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A Mean Convergence Theorem without Convexity for Finite Commutative Nonlinear Mappings in Reflexive Banach Spaces

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Abstract: This paper investigates the Bregman version of the Takahashi-type generic 2-generalized nonspreading mapping which includes the generic 2-generalized Bregman nonspreading mapping as a special case. Relative to the attractive points of nonlinear mapping, the Baillon-type nonlinear mean convergence theorem for finite commutative generic 2-generalized Bregman nonspreading mappings without the convexity assumption is proved in the setting of reflexive Banach spaces. Using this result, some new and well-known nonlinear mean convergence theorems for the finite generic generalized Bregman nonspreading mapping, the 2-generalized Bregman nonspreading mapping and the normally 2-generalized hybrid mapping, among others, are established. Our results extend and generalize many corresponding ones announced in the literature.

Keywords: attractive point; nonlinear mean convergence; generic 2-generalized Bregman nonspreading mapping; generic 2-generalized nonspreading mapping; normally 2-generalized hybrid mapping

MSC: 47H09; 47H25; 47J25

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1. Introduction

Let H be a real Hilbert space and C be a nonempty subset of H. Let $M : C \to H$ be a nonlinear mapping and denote the sets of the fixed and attractive points of M by $F(M)$ and $A(M)$, respectively, i.e., $F(M) = \{x \in C : Mx = x\}$ and $A(M) = \{x \in H : \|x - My\| \leq \|x - y\|, \forall y \in C\}$. A mapping M of C onto H is called an (α, β)-generalized hybrid mapping [1] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Mx - My\|^2 + (1-\alpha)\|x - My\|^2 \leq \beta\|Mx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall\, x, y \in C.$$

We call the mapping M nonexpansive if $\alpha = 1$ and $\beta = 0$. We call the mapping M hybrid [2,3] if $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. In addition, if $\alpha = 2$ and $\beta = 1$, the mapping M reduces to nonspreading [2,4], i.e., $2\|Mx - My\|^2 \leq \|Mx - y\|^2 + \|My - x\|^2\ \forall\, x, y \in C$. In 1975, Bailon [5] proved the first nonlinear mean convergence theorem. He proved that a sequence $\{S_n x\}$ of the Cesaro mean defined for all $x \in C$ by

$$S_n x = \frac{1}{n}\sum_{k=0}^{n} M^k x$$

converges weakly to an element $u \in F(M)$, where $M : C \to H$ is known to be nonexpansive with $F(M) \neq \emptyset$. Kocourek et al. [1] extended the work of Baillon by considering a larger class of mappings M more general than that of nonexpansive. They proved that for any $x \in C$, a sequence $\{S_n x\}$ defined by

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} M^k x$$

converges weakly to an element $v \in F(M)$, where M is a generalized hybrid mapping with $F(M)$ nonempty. It is worth mentioning that the nonempty subset C of H is assumed to be closed and convex in the works of both Baillon [5] and Kocourek et al. [1]. However, not all the cases are true in respect to C, for example, when C is a star-shaped (see Definition 1 below) subset of H.

Takahashi and Takeuchi [6] introduced the concept of an attractive point of a nonlinear mapping in the setting of Hilbert spaces. They proved the attractive point and nonlinear mean convergence theorem without a convexity assumption for a generalized hybrid mapping $M : C \to H$ in the space. In fact, they defined sequences $\{v_n\}$ and $\{b_n\}$ by

$$v_1 \in C,\ v_{n+1} = Mv_n,\ b_n = \frac{1}{n} \sum_{k=1}^{n} v_k,$$

for all $n \in \mathbb{N}$ and proved that if $\{v_n\}$ is bounded, then $\{b_n\}$ converges weakly to an element $u \in A(M)$. Takahashi et al. [7] defined a sequence $\{S_n x\}$ for all $n \in \mathbb{N}$ by

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} M^k x$$

and proved that $\{S_n x\}$ converges weakly to $q \in A(M)$, where $q = \lim_{n\to\infty} PM^n x$ and P is a metric projection. Another class of mappings which is said to include a special case, that of the generalized hybrid, was introduced. By considering two commutative 2-generalized hybrid mappings $M, N : C \to H$, Hojo et al. [8] defined a sequence $\{S_n x\}$ by

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} M^k N^l x$$

for all $n \in \mathbb{N} \cup \{0\}$. They proved that the sequence $\{S_n x\}$ converges weakly to an element $p \in A(M) \cap A(N)$. By considering two commutative normally 2-generalized hybrid mappings M and N and a bounded sequence $\{x_n\}$, Hojo et al. [9] defined a sequence $\{S_n x_n\}$ by

$$S_n x_n = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} M^k N^l x_n$$

for all $n \in \mathbb{N} \cup \{0\}$ and proved that every cluster point of $\{S_n x_n\}$ is a point in $A(M) \cap A(M)$.

