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Sharp Bounds for a Generalized Logarithmic Operator Mean and Heinz Operator Mean by Weighted Ones of Classical Operator Ones

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Abstract: In this paper, using a criteria for the monotonicity of the quotient of two power series, we present some sharp bounds for a generalized logarithmic operator mean and Heinz operator mean by weighted ones of classical operator ones.

Keywords: bounds; criteria for the monotonicity of quotient of two power series; a generalized logarithmic operator mean; Heinz operator mean; Heron operator mean; classical operator means

MSC: 26D15; 26D20; 47A63; 42A10; 47A64; 47A56

1. Introduction

Since scholars ([1–3]) found that the original weighted arithmetic-geometric mean inequality of two positive numbers can be extended to positive invertible operators, this research field has been developing in the direction of prosperity. In addition to establishing the relationships between various classical means, researchers have also introduced new means and generalized classical means, created new inequalities, and finally tried to extend these conclusions to positive invertible operators. In recent years, progress and achievements in this field have been very rich, as can be seen in the related body of literature [1–41].

The Heinz mean (see Bhatia and Davis [1]), denoted as \(H_\nu(a, b)\), is introduced and defined by

\[
H_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}
\]

for \(0 \leq \nu \leq 1\) and \(a, b \geq 0\). Since \(H_0(a, b) = H_1(a, b) = A(a, b)\), \(H_{1/2}(a, b) = G(a, b)\), we can assume \(0 < \nu < 1\) and \(\nu \neq 1/2\), and note that \(I^1 = (0, 1)\setminus\{1/2\} = \mathbb{U}(1/2, 1/2)\). We know that

\[
G(a, b) = \sqrt{ab} < H_\nu(a, b) < \frac{a + b}{2} = A(a, b).
\]

Bhatia [3] defined the weighted mean of the arithmetic mean and geometric mean as the Heron means by

\[
F_\alpha(a, b) = (1 - \alpha)G(a, b) + \alpha A(a, b) = (1 - \alpha)\sqrt{ab} + \alpha \frac{a + b}{2},
\]

and obtained the relationship between the Heinz mean and the Heron mean.

**Proposition 1.** Let \(a, b \geq 0, 0 \leq \nu \leq 1, \text{and} \ a(\nu) = 1 - 4(\nu - \nu^2)\). Then,

\[
H_\nu(a, b) \leq F_{a(\nu)}(a, b).
\]
Kittaneh, Moslehian and Sababheh [20] (Theorem 2.1) also obtained the above result in a very flexible way. Recently, in [32], the author of this paper sharpened the above inequality and obtained the following results.

**Proposition 2.** Let \( a, b > 0, a \neq b, v \in \mathbb{P}, \) and \( \theta = 1 - 2v. \) Then,

\[
H_v(a, b) < F_{\theta^2}(a, b)
\]

holds, where \( \theta^2 \) cannot be replaced by any smaller number.

Ref. [32] also gave a lower bound and a better upper bound for \( H_v(a, b), \) as follows.

**Proposition 3.** Let \( a, b > 0, a \neq b, v \in \mathbb{P}, \) and

\[
\frac{\sqrt{3} - 1}{2} = \kappa_1, \quad \frac{\sqrt{3} + 1}{2} = \kappa_2.
\]

Then

\[
\frac{a^{1-v}b^v - a^v b^{1-v}}{(1 - 2v)(\ln a - \ln b)} < H_v(a, b),
\]

and

\[
H_v(a, b) < \frac{a^{\kappa_2} - \sqrt{3}b^{\kappa_2} - \kappa_1}{\sqrt{3}(1 - 2v)(\ln a - \ln b)}
\]

hold.

In fact, the left-hand side of the inequality (7) is a mean of two positive numbers, \( a \) and \( b, \) and it is a generalization of the logarithmic mean, which was first introduced by Shi in [28]. For the sake of narration, we give it the sign \( L_v; \) that is,

\[
L_v(a, b) = \frac{a^{1-v}b^v - a^v b^{1-v}}{(1 - 2v)(\ln a - \ln b)}.
\]

Recently, Alsaafin and Burqan [41] produced richer results for the relationship between the Heinz mean \( H_v(a, b) \) and the above generalization of the logarithmic mean \( L_v(a, b). \)

Considering the particular mean \( L_v(a, b) \) has the following properties

\[
\begin{align*}
L_0(a, b) & = L_1(a, b) = \frac{a - b}{\ln a - \ln b} = L(a, b), \\
L_{1/2}(a, b) & = \lim_{v \to 1/2} L_v(a, b) = \sqrt{ab} = G(a, b), \\
L_v(a, b) & = L_{1-v}(a, b),
\end{align*}
\]

we come to the first conclusion of this paper, which is about its monotonicity.

