators with the variable coefficients (see, for example, [2–4]) that turned out to be particular cases of the hyper-Bessel differential operator in the form

$$(B\,y)(t) = t^{-\beta} \prod_{j=1}^{m} \left(\gamma_j + \frac{1}{\beta} t \frac{d}{dt} \right) y(t),\ \beta > 0,\ \gamma_j \in \mathbb{R},\ j = 1, \ldots, m. \tag{1}$$

An operational calculus of Mikusiński type for the hyper-Bessel differential operator (1) was constructed by Dimovski in [5].

A new stage in further development of operational calculi of Mikusiński type was initiated in the works by Luchko and his co-authors, where operational calculi for different fractional derivatives were constructed and applied for derivation of the closed form solution formulas for the fractional integral and differential equations. The first operational calculus for a fractional derivative has been proposed in [6,7], where an operational calculus of Mikusiński type for the multiple Erdélyi-Kober fractional derivative was developed. The two prominent particular cases of this fractional derivative are the hyper-Bessel differential operator (1) and the Riemann–Liouville fractional derivative. An extended version of the operational calculus for the Riemann–Liouville fractional derivative was suggested in [8,9]. An operational calculus for another basic fractional derivative, the Caputo derivative, was developed in [10]. It is worth mentioning that in [10], this operational calculus was applied for derivation of the closed form solutions formulas to the multi-term fractional differential equations involving the Caputo fractional derivatives with the commensurate and non-commensurate orders. An operational calculus of Mikusiński type for the Hilfer fractional derivative was worked out in [11]. In [12], the case of the Caputo-type fractional Erdélyi-Kober derivative was treated. In [13], a Mikusiński type operational calculus for the general fractional derivative (GFD) of the "generalized order" form the interval $(0, 1)$ in the Caputo sense was developed. This GFD is a composition of a Laplace convolution integral with a Sonine kernel with an integrable singularity of power function type at the point zero and the first order derivative. In [13], this calculus was applied for derivation of the closed form solution formulas for the initial-value problems for the multi-term fractional differential equations with the sequential fractional derivatives of this type. The case of the GFD of arbitrary order in the Riemann–Liouville sense was treated in the very recent paper [14], where the corresponding operational calculus was applied to solve the multi-term fractional differential equations with these derivatives and the suitably formulated initial conditions. In [15,16], a survey of the operational calculi for several different fractional derivatives was provided.

The rest of this paper is organized as follows: In Section 2, we present an overview of some important properties of the general fractional integrals (GFI) and the GFD of arbitrary order. Then we introduce the sequential GFDs, prove the 1st and the 2nd fundamental theorem of fractional calculus (FC) for these derivatives, and derive an explicit form for their projector operators. Section 3 is devoted to construction of an operational calculus of the Mikusiński type for the GFD of arbitrary order. In particular, the sequential GFDs of arbitrary order are represented as multiplication with certain elements of the constructed field of convolution quotients. The developed operational calculus can be applied for derivation of the closed form solution formulas for the ordinary and partial fractional differential equations with the sequential GFDs of arbitrary order.

2. General Fractional Derivatives of Arbitrary Order

The first publication devoted to the GFI and GFD was the paper [17] by Sonine published in 1884, even if no fractional integrals and fractional derivatives were explicitly mentioned there. In [17], Sonine generalized the solution method employed by Abel in [18,19] for the integro-differential equation (in slightly different notations)

$$f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(\tau)\,d\tau}{(t-\tau)^{\alpha}},\ 0 < \alpha < 1 \tag{2}$$

to the case of more general integral equations. In his derivations, Abel essentially used the relation

$$(h_\alpha * h_{1-\alpha})(t) = \{1\},\ t > 0,\ 0 < \alpha < 1, \tag{3}$$

where h_α denotes a power function

$$h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ t > 0,\ \alpha > 0, \tag{4}$$

the operation $*$ stands for the Laplace convolution

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau, \tag{5}$$

and $\{1\}$ is the function that is identically equal to 1 for $t > 0$. Abel's solution to the integro-differential Equation (2) under the condition $g(0) = 0$ is nowadays well-known as the Riemann–Liouville fractional integral:

$$g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ t > 0. \tag{6}$$

The brilliant idea of Sonine was to replace the power functions h_α and $h_{1-\alpha}$ in the relation (3) with the arbitrary functions $\kappa,\ k$ that satisfy the same relation:

$$(\kappa * k)(t) = \{1\},\ t > 0. \tag{7}$$

Nowadays such functions are called the Sonine kernels and the condition (7) is refereed to as the Sonine condition.

By employing the Abel method, Sonine solved the convolution type integral equation

$$f(t) = \int_0^t \kappa(t-\tau) g(\tau)\, d\tau\ =\ (\kappa * g)(t) \tag{8}$$

in explicit form:

$$g(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau\ =\ \frac{d}{dt}(k * f)(t), \tag{9}$$

provided the kernels $\kappa,\ k$ satisfy the Sonine condition (7).

It is worth mentioning that derivations of both Abel and Sonine were not rigorous from the modern viewpoint because they did not introduce the suitable spaces of functions and did not provide conditions for validity of their formal manipulations with integrals and derivatives. Only recently, the operators (8) and (9) became a subject of active research in FC and nowadays their mathematical theory is under intensive construction, see [13,20–28].

In this paper, we deal with the GFI and GFD of arbitrary order introduced in [24] for the first time. It is well-known that both the Riemann–Liouville fractional integral

$$(I^\alpha_{0+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0,\ t > 0 \tag{10}$$

and the Riemann–Liouville and Caputo fractional derivatives

$$(D^\alpha_{0+} f)(t) = \frac{d^n}{dt^n} (I^{n-\alpha}_{0+} f)(t),\ n-1 \le \alpha < n,\ n \in \mathbb{N}, \tag{11}$$

$$({}_*D^\alpha_{0+} f)(t) = \left(D^\alpha_{0+} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot) \right) \right)(t),\ t > 0,\ n-1 \le \alpha < n,\ n \in \mathbb{N} \tag{12}$$

are well-defined for any order $\alpha > 0$.