In 2013, Lin and Takahashi [10] extended the concept of the attractive point to smooth Banach spaces. By considering a generalized nonspreading mapping M of a nonempty subset C of a smooth and reflexive Banach space E onto itself, Lin et al. [11] defined a sequence $\{S_n x\}$ by

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} M^k x$$

for all $n \in \mathbb{N}$ and proved that if a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to p then $p \in A(M)$. Takahashi et al. [12] defined a sequence $\{S_n x\}$ for all $n \in \mathbb{N}$ by

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} M^k x$$

and proved that $\{S_n x\}$ converges weakly to $q \in A(M)$, where $q = \lim_{n\to\infty} RM^n x$ and R is a sunny generalized nonexpansive retraction. By considering two commutative 2-generalized nonspreading mappings M and N of a nonempty subset C of a smooth, strictly convex and reflexive Banach space E into itself, Takahashi et al. [13] (see also Alsulami et al. [14]) defined a sequence $\{S_n x\}$ by

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} M^k N^l x_n$$

for all $n \in \mathbb{N} \cup \{0\}$ and proved that $\{S_n x\}$ converges weakly to $p \in A(M) \cap A(N)$.

For the Bregman version of the generalized nonspreading mapping, the generic generalized nonspreading mapping and the 2-generalized nonspreading mapping, see [15,16]. By considering two commutative generic 2-generalized nonspreading mappings M and N of a nonempty subset C of a smooth, strictly convex and reflexive Banach space E into itself, Hojo and Takahashi [17] defined a sequence $\{S_n x\}$ by

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} M^k N^l x_n$$

for all $n \in \mathbb{N} \cup \{0\}$ and proved that $\{S_n x\}$ converges weakly to a point in $A(M) \cap A(N)$.

1.1. Our Contributions

Motivated and inspired by the corresponding results in [1,5,7–9,11,13–15,17–19], our contributions in this paper are:

- We first study the Bregman version of the Takahashi-type generic 2-generalized nonspreading mapping, which includes as a special case the Ali and Haruna-type [15] generic 2-generalized Bregman nonspreading mapping in reflexive Banach spaces.
- We then prove a nonlinear mean convergence theorem for finite commutative generic 2-generalized Bregman nonspreading mappings without convexity assumptions in the space.
- As an application of our main results, we establish some new and well-known mean convergence theorems for the finite generic generalized Bregman nonspreading mapping [16], the 2-generalized Bregman nonspreading mapping [15] and the normally 2-generalized hybrid mapping.
- Our results extend and generalize the corresponding ones in Ali and Haruna [15], Alsulami et al. [14], Baillon [5], Hojo and Takahashi [9,17], Hojo et al. [8], Kocourek et al. [1], Lin et al. [11] and Takahashi et al. [13].

1.2. Organization

We organize the rest of our paper as follows: Section 2 contains some basic definitions and related results which are needed in other subsequent sections. In Section 3, we present and discuss our main results.

2. Preliminaries

Definition 1. *Let C be a nonempty subset of H. Then C is called star-shaped if there exists a $z \in C$ such that for any $x \in C$ and $\lambda \in (0,1)$,*

$$\lambda z + (1-\lambda)x \in C.$$

Such a $z \in C$ is called a center of the star-shaped set C.

A mapping $M : C \to H$ is called a normally generalized hybrid [7] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Mx - My\|^2 + \beta\|x - My\|^2 + \gamma\|Mx - y\|^2 + \delta\|x - y\|^2 \le 0,\ \forall\, x, y \in C,$$

where (a) $\alpha + \beta + \gamma + \delta \geq 0$ and (b) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. Observe that if $\alpha + \beta = -\gamma - \delta = 1$, then M reduces to a generalized hybrid mapping.