**Theorem 1.** Let \( a, b > 0, a \neq b, v \in \mathbb{P}, \) and \( L_v(a, b) \) is defined by (9). Then, \( L_v(a, b) \) decreases as \( v \) increases on \((0, 1/2)\) and increases as \( v \) increases on \((1/2, 1)\).

The second topic of this paper is to introduce the weighted mean of the geometric mean \( G(a, b) \) and the logarithm mean \( L(a, b) \) and the weighted mean of the geometric mean \( A(a, b) \) and the logarithm mean \( L(a, b) \) by

\[
\begin{align*}
E_\beta(a, b) & = (1 - \beta)G(a, b) + \beta L(a, b), \\
J_\sigma(a, b) & = (1 - \sigma)A(a, b) + \sigma L(a, b),
\end{align*}
\]

respectively and to obtain the following results about the relationships between \( L_v(a, b) \) and \( E_\beta(a, b), L_v(a, b) \) and \( F_\alpha(a, b), L_v(a, b) \) and \( J_\sigma(a, b). \)
Theorem 2. Let \( a, b > 0, a \neq b, v \in I^0, \theta = 1 - 2v, \beta_1 = 0, \) and \( \beta_2 = \theta^2. \) Then, the double inequality
\[
G(a, b) = E_{\beta_1}(a, b) < L_v(a, b) < E_{\beta_2}(a, b)
\]
holds, where \( \beta_1 \) cannot be replaced by any larger number and \( \beta_2 \) cannot be replaced by any smaller number.

Theorem 3. Let \( a, b > 0, a \neq b, v \in I^0, \theta = 1 - 2v, \lambda_1 = 1 - \theta^2/3, \) and \( \lambda_2 = 1. \) Then, the double inequality
\[
G(a, b) = F_{1-\lambda_2}(a, b) < L_v(a, b) < F_{1-\lambda_1}(a, b)
\]
holds, where \( \lambda_1 \) cannot be replaced by any larger number and \( \lambda_2 \) cannot be replaced by any smaller number.

Theorem 4. Let \( a, b > 0, a \neq b, v \in I^0, \theta = 1 - 2v, \sigma_1 = (3 - \theta^2)/2, \) and \( \sigma_2 = 1. \) Then, the double inequality
\[
J_{\sigma_1}(a, b) < L_v(a, b) < J_{\sigma_2}(a, b) = L(a, b)
\]
holds, where \( \sigma_2 \) cannot be replaced by any larger number and \( \sigma_1 \) cannot be replaced by any smaller number.

Letting \( \theta = 1/2 \) in the above Theorems gives

Corollary 1. Let \( a, b > 0, a \neq b. \) Then, the following three double inequalities hold:
\[
\begin{align*}
G(a, b) &< L_{3/4}(a, b) < \frac{3}{4} G(a, b) + \frac{1}{4} A(a, b), \\
G(a, b) &< L_{3/4}(a, b) < \frac{11}{12} G(a, b) + \frac{1}{12} L(a, b), \\
-\frac{8}{3} A(a, b) + \frac{11}{3} L(a, b) &< L_{3/4}(a, b) < L(a, b).
\end{align*}
\]

The double inequality (16) is equivalent to
\[
L_{3/4}(a, b) < L(a, b) < \frac{3}{11} L_{3/4}(a, b) + \frac{8}{11} A(a, b).
\]

We know that Proposition 2 reveals the relationship between \( H_v(a, b) \) and \( F_\alpha(a, b). \)

The third goal of this paper is to determine the relationships between \( H_v(a, b) \) and \( E_\beta(a, b), \)
\( H_v(a, b) \) and \( J_\sigma(a, b). \)

Theorem 5. Let \( a, b > 0, a \neq b, v \in I^0, \theta = 1 - 2v, \) and \( \theta^2 \leq 3/5 \) or
\[
v_1 = \frac{1 - \sqrt{3/5}}{2} = 0.1127 \ldots \leq v \leq \frac{\sqrt{3/5} + 1}{2} = 0.88730 \ldots = 1 - v_1.
\]