However, the relation (3) is valid only under the restriction $0 < \alpha < 1$. Similarly, the Sonine operators defined by the right-hand sides of the Formulas (8) and (9) (GFI and GFD in the modern FC notations) have a "generalized order" between zero and one because of the Sonine condition (7). To define the GFI and GFD of arbitrary positive order, in [24], the Sonine condition (7) was extended and specified for the kernels from some suitable spaces of functions:

Definition 1 ([24])**.** *Let the functions κ and k satisfy the condition*

$$(\kappa * k)(t) = \{1\}^{<n>}(t),\ n \in \mathbb{N},\ t > 0, \tag{13}$$

where

$$\{1\}^{<n>}(t) := (\underbrace{\{1\} * \ldots * \{1\}}_{n\ times})(t) = h_n(t) = \frac{t^{n-1}}{(n-1)!}$$

and the inclusions $\kappa \in C_{-1}(0,+\infty)$ and $k \in C_{-1,0}(0,+\infty)$ hold true, where

$$C_{-1}(0,+\infty) \ := \ \{f:\ f(t) = t^p f_1(t),\ t > 0,\ p > -1,\ f_1 \in C([0,+\infty))\}, \tag{14}$$

$$C_{-1,0}(0,+\infty) \ = \ \{f:\ f(t) = t^p f_1(t),\ t > 0,\ -1 < p < 0,\ f_1 \in C([0,+\infty))\}. \tag{15}$$

The set of pairs $(\kappa,\ k)$ of such kernels is denoted by $\mathcal{L}_n$.

For a detailed treatment of the case $n = 1$, i.e., for a theory of the GFI and GFD with the kernels from $\mathcal{L}_1$ we refer to [23]. In this paper, our focus will be on the case $n > 1$, even if the case $n = 1$ is also included in all formulations and derivations.

It is worth mentioning that one cannot interchange the kernels κ and k in Definition 1 with $n > 1$ because of the non-symmetrical inclusions $\kappa \in C_{-1}(0,+\infty)$ and $k \in C_{-1,0}(0,+\infty)$ (in the case $n = 1$, Definition 1 is symmetrical and one can interchange the kernels κ and k). However, the kernel $\kappa(t) = h_\alpha(t),\ \alpha > 0$ of the Riemann–Liouville integral (10) and the kernel $k(t) = h_{n-\alpha}(t)$ of the Riemann–Liouville and Caputo fractional derivatives (11) and (12) of order $\alpha,\ n-1 < \alpha < n,\ n \in \mathbb{N}$ can be also not interchanged in the case $n > 1$. Evidently, the kernels $\kappa(t) = h_\alpha(t),\ \alpha > 0$ and $k(t) = h_{n-\alpha}(t),\ n-1 < \alpha < n,\ n \in \mathbb{N}$ belong to the kernel set $\mathcal{L}_n$. However, they are the Sonine kernels only in the case $n = 1$, i.e., only in the case of the fractional derivatives orders less than one.

For other examples of the kernels from $\mathcal{L}_n,\ n > 1$ and for the procedures how to construct them starting from the known Sonine kernels from the set $\mathcal{L}_1$ we refer to [24,28].

Now we define the GFI and GFD of arbitrary order, present their known properties and derive some new ones.

Definition 2 ([24])**.** *Let $(\kappa,\ k) \in \mathcal{L}_n$. The GFI with the kernel κ and the GFD of arbitrary order with the kernel k are defined as follows:*

$$(\mathbb{I}_{(\kappa)}\, f)(t) := \int_0^t \kappa(t-\tau) f(\tau)\, d\tau,\ t > 0, \tag{16}$$

$$({}_*\mathbb{D}_{(k)}\, f)(t) := \left(\mathbb{D}_{(k)}\left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot)\right)\right)(t),\ t > 0, \tag{17}$$

where the GFD $\mathbb{D}_{(k)}$ defined in the Riemann–Liouville sense is given by the relation

$$(\mathbb{D}_{(k)}\, f)(t) := \frac{d^n}{dt^n} \int_0^t k(t-\tau) f(\tau)\, d\tau,\ t > 0. \tag{18}$$

As already mentioned, the kernels of the Riemann–Liouville fractional integral (10) and the Caputo fractional derivative (12) belong to the kernel set $\mathcal{L}_n$ and thus these FC operators are particular cases of the GFI (16) and the GFD (17), respectively.

Another interesting particular case of the GFI (16) and the GFD (17) was presented in [24]. Let the condition $n-2 < \nu < n-1,\ n \in \mathbb{N}$ holds true. Then the operator

$$(\mathbb{I}_{(\kappa)}\, f)(t) = \int_0^t (t-\tau)^{\nu/2}\, J_\nu(2\sqrt{t-\tau}) f(\tau) d\tau,\ t > 0, \tag{19}$$

is a particular case of the GFI (16) and the corresponding GFD of arbitrary order takes the following form:

$$({}_*\mathbb{D}_{(k)}\, f)(t) := \left(\mathbb{D}_{(k)} \left(f(\cdot) - \sum_{j=0}^{n-1} f^{(j)}(0) h_{j+1}(\cdot)\right)\right)(t),\ t > 0, \tag{20}$$

where

$$(\mathbb{D}_{(k)}\, f)(t) = \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n/2-\nu/2-1} I_{n-\nu-2}(2\sqrt{t-\tau}) f(\tau)\, d\tau,\ t > 0 \tag{21}$$

and the functions J_ν and I_ν are the Bessel and the modified Bessel functions, respectively.

In this paper, we investigate the GFI and the GFD of arbitrary order on the space $C_{-1}(0,+\infty)$ defined by the Formula (14) and its sub-spaces. In particular, the sub-spaces

$$C_{-1}^m(0,+\infty) := \{f :\ f^{(m)} \in C_{-1}(0,+\infty)\}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

will be often used. These sub-spaces were introduced and investigated in [10] in connection with construction of an operational calculus of Mikusiński type for the Caputo fractional derivative.

As mentioned in [24], if the inclusion $k \in C_{-1}^{n-1}(0,+\infty)$ holds valid, the GFD (17) can be represented as follows:

$$({}_*\mathbb{D}_{(k)}\, f)(t) = (\mathbb{D}_{(k)}\, f)(t) - \sum_{j=0}^{n-1} f^{(j)}(0) \frac{d^{n-j-1}}{dt^{n-j-1}} k(t),\ t > 0, \tag{22}$$

where $\mathbb{D}_{(k)}$ is given by the relation (18). Moreover, for $f \in C_{-1}^n(0,+\infty)$, the GFD (17) takes the form

$$({}_*\mathbb{D}_{(k)}\, f)(t) = \int_0^t k(t-\tau) f^{(n)}(\tau)\, d\tau,\ t > 0 \tag{23}$$

that is often used in the case of the power law kernel $k(\tau) = h_{n-\alpha}(\tau),\ n-1 < \alpha < n,\ n \in \mathbb{N}$ of the Caputo fractional derivative.