A mapping M of C into H is called 2-generalized hybrid [20] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|M^2x - My\|^2 &+ \alpha_2\|Mx - My\|^2 + (1-\alpha_1-\alpha_2)\|x - My\|^2 \\ &\leq \beta_1\|M^2x - y\|^2 + \beta_2\|Mx - y\|^2 + (1-\beta_1-\beta_2)\|x-y\|^2, \quad \forall x, y \in C.\end{aligned}$$

Observe that if $\alpha_1 = \beta_1 = 0$, then the mapping reduces to a generalized hybrid.

As a unification of the normally generalized hybrid mapping and the 2-generalized hybrid mapping, a new nonlinear mapping is introduced. A mapping $M : C \to H$ is called a normally 2-generalized hybrid [21] if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|M^2x - My\|^2 &+ \alpha_2\|Mx - My\|^2 + \alpha_3\|x - My\|^2 \\ &+ \beta_1\|M^2x - y\|^2 + \beta_2\|Mx - y\|^2 + \beta_3\|x-y\|^2 \leq 0, \quad \forall\, x, y \in C,\end{aligned}$$

where $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$ and $\sum_{i=1}^{3}\alpha_i > 0$.

In another development, the class of generalized hybrid mappings was extended to that of generalized nonspreading mappings in Banach spaces more general than Hilbert. Let E be a smooth Banach space. A mapping $M : C \to E$ is called generalized nonspreading [22] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Mx, My) &+ (1-\alpha)\phi(x, My) + \gamma\{\phi(My, Mx) - \phi(My, x)\} \\ &\leq \beta\phi(Mx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Mx) - \phi(y, x)\}, \quad \forall\, x, y \in C,\end{aligned}$$

where a map $\phi : E \times E \to \mathbb{R}$ is a function defined by $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$ and $J : E \to 2^{E^*}$ is a duality map defined by $Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$. Observe that if $E = H$, then we have $\phi(x, y) = \|x - y\|^2$, and consequently, an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping reduces to an $(\alpha + \gamma, \beta + \delta)$-generalized hybrid mapping.

A mapping $M : C \to E$ is called generic generalized nonspreading [12] if there exist $\alpha, \beta, \gamma, \delta, \epsilon$ and $\zeta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Mx, My) &+ \beta\phi(x, My) + \gamma\phi(Mx, y) + \delta\phi(x, y) \\ &\leq \epsilon\{\phi(My, Mx) - \phi(My, x)\} + \zeta\{\phi(y, Mx) - \phi(y, x)\}, \quad \forall x, y \in C, \qquad (1)\end{aligned}$$

where (i) $\alpha + \beta + \gamma + \delta \geq 0$ and (ii) $\alpha + \beta > 0$. Observe that a generic generalized nonspreading mapping reduces to a generalized nonspreading mapping if $\alpha + \beta = -\gamma - \delta = 1$.

A mapping $M : C \to E$ is called 2-generalized nonspreading [19] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\phi(M^2x, My) &+ \alpha_2\phi(Mx, My) + (1-\alpha_1-\alpha_2)\phi(x, My) \\ &+ \gamma_1\{\phi(My, M^2x) - \phi(My, x)\} + \gamma_2\{\phi(My, Mx) - \phi(My, x)\} \\ &\leq \beta_1\phi(M^2x, y) + \beta_2\phi(Mx, y) + (1-\beta_1-\beta_2)\phi(x, y) \\ &+ \delta_1\{\phi(y, M^2x) - \phi(y, x)\} + \delta_2\{\phi(y, Mx) - \phi(y, x)\}, \quad \forall x, y \in C.\end{aligned}$$

Observe that if $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, then a 2-generalized nonspreading mapping reduces to a generalized nonspreading.

A mapping $M : C \to E$ is called generic 2-generalized nonspreading [23] if there exist $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$
\begin{aligned}
&\alpha_2\phi(M^2x,My)+\alpha_1\phi(Mx,My)+\alpha_0\phi(x,My)+\beta_2\phi(M^2x,y)+\beta_1\phi(Mx,y)+\beta_0\phi(x,y)\\
\le\ &\gamma_2\{\phi(My,M^2x)-\phi(My,Mx)\}+\gamma_1\{\phi(My,Mx)-\phi(My,x)\}+\gamma_0\{\phi(My,x)-\phi(My,M^2x)\}\\
+\ &\delta_2\{\phi(y,M^2x)-\phi(y,Mx)\}+\delta_1\{\phi(y,Mx)-\phi(y,x)\}+\delta_0\{\phi(y,x)-\phi(y,M^2x)\},\quad \forall x,y\in C,
\end{aligned}
$$

where $\alpha_0+\alpha_1+\alpha_2+\beta_0+\beta_1+\beta_2\ge 0$ and $\alpha_0+\alpha_1+\alpha_2>0$.