Then,
\[
G(a, b) = E_0(a, b) < H_v(a, b) < E_{3\theta^2}(a, b)
\]
holds with the best constants 0 and \( 3\theta^2. \)

Theorem 6. Let \( a, b > 0, a \neq b, v \in I^0, \theta = 1 - 2v, \) and \( \theta^2 \leq 1/5 \) or \( \sqrt{1/5} = 0.4472 \)
\[
v_2 = \frac{1 - \sqrt{1/5}}{2} = 0.27639 \ldots \leq v \leq \frac{\sqrt{1/5} + 1}{2} = 0.72361 \ldots = 1 - v_2.
\]

Then,
\[
J_{3(1-\theta^2)/2}(a, b) < H_v(a, b) < J_1(a, b) = L(a, b)
\]
holds with the best constants \( 3(1 - \theta^2)/2 \) and 1.
Letting $\theta = 3/4$ and $\theta = 2/5$ in the above two Theorems, respectively, gives

**Corollary 2.** Let $a, b > 0$, $a \neq b$. Then, the following two double inequalities hold:

\[
G(a, b) < H_{1/8}(a, b) < \frac{27}{16} L(a, b) - \frac{11}{16} G(a, b),
\]

\[
-\frac{13}{50} A(a, b) + \frac{63}{50} L(a, b) < H_{3/10}(a, b) < L(a, b).
\]

The right-hand side of (20) and the left-hand side inequality of (21) are equivalent to

\[
\frac{11}{27} G(a, b) + \frac{16}{27} H_{1/8}(a, b) < L(a, b),
\]

\[
L(a, b) < \frac{50}{63} H_{3/10}(a, b) + \frac{13}{63} A(a, b).
\]

The last section of this paper gives the operator conclusions of the above results.

2. Lemmas

**Lemma 1** ([42]). Let $a_n$ and $b_n$ ($n = 0, 1, 2, \cdots$) be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$ ($R \leq +\infty$). If $b_n > 0$ for $n = 0, 1, 2, \cdots$, and if $\kappa_n = a_n/b_n$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \cdots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $0, R$ ($R \leq +\infty$).

**Lemma 2.** Let $t \neq 0$, and $0 < |\theta| < 1$. Then, the double inequality

\[
0 < \frac{\sinh \theta t}{\sinh t} - 1 < \theta^2
\]

holds with the best constants $0$ and $\theta^2$.

**Proof.** Let

\[
A_1(t) = \frac{\sinh \theta t}{\theta t} - 1 = \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n+1)!} t^{2n} = \sum_{n=1}^{\infty} a_n t^{2n},
\]

\[
B_1(t) = \frac{\sinh t}{t} - 1 = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} t^{2n} = \sum_{n=1}^{\infty} b_n t^{2n},
\]

where

\[
a_n = \frac{\theta^{2n}}{(2n+1)!}, \quad b_n = \frac{1}{(2n+1)!} > 0.
\]

Then,

\[
\kappa_n := a_n/b_n = \theta^{2n}.
\]

Since $\{\kappa_n\}_{n \geq 1}$ is decreasing for $0 < |\theta| < 1$, from Lemma 1, we know that the function $A_1(t)/B_1(t)$ is decreasing on $(0, \infty)$ for $0 < |\theta| < 1$. In view of

\[
\lim_{t \to 0^+} A_1(t) = B_1(t) = \kappa_1 = \theta^2,
\]

\[
\lim_{t \to \infty} A_1(t) = B_1(t) = \lim_{n \to \infty} \kappa_n = \lim_{n \to \infty} \theta^{2n} = 0,
\]

the proof of Lemma 2 is complete. ☐
Lemma 3. Let \( t \neq 0, \) and \( 0 < |\theta| < 1. \) Then, the double inequality

\[
1 - \frac{1}{3} \theta^2 < \frac{\cosh t - \frac{\sinh \theta t}{\theta t}}{\cosh t - 1} < 1 \tag{25}
\]

holds with the best constants \( 1 - (\theta^2/3) \) and \( 1. \)

**Proof.** Let

\[
A_2(t) = \cosh t - \frac{\sinh \theta t}{\theta t} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n} - \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n + 1)!} t^{2n} = \sum_{n=1}^{\infty} c_n t^{2n},
\]

\[
B_2(t) = \cosh t - 1 = \sum_{n=1}^{\infty} \frac{1}{(2n)!} t^{2n} = \sum_{n=1}^{\infty} d_n t^{2n},
\]

where

\[
c_n = \frac{(2n + 1) - \theta^{2n}}{(2n + 1)!}, \quad d_n = \frac{1}{(2n)!} > 0.
\]