The basic properties of the GFI (16) of arbitrary order on the space $C_{-1}(0,+\infty)$ immediately follow from the well-known properties of the Laplace convolution [23]:

$$\mathbb{I}_{(\kappa)} :\ C_{-1}(0,+\infty)\ \to C_{-1}(0,+\infty)\ \text{(mapping property)}, \tag{24}$$

$$\mathbb{I}_{(\kappa_1)}\, \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)}\, \mathbb{I}_{(\kappa_1)}\ \text{(commutativity law)}, \tag{25}$$

$$\mathbb{I}_{(\kappa_1)}\, \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)}\ \text{(index law)}. \tag{26}$$

We also mention the following important theorem:

Theorem 1 ([10]). *The triple $\mathcal{R}_{-1} = (C_{-1}(0,+\infty),+,*)$ with the usual addition $+$ and multiplication $*$ in form of the Laplace convolution is a commutative ring without unity with respect to multiplication and without divisors of zero.*

Furthermore, in [24], two fundamental theorems of FC for the GFI and the GFD of arbitrary order have been proved. The formulations of these theorems are provided below, for the proofs see [24].

Theorem 2 ([24]). *Let* $(\kappa,\, k) \in \mathcal{L}_n$.

The GFD (17) *of arbitrary order is a left inverse operator to the GFI* (16) *on the space* $C_{-1,(k)}(0,+\infty)$*:*

$$({}_*\mathbb{D}_{(k)}\, \mathbb{I}_{(\kappa)}\, f)(t) = f(t),\ f \in C_{-1,(k)}(0,+\infty),\ t > 0, \tag{27}$$

where $C_{-1,(k)}(0,+\infty) := \{f :\ f(t) = (\mathbb{I}_{(k)}\, \phi)(t),\ \phi \in C_{-1}(0,+\infty)\}$.

Theorem 3 ([24]). *Let* $(\kappa,\, k) \in \mathcal{L}_n$.

Then the relation

$$(\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)}\, f)(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0)\, h_{j+1}(t) \tag{28}$$

holds true on the space $C^n_{-1}(0,+\infty)$.

It is worth mentioning that the result of Theorem 3 (2nd fundamental theorem of FC for the GFD of arbitrary order) can be reformulated in terms of the so-called projector operator P of the GFD (17):

$$(P\, f)(t) := f(t) - (\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)}\, f)(t) = \sum_{j=0}^{n-1} f^{(j)}(0)\, h_{j+1}(t). \tag{29}$$

The form of the projector operator P determines the natural initial conditions for the initial-value problems for the fractional differential equations with the GFD (17). According to the representation (29), they are provided in terms of the integer order derivatives $f^{(j)}(0),\ j = 0, 1, \ldots, n-1$ of the unknown function as it is the case for the ordinary differential equations and for the fractional differential equations with the Caputo derivatives.

In the rest of this section, we define the m-fold sequential GFIs and GFDs of arbitrary order with the kernels $(\kappa,\, k) \in \mathcal{L}_n,\ n \in \mathbb{N}$ and investigate their properties. In the case $n = 1$, the m-fold GFIs and the m-fold sequential GFDs were introduced and studied in [25,29]. In [14], the m-fold sequential GFDs in the Riemann–Liouville sense with the kernels $(\kappa,\, k) \in \mathcal{L}_n$ have been considered based on the GFD of arbitrary order defined in the Riemann–Liouville sense by the Formula (18). In this paper, we deal with the case of the GFD of arbitrary order in form (17).

First we define the convolution powers $f^{<m>},\ m \in \mathbb{N}_0$ of a function f as follows:

$$f^{<m>}(t) := \begin{cases} \{1\}, & m = 0, \\ f(t), & m = 1, \\ (\underbrace{f * \ldots * f}_{m\ \text{times}})(t), & m = 2, 3, \ldots. \end{cases} \tag{30}$$

Definition 3. *Let* $(\kappa,\, k) \in \mathcal{L}_n,\ n \in \mathbb{N}$.

The m-fold GFI is defined as a composition of m GFIs with the kernel κ*:*

$$(\mathbb{I}^{<m>}_{(\kappa)}\, f)(t) := (\underbrace{\mathbb{I}_{(\kappa)} \ldots \mathbb{I}_{(\kappa)}}_{m\ \text{times}})\, f)(t) = (\kappa^{<m>} * f)(t),\ t > 0. \tag{31}$$

The corresponding m-fold sequential GFD of arbitrary order is defined as follows:

$$({}_*\mathbb{D}_{(k)}^{<m>}\, f)(t) := (\underbrace{{}_*\mathbb{D}_{(k)} \ldots {}_*\mathbb{D}_{(k)}}_{m \text{ times}}\, f)(t),\ t > 0. \tag{32}$$

In analogy to the case of the Riemann–Liouville fractional integral and the Caputo fractional derivative, the operators $\mathbb{I}_{(\kappa)}^{<0>}$ and ${}_*\mathbb{D}_{(k)}^{<0>}$ are interpreted as the identity operator Id.

Due to Theorem 1, the kernel $\kappa^{<m>},\ m \in \mathbb{N}$ from the Formula (31) is from the space $C_{-1}(0,+\infty)$ and thus the m-fold GFI can be represented as a GFI with the kernel $\kappa^{<m>}$:

$$(\mathbb{I}_{(\kappa)}^{<m>}\, f)(t) = (\kappa^{<m>} * f)(t) = (\mathbb{I}_{(\kappa)^{<m>}}\, f)(t),\ t > 0. \tag{33}$$

The m-fold sequential GFD (32) of arbitrary order is a direct generalization of the sequential fractional derivative in the Caputo sense to the case of the integro-differential operators with the kernels from $\mathcal{L}_n$.

The 1st fundamental theorem of FC (Theorem 2) for the GFI (16) and the GFD (17) of arbitrary order immediately leads to the following important result:

Theorem 4 (1st fundamental theorem of FC for the m-fold sequential GFD of arbitrary order)**.** *Let $(\kappa,\ k) \in \mathcal{L}_n,\ n \in \mathbb{N}$.*

The m-fold sequential GFD (32) of arbitrary order is a left inverse operator to the m-fold GFI (31) on the space $C_{-1,(k)}(0,+\infty)$:

$$(\mathbb{D}_{(k)}^{<m>}\, \mathbb{I}_{(\kappa)}^{<m>}\, f)(t) = f(t),\ f \in C_{-1,(k)}(0,+\infty),\ t > 0. \tag{34}$$

To generalize Theorem 3 to the case of the m-fold sequential GFDs, we first introduce the suitable spaces of functions in the form

$$C_{-1,(k)}^{n,m}(0,+\infty) := \{f \in C_{-1}^{n}(0,+\infty) :\ {}_*D_{(k)}^{<i>} f \in C_{-1}^{n}(0,+\infty),\ i = 1,\ldots,m-1\}. \tag{35}$$

For $m = 1$, we set $C_{-1,(k)}^{n,1}(0,+\infty) := C_{-1}^{n}(0,+\infty)$.