Let E be a real Banach space and $f:E\to\mathbb{R}$ be a function. The gradient of f at x is the function $\nabla f(x):E\to(-\infty,+\infty]$ defined by $\langle\nabla f(x),y\rangle = f^\circ(x,y)$, for any $x\in\text{int}(\text{dom}(f))$ and $y\in E$, where $f^\circ(x,y)$ is the derivative of f at x in the direction y which is defined as

$$f^\circ(x,y):=\lim_{t\to 0}\frac{f(x+ty)-f(x)}{t}. \tag{2}$$

The function f is said to be Gâteaux-differentiable at x if the limit in (2) exists for any y. In addition, f is said to be Gâteaux-differentiable if it is Gâteaux-differentiable at every $x\in\text{int}(\text{dom}(f))$. The function f is said to be Fréchet-differentiable at x if the limit in (2) is attained uniformly in y with $\|y\|=1$. In addition, f is said to be Fréchet-differentiable on a subset C of X if the limit (2) is attained uniformly for $x\in X$ and $\|y\|=1$. It is known from [24] that if a continuous convex function f is Fréchet-differentiable (resp. Gâteaux-differentiable) in int(dom(f)), then ∇f is continuous (resp. norm-to-weak* continuous) in int(dom(f)).

Let $f:E\to(-\infty,+\infty]$ be a convex and Gâteaux-differentiable function. The Bregman distance with respect to f [25,26] denoted by D_f is a function $D_f:\text{dom}f\times\text{int}(\text{dom}(f))\to[0,+\infty)$, defined by

$$D_f(x,y):=f(x)-f(y)-\langle\nabla f(y),x-y\rangle. \tag{3}$$

Remark 1. *If E is a smooth Banach space and $f(x)=\|x\|^2$ for all $x\in E$, then the gradient $\nabla f(x)$ of f reduces to $2Jx$ for all $x\in E$, and subsequently, $D_f(x,y)=\phi(x,y)$. In addition, if $E=H$ is a real Hilbert space, then $\phi(x,y)=\|x-y\|^2$, $\forall x,y\in E$.*

For any $x\in domf$ and $y,z\in int(dom(f))$, the three-point identity can easily be obtained from (3) and is given by

$$D_f(x,z)=D_f(x,y)+D_f(y,z)+\langle\nabla f(y)-\nabla f(z),x-y\rangle. \tag{4}$$

We now define the Bregman version of a Takahashi-type generic 2-generalized nonspreading mapping [23] in a reflexive Banach space E.

Definition 2. *A mapping $M:C\to E$ is called generic 2-generalized Bregman nonspreading if there exist $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2,\gamma_0,\gamma_1,\gamma_2,\delta_0,\delta_1,\delta_2\in\mathbb{R}$ such that $\alpha_0+\alpha_1+\alpha_2+\beta_0+\beta_1+\beta_2\ge 0$, $\alpha_0+\alpha_1+\alpha_2>0$ and*

$$
\begin{aligned}
&\alpha_2D_f(M^2x,My)+\alpha_1D_f(Mx,My)+\alpha_0D_f(x,My)+\beta_2D_f(M^2x,y)+\beta_1D_f(Mx,y)+\beta_0D_f(x,y)\\
\le\ &\gamma_2\{D_f(My,M^2x)-D_f(My,Mx)\}+\gamma_1\{D_f(My,Mx)-D_f(My,x)\}+\gamma_0\{D_f(My,x)-D_f(My,M^2x)\}\\
+\ &\delta_2\{D_f(y,M^2x)-D_f(y,Mx)\}+\delta_1\{D_f(y,Mx)-D_f(y,x)\}+\delta_0\{D_f(y,x)-D_f(y,M^2x)\},\quad\forall x,y\in C.
\end{aligned}
$$