Then,

\[
\epsilon_n := \frac{c_n}{d_n} = \frac{\frac{(2n + 1) - \theta^{2n}}{(2n + 1)!}}{\frac{1}{(2n)!}} = \frac{(2n + 1) - \theta^{2n}}{(2n + 1)} = 1 - \frac{\theta^{2n}}{2n + 1},
\]

which is increasing for \( 0 < |\theta| < 1. \) From Lemma 1 we know that the function \( A_2(t)/B_2(t) \) is increasing on \((0, \infty)\) for \( 0 < |\theta| < 1. \) In view of

\[
\lim_{t \to 0} \frac{A_2(t)}{B_2(t)} = \epsilon_1 = 1 - \frac{1}{3} \theta^2,
\]

\[
\lim_{t \to \infty} \frac{A_2(t)}{B_2(t)} = \lim_{n \to \infty} \epsilon_n = 1,
\]

the proof of Lemma 3 is complete. \( \square \)

Lemma 4. Let \( t \neq 0, \) and \( 0 < |\theta| < 1. \) Then, the double inequality

\[
1 < \frac{\cosh t - \frac{\sinh \theta t}{\theta t}}{\cosh t - \frac{\sinh t}{t}} < \frac{3 - \theta^2}{2} \tag{26}
\]

holds with the best constants \( 1 \) and \((3 - \theta^2)/2. \)

**Proof.** Let

\[
A_3(t) = \cosh t - \frac{\sinh \theta t}{\theta t} = \sum_{n=1}^{\infty} \left[ \frac{1}{(2n)!} - \frac{\theta^{2n}}{(2n + 1)!} \right] t^{2n} = \sum_{n=1}^{\infty} f_n t^{2n},
\]

\[
B_3(t) = \cosh t - \frac{\sinh t}{t} = \sum_{n=1}^{\infty} \frac{2n}{(2n + 1)!} t^{2n} = \sum_{n=1}^{\infty} g_n t^{2n},
\]

where

\[
f_n = \frac{1}{(2n)!} - \frac{\theta^{2n}}{(2n + 1)!}, \quad g_n = \frac{2n}{(2n + 1)!} > 0.
\]

Then,

\[
\epsilon_n := \frac{f_n}{g_n} = \frac{\frac{1}{(2n)!} - \frac{\theta^{2n}}{(2n + 1)!}}{\frac{2n}{(2n + 1)!}} = \frac{2n + 1 - \theta^{2n}}{2n}, \quad n \geq 1.
\]
We can prove that \( \epsilon_n > \epsilon_{n+1} \) is based on the following facts
\[
\epsilon_n > \epsilon_{n+1} \iff \frac{2n + 1 - \theta^{2n}}{2n} > \frac{2n + 3 - \theta^{2n+2}}{2n + 2} \\
\iff (n + 1)(2n + 1 - \theta^{2n}) > n(2n + 3 - \theta^{2n+2}) \\
\iff n\theta^{2(n+1)} - (n + 1)\theta^{2n} + 1 > 0.
\]

Let \( \theta^2 = x \). Then, \( 0 < x < 1 \), and the last inequality above is equivalent to
\[n x^{n+1} - (n + 1)x^n + 1 > 0,
\]
which is true for
\[n x^{n+1} - (n + 1)x^n + 1 = \left(1 + 2x + 3x^2 + \cdots + nx^{n-1}\right)(1 - x)^2.
\]

Since \( \{\epsilon_n\}_{n \geq 1} \) is decreasing for \( 0 < |\theta| < 1 \), from Lemma 1, we know that the function \( A_3(t)/B_3(t) \) is decreasing on \( (0, \infty) \) for \( 0 < |\theta| < 1 \). In view of
\[
\lim_{t \to 0^+} \frac{A_3(t)}{B_3(t)} = \epsilon_1 = \frac{3 - \theta^2}{2},
\]
\[
\lim_{t \to \infty} \frac{A_3(t)}{B_3(t)} = \lim_{n \to \infty} \epsilon_n = 1,
\]
the proof of Lemma 4 is complete. \( \square \)

**Lemma 5.** Let \( t \neq 0 \), and \( 0 < \theta^2 \leq 3/5 \). Then, the double inequality
\[
0 < \frac{\cosh \theta t - 1}{\sinh \frac{t}{\theta^2}} < 3\theta^2 \tag{27}
\]
holds with the best constants 0 and \( 3\theta^2 \).