Theorem 5 (2nd fundamental theorem of FC for the m-fold sequential GFD of arbitrary order)**.** *Let $(\kappa,\ k) \in \mathcal{L}_n$.*

Then the relation

$$(\mathbb{I}_{(\kappa)}^{<m>}\, {}_*\mathbb{D}_{(k)}^{<m>}\, f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{d^j f}{dt^j}(0)\, h_{j+1}(t) - \sum_{j=0}^{n-1}\sum_{i=1}^{m-1} \left(\frac{d^j}{dt^j}\, {}_*\mathbb{D}_{(k)}^{<i>} f\right)(0)\, (\kappa^{<i>} * h_{j+1})(t) \tag{36}$$

holds true on the space $C_{-1,(k)}^{n,m}(0,+\infty)$.

Proof. In the case $m = 1$, the statement of Theorem 5 is Theorem 3 that was proved in [24]. Now we proceed with the case $m = 2$. Then we have the inclusion $f \in C_{-1,(k)}^{n,2}(0,+\infty)$ and the representation

$$(\mathbb{I}_{(\kappa)}^{<2>}\, {}_*\mathbb{D}_{(k)}^{<2>}\, f)(t) = (\mathbb{I}_{(\kappa)}\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)}\, {}_*\mathbb{D}_{(k)}\, f)(t) =$$

$$(\mathbb{I}_{(\kappa)}(\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)}(\, {}_*\mathbb{D}_{(k)} f)))(t).$$

For a function $f \in C_{-1,(k)}^{n,2}(0,+\infty)$, the inclusion ${}_*D_{(k)} f \in C_{-1}^{n}(0,+\infty)$ holds true and thus we can apply Theorem 3 to the inner composition $\mathbb{I}_{(\kappa)}\, {}_*\mathbb{D}_{(k)}$ acting on the function ${}_*\mathbb{D}_{(k)}\, f$ and get the formula

$$(\mathbb{I}_{(\kappa)}^{<2>}\,{}_*\mathbb{D}_{(k)}^{<2>}\,f)(t) = \left(\mathbb{I}_{(\kappa)}\left[({}_*\mathbb{D}_{(k)}\,f)(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}f\right)(0)\,h_{j+1})(t)\right]\right)(t) =$$

$$(\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}\,f)(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}f\right)(0)\,(\kappa * h_{j+1})(t).$$

The final result immediately follows by applying Theorem 3 to the composition $\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}\,f$ at the right-hand side of the last formula:

$$(\mathbb{I}_{(\kappa)}^{<2>}\,\mathbb{D}_{(k)}^{<2>}\,f)(t) = f(t) - \sum_{j=0}^{n-1}\frac{d^j f}{dt^j}(0)\,h_{j+1}(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}f\right)(0)\,(\kappa * h_{j+1})(t).$$

For $m = 3, 4, \ldots$, we employ the recurrent formula

$$(\mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>}\,f)(t) = (\mathbb{I}_{(\kappa)}^{<m-1>}(\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}({}_*\mathbb{D}_{(k)}^{<m-1>}\,f)))(t) =$$

$$\left(\mathbb{I}_{(\kappa)}^{<m-1>}\left[({}_*\mathbb{D}_{(k)}^{<m-1>}\,f)(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}^{<m-1>}f\right)(0)\,h_{j+1}(t)\right]\right)(t) =$$

$$\left(\mathbb{I}_{(\kappa)}^{<m-1>}\,{}_*\mathbb{D}_{(k)}^{<m-1>}\,f\right)(t) - \sum_{j=0}^{n-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}^{<m-1>}f\right)(0)\,(\kappa^{<m-1>} * h_{j+1})(t)$$

and the principle of the mathematical induction to complete the proof of the representation (36). □

Remark 1. *The Formula (36) can be rewritten in terms of the projector operator P_m of the m-fold sequential GFD (32) as follows:*

$$(P_m\,f)(t) := f(t) - (\mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>}\,f)(t) =$$

$$\sum_{j=0}^{n-1}\frac{d^j f}{dt^j}(0)\,h_{j+1}(t) + \sum_{j=0}^{n-1}\sum_{i=1}^{m-1}\left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}^{<i>}f\right)(0)\,(\kappa^{<i>} * h_{j+1})(t),\ t > 0. \tag{37}$$

According to Theorem 5, the Formula (37) is valid on the space $C_{-1,(k)}^{n,m}(0, +\infty)$.

It is worth mentioning that the Formula (37) specifies the natural form of the initial conditions while dealing with the fractional differential equations that contain the sequential GFDs of the orders up to m. These initial conditions should be formulated in terms of the values of the integer order derivatives applied to the sequential GFDs of the orders up to $(m-1)$ evaluated at the point zero:

$$\frac{d^j f}{dt^j}(0) = a_{j0},\ j = 0, \ldots, n-1,\ \left(\frac{d^j}{dt^j}\,{}_*\mathbb{D}_{(k)}^{<i>}f\right)(0) = a_{ji},\ j = 0, \ldots, n-1,\ i = 1, \ldots, m-1.$$

Remark 2. *Evidently, the identity operator Id on the space $C_{-1,(k)}^{n,m}(0, +\infty)$ can be represented as follows:*

$$\begin{aligned}
Id =& (Id - \mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}) + (\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)} - \mathbb{I}_{(\kappa)}^{<2>}\,{}_*\mathbb{D}_{(k)}^{<2>}) + \ldots \\
&+ (\mathbb{I}_{(\kappa)}^{<m-1>}\,{}_*\mathbb{D}_{(k)}^{<m-1>} - \mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>}) + \mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>} \\
=& (Id - \mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)}) + (\mathbb{I}_{(\kappa)}\,({}_*\mathbb{D}_{(k)} - (\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)})\,{}_*\mathbb{D}_{(k)})) + \ldots \\
&+ (\mathbb{I}_{(\kappa)}^{<m-1>}\,({}_*\mathbb{D}_{(k)}^{<m-1>} - (\mathbb{I}_{(\kappa)}\,{}_*\mathbb{D}_{(k)})\,{}_*\mathbb{D}_{(k)}^{<m-1>})) + \mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>} \\
=& P + (\mathbb{I}_{(\kappa)}\,P({}_*\mathbb{D}_{(k)})) + \cdots + (\mathbb{I}_{(\kappa)}^{<m-1>}\,P({}_*\mathbb{D}_{(k)}^{<m-1>})) + \mathbb{I}_{(\kappa)}^{<m>}\,{}_*\mathbb{D}_{(k)}^{<m>},
\end{aligned}$$

where P stands for the projector operator of the GFD (17) given by the Formula (29).