Remark 2. *Observe that by setting $\gamma_2=\delta_2=0$, the mapping in Definition 2 reduces to a generic 2-generalized Bregman nonspeading mapping in the sense of Ali and Haruna [16]. In addition, if E is smooth and $f(x)=||x||^2$, then the mapping reduces to generic 2-generalized nonspreading in the sense of Takahashi [23]. Furthermore, if $E=H$ is a real Hilbert space, the mapping reduces to a normally 2-generalized hybrid mapping in the sense of Kondo and Takahashi [21].*

Example 1. *Let $E = \mathbb{R}$ and $C = [0, 2]$. Let $f(x) = x^2$ and $M : C \to C$ be defined by*

$$Mx = \begin{cases} 0, & x \in [0, 2) \\ 1, & x = 2. \end{cases}$$

Observe that for the choice of real numbers $\alpha_2 = \alpha_1 = \alpha_0 = 1$, $\beta_2 = \beta_1 = \beta_0 = \gamma_1 = \gamma_0 = \delta_1 = \delta_0 = -1$ and $\gamma_2 = \delta_2 = 0$, we see that $\sum_{i=1}^{3} \alpha_i > 0$; $\sum_{i=1}^{3} (\alpha_i + \beta_i) \geq 0$ and $h(x, y) \leq 0$, for all $x, y \in C$, where

$$\begin{aligned} h(x,y) = {} & \alpha_2(M^2x - My)^2 + \alpha_1(Mx - My)^2 + \alpha_0(x - My)^2 + \beta_2(M^2x - y)^2 + \beta_1(Mx - y)^2 + \beta_0(x - y)^2 \\ & - \gamma_2\{(My - M^2x)^2 - (My - Mx)^2\} - \gamma_1\{(My - Mx)^2 - (My - x)^2\} - \gamma_0\{(My - x)^2 - (My - M^2x)^2\} \\ & - \delta_2\{(y - M^2x)^2 - (y - Mx)^2\} - \delta_1\{(y - Mx)^2 - (y - x)^2\} - \delta_0\{(y - x)^2 - (y - M^2x)^2\}. \end{aligned}$$

Therefore, M is a generic 2-generalized Bregman nonspreading mapping.

Let E be a reflexive Banach space and T be a mapping of a nonempty subset C of $int(domf)$ into E. We denote the set of Bregman attractive points of T by $A_f(T)$ and that of Bregman skew-attractive points [27] by $B_f T$, i.e., $A_f(T) = \{x \in E : D_f(x, Ty) \leq D_f(x, y), \ \forall y \in C\}$ and $B_f(T) = \{x \in E : D_f(Ty, x) \leq D_f(y, x), \ \forall y \in C\}$.

Lemma 1 ([27])**.** *Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ a convex, continuous, strongly coercive and Gâteaux-differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty subset of E and $T : C \to E$ be a mapping. Then $B_f(T)$ is closed and convex.*

Lemma 2 ([27])**.** *Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ a convex, continuous, strongly coercive and Gâteaux-differentiable function which is bounded on bounded sets, uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty subset of E and $T : C \to E$ be a mapping. Let $T^* : \nabla f C \to E^*$ be the duality mapping of T. Then the following assertions hold:*

(1) $\nabla g B_f(T) = A_f(T^*)$;
(2) $\nabla g A_f(T) = B_f(T^*)$.

In particular, $\nabla g B_f(T)$ is closed and convex.

Let C be a nonempty subset of a Banach space X. A mapping $\mathcal{R}$ of X into C is said to be sunny [28] if

$$\mathcal{R}(\mathcal{R}x + r(x - \mathcal{R}x)) = \mathcal{R}x,$$

for each $x \in X$ and $r \geq 0$. A mapping $\mathcal{R} : X \to C$ is said to be retraction [28] if $\mathcal{R}x = x$ for all $x \in C$. A nonempty subset C of X is said to be a sunny Bregman generalized nonexpansive retract (resp. a Bregman generalized nonexpansive retract) of X if there exists a sunny Bregman generalized nonexpansive retraction (resp. a Bregman generalized nonexpansive retraction) of X onto C, see [29] for details.

Lemma 3 ([30])**.** *Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ a convex, continuous, strongly coercive function which is bounded on bounded sets and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E. Then the following statements are equivalent*

(1) *C is a sunny Bregman generalized nonexpansive retract of E;*
(2) *C is a Bregman generalized nonexpansive retract of E;*
(3) *$\nabla g C$ is closed and convex.*

Using Lemma 3, the following result can be established.