**Proof.** Let
\[
A_4(t) = \cosh \theta t - 1 = \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n)!} t^{2n} =: \sum_{n=0}^{\infty} s_n t^{2n},
\]
\[
B_4(t) = \frac{t}{\sinh t / \theta^2} - 1 = \sum_{n=1}^{\infty} \frac{1}{(2n + 1)!} x^{2n} =: \sum_{n=0}^{\infty} t_n x^{2n}.
\]

Then,
\[
s_n = \frac{\theta^{2n}}{(2n)!}, \quad t_n = \frac{1}{(2n + 1)!} > 0,
\]
and
\[
\zeta_n = \frac{s_n}{t_n} = (2n + 1)\theta^{2n}.
\]

Since
\[
\zeta_{n+1} - \zeta_n = (2n + 3)\theta^{2n+2} - (2n + 1)\theta^{2n} = (2n + 3)\theta^{2n}\left(\theta^2 - \frac{2n + 1}{2n + 3}\right),
\]
and the sequence \( \{(2n + 1)/(2n + 3)\}_{n \geq 1} \) is increasing, we have
\[
\frac{3}{5} = \min_{n \geq 1} \frac{2n + 1}{2n + 3} \leq \frac{2n + 1}{2n + 3} < \max_{n \geq 1} \frac{2n + 1}{2n + 3} = 1.
\]
When
\[ \theta^2 \leq \min_{n \geq 1} \frac{2n + 1}{2n + 3} = \frac{3}{5}, \]
the sequence \( \{\xi_n\}_{n \geq 1} \) is decreasing. From Lemma 1, we know that the function \( A_4(t)/B_4(t) \) is decreasing on \((0, \infty)\). Notice that
\[
\lim_{t \to 0} \frac{\cosh \theta t - 1}{\sinh \theta t} = \xi_1 = 3\theta^2,
\]
\[
\lim_{t \to \infty} \frac{\cosh \theta t - 1}{\sinh \theta t} = \lim_{n \to \infty} \xi_n = \lim_{n \to \infty} (2n + 1)\theta^{2n} = 0,
\]
so the proof of Lemma 5 is complete. \( \square \)

**Lemma 6.** Let \( t \neq 0 \), and \( 0 < \theta^2 \leq 1/5 \). Then, the double inequality
\[
1 < \frac{\cosh t - \cosh \theta t}{\cosh t - \sinh \theta t} < \frac{3}{2} \left(1 - \theta^2\right)
\]
holds with the best constants 1 and \(3(1 - \theta^2)/2\).