As a consequence, the projector operator P_m of the m-fold sequential GFD (32) can be also represented in a more compact form:

$$(P_m f)(t) = \sum_{i=0}^{m-1} (\mathbb{I}_{(\kappa)}^{<i>} (P ({}_*\mathbb{D}_{(k)}^{<i>}) f))(t), \; t > 0. \tag{38}$$

It is worth mentioning that for any function $f \in C_{-1,(k)}^{n,m}(0, +\infty)$, its image $P_m f$ by the projector operator is from the kernel of the m-fold sequential GFD:

$$\begin{aligned}({}_*\mathbb{D}_{(k)}^{<m>} (P_m f))(t) =& ({}_*\mathbb{D}_{(k)}^{<m>} f)(t) - ({}_*\mathbb{D}_{(k)}^{<m>} \mathbb{I}_{(\kappa)}^{<m>} {}_*\mathbb{D}_{(k)}^{<m>} f)(t) \\ =& ({}_*\mathbb{D}_{(k)}^{<m>} f)(t) - ({}_*\mathbb{D}_{(k)}^{<m>} f)(t) = 0.\end{aligned}$$

We also mention the inclusions

$$\mathrm{Ker}\, {}_*\mathbb{D}_{(k)} \subset \mathrm{Ker}\, {}_*\mathbb{D}_{(k)}^{<2>} \subset \cdots \subset \mathrm{Ker}\, {}_*\mathbb{D}_{(k)}^{<m>}$$

that immediately follow from the definition of the m-fold sequential GFD.

3. Operational Calculus for the GFD of Arbitrary Order

As already mentioned in Introduction, an operational calculus of the Mikusiński type for the GFI and GFD with the kernels $(\kappa,\, k) \in \mathcal{L}_1$ has been constructed in [13]. This case corresponds to the GFI and GFD of the "generalized order" between zero and one. In this paper, we extend the constructions presented in [13] to the case of the kernels $(\kappa,\, k) \in \mathcal{L}_n$, $n \in \mathbb{N}$, i.e., for the GFI and GFD of arbitrary order.

The first important component of any operational calculus of Mikusiński type is a suitable ring of functions. For the operational calculus for the GFI and GFD with the kernels $(\kappa,\, k) \in \mathcal{L}_n$, this ring is described in Theorem 1 that says that the triple $\mathcal{R}_{-1} = (C_{-1}(0, +\infty), +, *)$ with the usual addition $+$ and multiplication $*$ in form of the Laplace convolution is a commutative ring without divisors of zero.

Furthermore, Definition 1 ensures that the kernels κ and k from $\mathcal{L}_n$ are elements of this ring:

$$\kappa \in \mathcal{R}_{-1},\; k \in \mathcal{R}_{-1} \text{ if } (\kappa,\, k) \in \mathcal{L}_n.$$

In particular, this means that the GFI with the kernel κ is reduced to a conventional multiplication on the ring $\mathcal{R}_{-1}$:

$$(\mathbb{I}_{(\kappa)} f)(t) = (\kappa * f)(t), \; t > 0, \; f \in \mathcal{R}_{-1}. \tag{39}$$

As to the GFD of arbitrary order, it cannot be reduced to the algebraic operations on the ring $\mathcal{R}_{-1}$. The reason is that the GFD is a left-inverse operator to the GFI and the ring $\mathcal{R}_{-1}$ does not possess a unity element with respect to multiplication. Thus, no inverse element to $\kappa \in \mathcal{R}_{-1}$ does exist in $\mathcal{R}_{-1}$.

To demonstrate the last statement, let us assume that the GFD (17) of arbitrary order can be represented as a convolution with a certain element $\kappa^{-1} \in \mathcal{R}_{-1}$:

$$({}_*\mathbb{D}_{(k)} f)(t) = (\kappa^{-1} * f)(t), \; t > 0, \; f \in \mathcal{R}_{-1}.$$

According to Theorem 2, the GFD (17) of arbitrary order is a left inverse operator to the GFI (16). Combining the last equation with the representation (39), we arrive at the relation

$$({}_*\mathbb{D}_{(k)}\, \mathbb{I}_{(\kappa)} f)(t) = (\kappa^{-1} * (\kappa * f))(t) = ((\kappa^{-1} * \kappa) * f)(t) \;=\; f(t), \; t > 0, \; f \in \mathcal{R}_{-1}$$

that contradicts to the fact that the ring $\mathcal{R}_{-1}$ does not possess a unity element with respect to multiplication.

To resolve the problem mentioned above, the ring $\mathcal{R}_{-1}$ is extended to a field of convolution quotients. Let us remind that $\mathcal{R}_{-1}$ does not have any divisors of zero (Theorem 1) and thus this extension follows a standard procedure, see, e.g., [10,13].

First, an equivalence relation on the set

$$C^2_{-1}(0,+\infty) := C_{-1}(0,+\infty) \times (C_{-1}(0,+\infty) \setminus \{0\})$$

is introduced:

$$(f_1, g_1) \sim (f_2, g_2) \Leftrightarrow (f_1 * g_2)(t) = (f_2 * g_1)(t), \ (f_1, g_1), \ (f_2, g_2) \in C^2_{-1}(0,+\infty).$$

Then we consider the equivalences classes $C^2_{-1}(0,+\infty)/\sim$ and denote them as quotients:

$$\frac{f}{g} := \{(f_1, g_1) \in C^2_{-1}(0,+\infty) : \ (f_1, g_1) \sim (f, g)\}.$$

On the space $C^2_{-1}(0,+\infty)/\sim$, usual operations of addition and multiplication are introduced:

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} := \frac{f_1 * g_2 + f_2 * g_1}{g_1 * g_2},$$

$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} := \frac{f_1 * f_2}{g_1 * g_2}.$$

These operations are correctly defined (they do not depend on the representatives of the equivalence classes). Theorem 1 and the definitions provided above immediately lead to the following important result:

Theorem 6 ([10])**.** *The triple $\mathcal{F}_{-1} = (C^2_{-1}(0,+\infty)/\sim, +, \cdot)$ is a field that is usually referred to as the field of convolution quotients.*

As usual, the ring $\mathcal{R}_{-1}$ can be embedded into the field $\mathcal{F}_{-1}$:

$$f \mapsto \frac{f * \kappa}{\kappa}, \tag{40}$$

where κ is the kernel of the GFI (16).

The set $C^2_{-1}(0,+\infty)/\sim$ of the equivalence classes is also a vector space with the addition operation mentioned above and multiplication with a scalar $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$ defined as follows [10]:

$$\lambda \frac{f}{g} := \frac{\lambda f}{g}, \ \frac{f}{g} \in C^2_{-1}(0,+\infty)/\sim.$$

On the other hand, the constant function $\{\lambda\}$ (the function that takes the value λ for any $t \geq 0$) is from the space $C_{-1}(0,+\infty)$ and thus it is an element of the ring $\mathcal{R}_{-1}$. According to our definitions, multiplication with $\{\lambda\}$ in the field $\mathcal{F}_{-1}$ is given by the following expression:

$$\{\lambda\} \cdot \frac{f}{g} = \frac{\{\lambda\} * f}{g}, \ \frac{f}{g} \in \mathcal{F}_{-1}.$$

As we see, one has to differentiate between multiplication of an element from $\mathcal{F}_{-1}$ with a scalar λ and with the constant function $\{\lambda\}$.

Even if the ring $\mathcal{R}_{-1}$ is embedded into the field $\mathcal{F}_{-1}$, some elements of the field of convolution quotients cannot be reduced to the conventional functions from the ring. One of them is the unity element $I = \frac{\kappa}{\kappa}$ of the field $\mathcal{F}_{-1}$ with respect to multiplication (see [13]). Such elements can be interpreted as a kind of generalized functions (hyper-functions in the terminology of [30]). The inverse element to the kernel κ of the GFI (16) is another important hyper-function.

Definition 4 ([13]). *The inverse element to the kernel $\kappa \in \mathcal{R}_{-1}$ in the field $\mathcal{F}_{-1}$ in form*

$$S_\kappa := \frac{\kappa}{\kappa * \kappa} \tag{41}$$

is called an algebraic inverse element to the GFI (16)*.*

By definition, the relation

$$\kappa \cdot S_\kappa = \frac{\kappa * \kappa}{\kappa} \cdot \frac{\kappa}{\kappa * \kappa} = \frac{\kappa^{<3>}}{\kappa^{<3>}} = I \tag{42}$$

holds true, where I is the unity of the field $\mathcal{F}_{-1}$ with respect to multiplication.

Then, following [13], we introduce an important notion of an algebraic GFD of arbitrary order.

Definition 5. *Let $(\kappa, k) \in \mathcal{L}_n$, $n \in \mathbb{N}$ and $f \in C^{n-1}[0, +\infty)$.*

The algebraic GFD of arbitrary order is defined as follows:

$$_*\mathbb{D}_{(k)}\, f = S_\kappa \cdot f - S_\kappa \cdot (P\, f), \tag{43}$$

where the function f and the projector operator $P\, f$ given by the Formula (29) *are interpreted as elements of the convolution quotients field $\mathcal{F}_{-1}$.*

As we see, the algebraic GFD of arbitrary order is defined for any function $f \in C^{n-1}[0, +\infty)$. However, the GFD of arbitrary order given by the Formula (17) evidently does not always exist on the space $C^{n-1}[0, +\infty)$. Thus, the algebraic GFD (43) can be interpreted as a kind of a generalized derivative that assigns a certain element of the field $\mathcal{F}_{-1}$ to any function $f \in C^{n-1}[0, +\infty)$. However, the algebraic GFD (43) coincides with the GFD (17) on the space $C^n_{-1}(0, +\infty)$ (the inclusion $C^n_{-1}(0, +\infty) \subset C^{n-1}[0, +\infty)$ holds valid, see [10]).

Theorem 7. *Let $(\kappa, k) \in \mathcal{L}_n$ and $f \in C^n_{-1}(0, +\infty)$.*

Then the algebraic GFD (43) *coincides with the GFD* (17) *of arbitrary order:*

$$(_*\mathbb{D}_{(k)}\, f)(t) \;=\; _*\mathbb{D}_{(k)}\, f = S_\kappa \cdot f - S_\kappa \cdot (P\, f), \tag{44}$$

where the projector operator $P\, f$ is defined by the Formula (29) *and the functions f, $_*\mathbb{D}_{(k)}\, f$, and $P\, f$ are interpreted as elements of the field $\mathcal{F}_{-1}$.*

Proof. On the space $C^n_{-1}(0, +\infty)$, the projector operator P is given by the Formula (29). Thus, we have the following representation:

$$(\mathbb{I}_{(\kappa)}\, _*\mathbb{D}_{(k)}\, f)(t) = (\kappa * (_*\mathbb{D}_{(k)}\, f))(t) = f(t) - (P\, f)(t), \; t > 0. \tag{45}$$

For any $f \in C^n_{-1}(0, +\infty)$, both the function at the right- and the function at the left-hand side of the Formula (45) evidently belong to the space $C_{-1}(0, +\infty)$. Because the ring $\mathcal{R}_{-1}$ is embedded into the convolution quotients field $\mathcal{F}_{-1}$, the Formula (45) can be interpreted as an equality of two elements from $\mathcal{F}_{-1}$. By multiplying this equality with the element S_κ and using the relation (42) we immediately arrive at the Formula (44). □

Remark 3. *In the case of the kernels $\kappa(t) = h_\alpha(t)$, $\alpha > 0$ and $k(t) = h_{n-\alpha}(t)$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, the GFI* (16) *is the well-known Riemann–Liouville fractional integral and the GFD* (17) *is the Caputo fractional derivative of order α. In* [10]*, a formula of type* (44) *for the Caputo fractional derivative has been derived for the first time.*

The constructions presented above can be extended to the case of the m-fold sequential GFD (32) of arbitrary order.

Definition 6. *Let $(\kappa, k) \in \mathcal{L}_n$, $n \in \mathbb{N}$ and ${}_*\mathbb{D}_{(k)}^{<i>} f \in C^{n-1}[0, +\infty)$, $i = 0, \ldots, m-1$.*

The m-fold sequential algebraic GFD of arbitrary order is defined by the expression

$$ {}_*\mathbb{D}_{(k)}^{<m>} f = S_\kappa^m \cdot f - S_\kappa^m \cdot (P_m f), \tag{46}$$

where the function f and the projector operator $P_m f$ given by the Formula (37) are interpreted as the elements of the convolution quotients field $\mathcal{F}_{-1}$.

According to the Formula (37), the m-fold sequential algebraic GFD of arbitrary order is well defined for any function f that satisfies the conditions ${}_*\mathbb{D}_{(k)}^{<i>} f \in C^{n-1}[0, +\infty)$, $i = 0, \ldots, m-1$. Evidently, these conditions do not ensure existence of the m-fold sequential GFD (32) of arbitrary order. Therefore, the m-fold sequential algebraic GFD of arbitrary order can be interpreted as a kind of a generalized derivative. Still, for the functions from the space $C_{-1,(k)}^{n,m}(0, +\infty)$ given by the Formula (35), Theorem 5 and the same arguments that were used in the case $m = 1$ (Theorem 7 and its proof) lead to the following important result:

Theorem 8. *Let $(\kappa, k) \in \mathcal{L}_n$, $n \in \mathbb{N}$ and $f \in C_{-1,(k)}^{n,m}(0, +\infty)$.*

Then the m-fold sequential algebraic GFD (46) coincides with the m-fold sequential GFD (32):

$$ ({}_*\mathbb{D}_{(k)}^{<m>} f)(t) \ = \ {}_*\mathbb{D}_{(k)}^{<m>} f = S_\kappa^m \cdot f - S_\kappa^m \cdot (P_m f), \tag{47}$$

where the projector operator P_m is defined by the Formula (37) and the functions f, $\mathbb{D}_{(k)}^{<m>} f$, and $P_m f$ are interpreted as the elements of the field $\mathcal{F}_{-1}$.