**Proof.** Let
\[
A_5(t) = \cosh t - \cosh \theta t = \sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n} - \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} t^{2n} = \sum_{n=1}^{\infty} u_n (2n)! t^{2n},
\]
\[
B_5(t) = \cosh t - \sinh \theta t = \sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+1} = \sum_{n=1}^{\infty} v_n (2n)! t^{2n+1}.
\]
Then,
\[
u_n = \frac{1 - \theta^{2n}}{(2n)!}, \quad v_n = \frac{2n}{(2n+1)!} > 0,
\]
and
\[
\eta_n = \frac{\nu_n}{\nu_{n+1}} = \frac{1 - \theta^{2n}}{2n(2n+3)!} = \frac{2n + 1}{2n} \left(1 - \theta^{2n}\right).
\]
Since
\[
\eta_{n+1} - \eta_n = \frac{2n + 3}{2n + 2} \left(1 - \theta^{2n+2}\right) - \frac{2n + 1}{2n} \left(1 - \theta^{2n}\right)
\]
\[
= \frac{1}{2n(n+1)} \left[ (n + 1)(2n + 1)\theta^{2n} - n(2n + 3)\theta^{2n+2} - 1 \right]
\]
\[
: = \frac{1}{2n(n+1)} f(x),
\]
where \( \theta^2 = x \) and
\[
f(x) = (n + 1)(2n + 1)x^n - n(2n + 3)x^{n+1} - 1, \quad 0 < x < 1, \quad n = 1, 2, \ldots.
\]
Next, we prove that the function \( f(x) \) is not positive on \((0, 1/5]\). Since \( f(0) = -1 \) and
\[
\frac{d}{dx} f(x) = n(n + 1)(2n + 3)x^{n-1}\left(\frac{2n+1}{2n+3} - x\right) > 0, \quad 0 < x \leq \frac{1}{5},
\]
the partial task is completed as long as we can prove
\[
f\left(\frac{1}{5}\right) = (n + 1)(2n + 1)\left(\frac{1}{5}\right)^n - n(2n + 3)\left(\frac{1}{5}\right)^{n+1} - 1 \leq w_n \leq 0. \tag{29}
\]
which can be proven for two reasons. First, \( w_1 = 0 \). Second, \( w_n \) is decreasing due to
\[
\begin{align*}
w_n &> w_{n+1} \iff \\
(n + 1)(2n + 1)\left(\frac{1}{5}\right)^n - n(2n + 3)\left(\frac{1}{5}\right)^{n+1} - 1 &> (n + 2)(2n + 3)\left(\frac{1}{5}\right)^{n+1} - (n + 1)(2n + 5)\left(\frac{1}{5}\right)^{n+2} - 1 \iff \\
(n + 1)(2n + 1)\left(\frac{1}{5}\right)^n - n(2n + 3)\left(\frac{1}{5}\right)^{n+1} &> (n + 2)(2n + 3)\left(\frac{1}{5}\right)^{n+1} - (n + 1)(2n + 5)\left(\frac{1}{5}\right)^{n+2} \iff \\
(n + 1)(2n + 1) - \frac{1}{5}n(2n + 3) &> \frac{1}{5}(n + 2)(2n + 3) - \frac{1}{25}(n + 1)(2n + 5) \iff \\
25(n + 1)(2n + 1) - 5n(2n + 3) &> 5(n + 2)(2n + 3) - (n + 1)(2n + 5).
\end{align*}
\]

In fact,
\[
25(n + 1)(2n + 1) - 5n(2n + 3) - [5(n + 2)(2n + 3) - (n + 1)(2n + 5)] = 32n(n + 1) > 0.
\]

In a word, when \( 0 < x = \theta^2 \leq 1/5 \), the sequence \( \{\eta_n\} \), \( n \geq 1 \) is decreasing. From Lemma 1, we know that the function \( A_3(t)/B_3(t) \) is decreasing on \((0, \infty)\). Note that
\[
\begin{align*}
\lim_{t \to 0} \frac{\cosh t - \cosh \theta t}{\cosh t - \sinh t} &\approx \frac{\eta_1}{2} = \frac{3}{2} (1 - \theta^2), \\
\lim_{t \to \infty} \frac{\cosh t - \cosh \theta t}{\cosh t - \sinh t} &\approx \lim_{n \to \infty} \eta_n = 1,
\end{align*}
\]
so the proof of Lemma 6 is complete. \( \square \)

**Remark 1.** The difficulty of this paper lies in the proofs of Lemmas 2–6. The difficulty of the proofs can be divided into three levels. The proofs of Lemmas 2 and 3 are relatively concise, while the proof of Lemma 4 is concise and ingenious. The proofs of Lemmas 5 and 6 are difficult. We know that the limiting condition comes from the problem itself. From the proof processes of Lemmas 5 and 6, it can be seen that these proof methods are of great benefit to future research in this area. At the same time, the author encourages readers and scholars to do some work to relax the conditions of Lemmas 5 and 6.
3. Proofs of the Main Results

Proof of Theorem 1. Since \( L_v(a, b) \) has the properties (10), we just need to prove that \( L_v(a, b) \) decreases as \( v \) increases on \((0, 1/2)\). Through calculations, we can get

\[
\frac{d}{dv}L_v(a, b) = \frac{d}{dv} \frac{a^{1-v}b^v - a^vb^{1-v}}{(1-2v)(\ln a - \ln b)} = \frac{2}{a^b(1-2v)} \left[ \frac{ab^{2v} - a^{2v}b}{(\ln a - \ln b)(1-2v)} - \frac{ab^{2v} + a^{2v}b}{2} \right].
\]