Remark 4. *The representation (37) of the projector operator P_m along with the formulas $(I_{(\kappa)}^{<i>} f)(t) = (\kappa^{<i>} * f)(t)$ and $S_\kappa \cdot \kappa = I$ lead to another form of the Formula (47):*

$$ ({}_*\mathbb{D}_{(k)}^{<m>} f)(t) \ = \ {}_*\mathbb{D}_{(k)}^{<m>} f = S_\kappa^m \cdot f - \sum_{i=0}^{m-1} S_\kappa^{m-i} \cdot (P\,({}_*\mathbb{D}_{(k)}^{<i>} f)), \tag{48}$$

where the projector operator P is given by Equation (29).

According to the representations (44) and (47) (or (48)), the GFD (17) of arbitrary order and the m-fold sequential GFD (32) of arbitrary order can be represented as algebraic operations (multiplications) on the field $\mathcal{F}_{-1}$ of convolution quotients. These representations can be used to reduce the initial-value problems for the fractional differential equations with the GFDs of arbitrary order and the m-fold sequential GFDs of arbitrary order to some algebraic equations on the field $\mathcal{F}_{-1}$ of convolution quotients. The solutions to these equations are some elements of $\mathcal{F}_{-1}$ that in general are generalized functions or hyperfunctions. However, usually one looks for solutions in form of conventional functions, say, from the space $C_{-1}(0, +\infty)$. That is why the so-called operational relations (representations of some elements of the field $\mathcal{F}_{-1}$ as conventional functions from the ring $\mathcal{R}_{-1}$) are another important component of any operational calculus of Mikusiński type. In the rest of this section, we provide operational relations for the elements of the field $\mathcal{F}_{-1}$ in form of the rational functions $R(S_\kappa) = Q(S_\kappa)/P(S_\kappa)$ with $\deg(Q) < \deg(P)$. The result formulated in the next theorem is a basis for derivation of these operational relations.

Theorem 9 ([14])**.** *Let a function $\kappa \in C_{-1}(0, +\infty)$ be represented in the form*

$$ \kappa(t) = h_p(t)\kappa_1(t), \ t > 0, \ p > 0, \ \kappa_1 \in C[0, +\infty) \tag{49}$$

and the convergence radius of the power series

$$\Sigma(z) = \sum_{j=0}^{+\infty} a_j\, z^j,\ a_j \in \mathbb{C},\ z \in \mathbb{C} \tag{50}$$

be non-zero. Then the convolution series

$$\Sigma_\kappa(t) = \sum_{j=0}^{+\infty} a_j\, \kappa^{<j+1>}(t) \tag{51}$$

is convergent for all $t > 0$ and defines a function from the space $C_{-1}(0, +\infty)$. Moreover, the series

$$t^{1-\alpha}\, \Sigma_\kappa(t) = \sum_{j=0}^{+\infty} a_j\, t^{1-\alpha}\, \kappa^{<j+1>}(t),\ \ \alpha = \min\{p,\, 1\} \tag{52}$$

is uniformly convergent for $t \in [0,\, T]$ for any $T > 0$.

It is worth mentioning that the convolution series (51) is a far reaching generalization of the power series (50) that is also a convolution series generated by the kernel $\kappa = \{1\}$ (Mikusiński's operational calculus for the first order derivative).

As an example, let us consider the geometric series

$$\Sigma(t) = \sum_{j=1}^{+\infty} \lambda^{j-1} t^j,\ \lambda \in \mathbb{C},\ \lambda \neq 0,\ t \in \mathbb{C}. \tag{53}$$

Its convergence radius is $r = 1/|\lambda| > 0$. Theorem 9 ensures that the convolution series

$$l_{\kappa,\lambda}(t) = \sum_{j=1}^{+\infty} \lambda^{j-1} \kappa^{<j>}(t),\ \lambda \in \mathbb{C},\ t > 0 \tag{54}$$

is a function that belongs to the space $C_{-1}(0, +\infty)$.

In the framework of the Mikusiński operational calculus for the first order derivative, the kernel function $\kappa = \{1\}$ was employed. It is easy to verify that for this kernel the relation $\kappa^{<j>}(t) = \{1\}^{<j>}(t) = h_j(t)$ holds valid. Thus, the convolution series (54) is an exponential function:

$$l_{\kappa,\lambda}(t) = \sum_{j=1}^{+\infty} \lambda^{j-1} h_j(t) = \sum_{j=0}^{+\infty} \frac{(\lambda\, t)^j}{j!} = e^{\lambda\, t}. \tag{55}$$

Another important example is the operational calculus of Mikusiński type for the Caputo fractional derivative ([10]). In this case, the kernel κ is the power function h_α and the formula $\kappa^{<j>}(t) = h_\alpha^{<j>}(t) = h_{j\alpha}(t)$ holds valid. Thus, the convolution series (54) takes the form

$$l_{\kappa,\lambda}(t) = \sum_{j=1}^{+\infty} \lambda^{j-1} h_{j\alpha}(t) = t^{\alpha-1} \sum_{j=0}^{+\infty} \frac{\lambda^j\, t^{j\alpha}}{\Gamma(j\alpha + \alpha)} = t^{\alpha-1} E_{\alpha,\alpha}(\lambda\, t^\alpha), \tag{56}$$

where the two-parameters Mittag–Leffler function $E_{\alpha,\beta}$ is defined by the following absolutely convergent series:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha\, j + \beta)},\ \Re(\alpha) > 0,\ z, \beta \in \mathbb{C}. \tag{57}$$

Some other particular cases of the convolution series (54) were presented in [13].

A very important operational relation in terms of the convolution series $l_{\kappa,\lambda}$ is presented in the next theorem.

Theorem 10. *For any $\lambda \in \mathbb{C}$, the operational relation*

$$\frac{I}{S_\kappa - \lambda} = l_{\kappa,\lambda}(t),\ t > 0 \tag{58}$$

holds true, where by $\frac{I}{e}$ we denote the element of the field $\mathcal{F}_{-1}$ inverse to the element $e \in \mathcal{F}_{-1}$.

Proof. In the case $\lambda = 0$, the convolution series (54) is just the kernel function $\kappa(t)$ and the operational relation (58) in the form $\frac{I}{S_\kappa} = \kappa$ holds true by definition of S_κ (see the Formula (42)).

For $\lambda \neq 0$, the GFI $\mathbb{I}_{(\kappa)}$ can be applied to the convolution series $l_{\kappa,\lambda}$ term by term due to Theorem 9 and we get the following chain of equations:

$$(I - \lambda\kappa) \cdot l_{\kappa,\lambda} = l_{\kappa,\lambda} - \lambda(\mathbb{I}_{(\kappa)} \sum_{j=1}^{+\infty} \lambda^{j-1}\kappa^{j})(t) = l_{\kappa,\lambda} - \lambda \sum_{j=1}^{+\infty} \lambda^{j-1}(\mathbb{I}_{(\kappa)}\, \kappa^{j})(t) =$$

$$l_{\kappa,\lambda} - \sum_{j=1}^{+\infty} \lambda^{j}\kappa^{j+1}(t) = \sum_{j=1}^{+\infty} \lambda^{j-1}\kappa^{j}(t) - \sum_{j=2}^{+\infty} \lambda^{j-1}\kappa^{j}(t) = \kappa.$$

Otherwise, the element $I - \lambda\kappa$ of the field $\mathcal{F}_{-1}$ can be represented as follows:

$$I - \lambda\kappa = \kappa \cdot S_\kappa - \lambda\kappa = \kappa \cdot (S_\kappa - \lambda).$$

The last two representations lead to the relation

$$\kappa \cdot (S_\kappa - \lambda) \cdot l_{\kappa,\lambda} = (I - \lambda\kappa) \cdot l_{\kappa,\lambda} = \kappa.$$

Thus, the element $(S_\kappa - \lambda) \cdot l_{\kappa,\lambda}$ is the unity I of the field $\mathcal{F}_{-1}$ and the proof of the operational relation (58) is completed. □

In the case of the kernels $(\kappa, k) \in \mathcal{L}_1$, Theorem (10) has been formulated and proved in [13].

As an example, we consider the kernel $\kappa = \{1\}$ (Mikusiński's operational calculus for the first order derivative). According to the Formula (55), the operational relation (58) can be represented in the well-known form:

$$\frac{I}{S_\kappa - \lambda} = l_{\kappa,\lambda}(t) = e^{\lambda t}. \tag{59}$$

In the case of the kernel $\kappa(t) = h_\alpha(t),\ t > 0$ (operational calculus for the Caputo fractional derivative [10]), the Formula (56) leads to the following operational relation:

$$\frac{I}{S_\kappa - \lambda} = l_{\kappa,\lambda}(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda\, t^\alpha), \tag{60}$$

where the two-parameters Mittag–Leffler function $E_{\alpha,\beta}$ is defined by (57). This operational relation has been deduced in [6] for the fist time.

Other particular cases of the operational relation (58) have been presented in [13].

As already mentioned, the operational relation (58) can be used to deduce other useful operational relations. The embedding of the ring $\mathcal{R}_{-1}$ into the field $\mathcal{F}_{-1}$ means in particular that a convolution of any ring elements complies with multiplication of the corresponding elements of the field of convolution quotients. Thus, we get the following operational relation:

$$\frac{I}{(S_\kappa - \lambda)^m} = l_{\kappa,\lambda}^{<m>}(t),\ t > 0,\ m \in \mathbb{N}. \tag{61}$$

For a representation of the convolution powers $l_{\kappa,\lambda}^{<m>}$ in terms of the convolution series see [13].

As an example, we consider the kernel $\kappa = \{1\}$ (Mikusiński's operational calculus for the first order derivative). The operational relation (61) takes the well-known form [1]:

$$\frac{I}{(S_\kappa - \lambda)^m} = h_m(t)\, e^{\lambda\, t}. \tag{62}$$

In the case of the kernel $\kappa(t) = h_\alpha(t),\ t > 0$ (operational calculus for the Caputo fractional derivative), we get the following operational relation [6,10]:

$$\frac{I}{(S_\kappa - \lambda)^m} = t^{m\alpha-1} E_{\alpha,m\alpha}^m(\lambda t^\alpha),\ t > 0,\ m \in \mathbb{N}, \tag{63}$$

where the Mittag–Leffler type function $E_{\alpha,\beta}^m$ is defined via the following convergent series:

$$E_{\alpha,\beta}^m(z) := \sum_{j=0}^{\infty} \frac{(m)_j z^j}{j!\Gamma(\alpha j + \beta)},\ \alpha, \beta > 0,\ z \in \mathbb{C},\ (m)_j = \prod_{i=0}^{j-1}(m+i).$$

Combining the operational relations (58) and (61), we deduce another important operational relation.

Let $R(S_\kappa) = Q(S_\kappa)/P(S_\kappa)$, where Q and P are polynomials and $\deg(Q) < \deg(P)$. In this case, the rational function $R(S_\kappa)$ can be represented as a sum of the partial fractions:

$$R(S_\kappa) = \sum_{j=1}^{J} \sum_{i=1}^{m_j} \frac{a_{ij}}{(S_\kappa - \lambda_j)^i},\ \sum_{j=1}^{J} m_j = \deg(P). \tag{64}$$

Then the operational relation

$$R(S_\kappa) = \sum_{j=1}^{J} \sum_{i=1}^{m_j} a_{ij}\, l_{\kappa,\lambda_j}^{<i>}(t),\ t > 0,\ \sum_{j=1}^{J} m_j = \deg(P), \tag{65}$$

holds true, where the constants λ_j and m_j, $j = 1, \ldots, J$ are uniquely determined by representation of the rational function $R(S_\kappa)$ as a sum of the partial fractions in form (64).

The operational relation (65) is a direct consequence from the Formula (64) and the operational relation (61).

4. Discussion

In this paper, we first discussed some important properties of the general fractional integrals (GFI) and the GFD of arbitrary order introduced recently in the works of the third named co-author. The new objects defined for the first time in this paper are the m-fold GFIs and the sequential GFDs of arbitrary order. For the m-fold GFIs and the sequential GFDs of arbitrary order we proved the 1st and the 2nd fundamental theorems of FC and derived an explicit form for their projector operator. The main contribution of this paper is an operational calculus of Mikusiński type for the GFDs of arbitrary order. In the field of the convolution quotients, the GFDs of arbitrary order and the sequential GFDs of arbitrary order are represented as multiplication with certain elements of the field. We also derived several important operational relations that provide useful representations of some field elements as conventional functions expressed in terms of the so-called convolution series.

In conclusion, we mention that the operational calculus for the GFDs of arbitrary order that we constructed in this paper can be applied for derivation of the closed form formulas for the solutions to the ordinary and partial fractional differential equations containing the m-fold sequential GFDs. These matters will be discussed elsewhere.

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