The proof of Theorem 1 is complete when proving

\[
\frac{ab^{2v} - a^{2v}b}{(\ln a - \ln b)(1-2v)} < \frac{ab^{2v} + a^{2v}b}{2}
\]

holds for \(0 < v < 1/2\). If we let \( b/a = e\), then \( s > 0 \) and (30) is equivalent to

\[
\frac{a^s - (\frac{a}{e})^{2v}}{(\ln a - \ln b)(1-2v)} < \frac{s}{2}, \quad \frac{e^{2v} - e^{2vs}}{s(1-2v)} < \frac{2}{2}, \quad \frac{e^{2vs}(e^{(1-2v)s} - 1)}{2}, \quad \frac{e^{(1-2v)s} - 1}{s(1-2v)} < \frac{2}{2}, \quad \frac{e^{(1-2v)s} - 1}{e^{(1-2v)s} + 1} < \frac{s(1-2v)}{2}.
\]

Let \( s(1-2v)/2 = y \). Then, \( y > 0 \), and the last inequality above is equivalent to \( \tanh y < y \), which is true for all \( y > 0 \).

Proof of Theorem 2–6. Let \( \sqrt{b/a} = u = e^t \). Then, \( t > 0 \) and

\[
H_v(a, b) = \frac{\left(\frac{b}{a}\right)^{1-v} + \left(\frac{b}{a}\right)^v}{\frac{2}{\sqrt{ab}}} = \frac{(b/a)^{1-v} + (b/a)^v}{\frac{2}{\sqrt{b/a}}} = \frac{u^{2-2v} + u^{2v}}{2u} = \frac{u^{1-2v} + u^{2v}}{2} = \cosh(1-2v)t = \cosh \theta t,
\]

\[
L_v(a, b) = \frac{\sqrt{ab}}{\frac{2}{(1-2v)(\ln a - \ln b)}} = \frac{\left(\frac{b}{a}\right)^{1-v} - \left(\frac{b}{a}\right)^v}{\frac{2}{(1-2v)(\ln a - \ln b)}} = \frac{e^{2(1-2v)t} - e^{2v}}{2t(1-2v)} = \frac{e^{(1-2v)t} - e^{(2v-1)t}}{2t} = \frac{t}{(1-2v)t} = \sinh(1-2v)t = \sinh \theta t,
\]

which gives

\[
\frac{A(a, b)}{G(a, b)} = \frac{H_0(a, b)}{G(a, b)} = \frac{H_1(a, b)}{G(a, b)} = \cosh t,
\]

\[
\frac{L(a, b)}{G(a, b)} = \frac{L_0(a, b)}{G(a, b)} = \frac{L_1(a, b)}{G(a, b)} = \sinh t.
\]

Through the above relationships, we know that the double inequalities (24)–(28) are equivalent to (11)–(19). This completes the proofs of Theorems 2–6.
4. Inequalities Related to a Generalized Logarithmic Operator Mean and the Heinz Operator Mean

4.1. Inequalities Related to a Generalized Logarithmic Operator Mean

Let $B^+$ denote the set of all positive invertible operators on a Hilbert space $\mathcal{H}$. For $A, B \in B^+$ and $\nu \in [0, 1]$, the weighted arithmetic operator mean $A \nabla_\nu B$, geometric mean $A^{\#\nu} B$, and the Heinz operator mean $H_\nu(A, B)$ are defined as

\[ A \nabla_\nu B = (1 - \nu)A + \nu B, \]
\[ A^{\#\nu} B = A^{1/2} \left( A^{-1/2}BA^{-1/2} \right)^\nu A^{1/2}, \]
\[ H_\nu(A, B) = (A^{\#\nu} B + A^{\#1-\nu} B)/2. \]

Let $\nu \in [0, 1]$ and define the function $K_\nu : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

\[ K_\nu(x) = \begin{cases} \frac{x^{1-\nu} - x^{1-\nu}}{\ln x}, & x > 0 \text{ and } x \neq 1 \\ \frac{1}{2}, & x = 1 \end{cases}. \]

The function above was first introduced by Kittaneh and Krmic in [8]. Then, using Theorems 1–4, we can obtain the following results for the generalized logarithmic operator mean

\[ L_\nu(A, B) = \frac{1}{2t - 1} A^{1/2} K_\nu \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}. \]

**Theorem 7.** Let $A$ and $B$ be two different positive and invertible operators, $t, s \in I^0$. Then,

(i) when $0 < t < s < 1/2$, we have

\[ \frac{1}{2t - 1} A^{1/2} K_t \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} > \frac{1}{2s - 1} A^{1/2} K_s \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}; \] (31)

(ii) when $1/2 < t < s < 1$, we have

\[ \frac{1}{2t - 1} A^{1/2} K_t \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} < \frac{1}{2s - 1} A^{1/2} K_s \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}. \] (32)

**Theorem 8.** Let $A$ and $B$ be two different positive and invertible operators, $\nu \in I^0$, $\theta = 1 - 2\nu$, $\beta_1 = 0$ and $\beta_2 = \theta^2$. Then, the double inequality

\[ (1 - \beta_1)A^{\#1/2}_1 B + \beta_1 A^{1/2} K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} \]
\[ < \frac{1}{2\nu - 1} A^{1/2} K_\nu \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} \]
\[ < (1 - \beta_2)A^{\#1/2}_2 B + \beta_2 A^{1/2} K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} \] (33)

holds with the best constants $\beta_1$ and $\beta_2$.

**Theorem 9.** Let $A$ and $B$ be two different positive and invertible operators, $\nu \in I^0$, $\theta = 1 - 2\nu$, $\lambda_1 = 1 - \theta^2/3$, and $\lambda_2 = 1$. Then, the double inequality

\[ A \nabla_{1/2} B = (1 - \lambda_2)A^{\#1/2}_2 B + \lambda_2 A \nabla_{1/2} B \]
\[ < \frac{1}{2\nu - 1} A^{1/2} K_\nu \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} \]
\[ < (1 - \lambda_1)A^{\#1/2}_1 B + \lambda_1 A \nabla_{1/2} B \] (34)

holds with the best constants $\lambda_1$ and $\lambda_2$. 
Theorem 10. Let $A$ and $B$ be two different positive and invertible operators, $\nu \in I^0$, $\theta = 1 - 2\nu$, $\sigma_1 = (3 - \theta^2) / 2$, and $\sigma_2 = 1$. Then, the double inequality

$$(1 - \sigma_1)A^\nu B + \sigma_1 A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

$$< \frac{1}{2\nu - 1} A^{1/2}K_2 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

$$< A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} = (1 - \sigma_2)A^\nu B + \sigma_2 A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

holds with the best constants $\sigma_2$ and $\sigma_1$.

4.2. Inequalities Related to the Heinz Operator Mean

From Theorems 5 and 6, we can obtain the following:

Theorem 11. Let $A$ and $B$ be two different positive and invertible operators, $\nu \in I^0$, $\theta = 1 - 2\nu$, and $\theta^2 \leq 3/5$ or

$$\nu_1 = \frac{1 - \sqrt{3/5}}{2} = 0.1127 \ldots \leq \nu \leq \frac{\sqrt{3/5} + 1}{2} = 0.88730 \ldots = 1 - \nu_1.$$  

Then,

$$A^{\nu_1/2}B < H_\nu(A, B) < (1 - 2\theta^2)A^{\nu_1/2}B + 2\theta^2 A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

(36)

holds with the best constants 0 and $3\theta^2$.

Theorem 12. Let $A$ and $B$ be two different positive and invertible operators, $\nu \in I^0$, $\theta = 1 - 2\nu$, and $\theta^2 \leq 1/5$ or

$$\nu_2 = \frac{1 - \sqrt{1/5}}{2} = 0.27639 \ldots \leq \nu \leq \frac{\sqrt{1/5} + 1}{2} = 0.72361 \ldots = 1 - \nu_2.$$  

Then,

$$(1 - \frac{3(1 - \theta^2)}{2})A^\nu B + \left( \frac{3(1 - \theta^2)}{2} \right) A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

$$< H_\nu(A, B)$$

$$< A^{1/2}K_1 \left( A^{-1/2}BA^{-1/2} \right) A^{1/2}$$

(37)

holds with the best constants 1 and $3(1 - \theta^2) / 2$.

5. Conclusions

By using criteria for the monotonicity of the quotient of two power series, this paper presented some sharp bounds for a generalized logarithmic operator mean and a Heinz operator mean by weighted ones of classical operator ones. The classical means involved in this paper were limited to the arithmetic mean, geometric mean, and logarithmic mean, but we know that classical means also include the harmonic mean, exponential mean, and power mean. Therefore, determining how to define the bounds of the two functions $H_\nu(a, b)$ and $L_\nu(a, b)$ by weighted means of any two classical means and how to prove these results are the goals of future research.

Funding: This research received no external funding.

Acknowledgments: The author is thankful to anonymous for careful corrections and valuable comments on the original version of this paper.

Conflicts of Interest: The author declares that he has no conflict of interest.