Lemma 4. *Let E be a reflexive Banach space and let $\{C_i\}$ be a family of sunny Bregman generalized nonexpansive retracts of E such that $\cap_{i\in I}C_i$ is nonempty. Then $\cap_{i\in I}C_i$ is a sunny Bregman generalized nonexpansive retract of E.*

Proof. It is easy to see $\nabla g(\cap_{i\in I}C_i) = \cap_{i\in I}\nabla gC_i$. Indeed,

$$\begin{aligned} x \in \nabla g(\cap_{i\in I}C_i) &\iff (\nabla g)^{-1}x \in \cap_{i\in I}C_i \\ &\iff (\nabla g)^{-1}x \in C_i, \forall i \in I \\ &\iff x \in \nabla gC_i, \forall i \in I \\ &\iff x \in \cap_{i\in I}\nabla gC_i. \end{aligned}$$

Thus, from Lemma 3 above, ∇gC_i is closed and closed for each $i \in I$. Therefore, $\cap_{i\in I}\nabla gC_i$ is closed and closed. Hence, we have that $\cap_{i\in I}C_i$ is a sunny Bregman generalized nonexpansive retract of E. □

Lemma 5 ([30])**.** *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E and let $\mathcal{R}$ be a retraction from E onto C. Then the following assertions are equivalent:*

(1) $\mathcal{R}$ is sunny Bregman generalized nonexpansive;
(2) $\langle x - \mathcal{R}x, \nabla g(y) - \nabla g(\mathcal{R}x)\rangle \le 0, \forall(x, y) \in E \times C.$

Lemma 6 ([31])**.** *Let E be a Banach space and $r > 0$. Let ρ_r be the gauge function of uniform convexity of g where $g : E \to \mathbb{R}$ is a convex function which is uniformly convex on bounded subsets of E. Then the following hold:*

(1) For any $x, y \in B_r$ and $\alpha \in (0, 1)$, $g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)$ $\rho_r(||x - y||)$;
(2) For any $x, y \in B_r$, $\rho_r(||x - y||) \le D_g(x, y)$.

Lemma 7 ([32], Theorem 7.3 (vi))**.** *Suppose $u \in domf$ and $v \in int(dom(f))$. If f is strictly convex, then $D_f(u, v) = 0 \Leftrightarrow u = v$.*

3. Main Results

In this section, we prove a nonlinear mean convergence theorem without convexity for finite commutative generic 2-generalized Bregman nonspreading mappings. The following lemma will play a vital role.

Lemma 8. *Let $f : E \to \mathbb{R}$ be a convex and uniformly Fréchet-differentiable function which is bounded on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative generic 2-generalized Bregman nonspreading mappings such that the set $\{M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N}\cup\{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_nx\}$ by*

$$S_nx = \frac{1}{(n+1)^N}\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n} M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,$$

for all $n \in \mathbb{N}\cup\{0\}$. Then every weak cluster point of $\{S_nx\}$ is a point of $\cap_{l=1}^N A_f(M_l)$. Additionally, if f is strictly convex and C is closed and conve, then every weak cluster point of $\{S_nx\}$ is a point of $\cap_{l=1}^N F(M_l)$.

Proof. Since M_1 is a generic 2-generalized Bregman nonspreading mapping, then by Definition 2, we have

$$\begin{aligned}\alpha_2 D_f(M_1^2x, M_1y) &+ \alpha_1 D_f(M_1x, M_1y) + \alpha_0 D_f(x, M_1y) + \beta_2 D_f(M_1^2x, y) + \beta_1 D_f(M_1x, y) + \beta_0 D_f(x, y)\\ &\leq \gamma_2\{D_f(M_1y, M_1^2x) - D_f(M_1y, M_1x)\} + \gamma_1\{D_f(M_1y, M_1x) - D_f(M_1y, x)\}\\ &+ \gamma_0\{D_f(M_1y, x) - D_f(M_1y, M_1^2x)\} + \delta_2\{D_f(y, M_1^2x) - D_f(y, M_1x)\}\\ &+ \delta_1\{D_f(y, M_1x) - D_f(y, x)\} + \delta_0\{D_f(y, x) - D_f(y, M_1^2x)\}, \quad \forall x, y \in C.\end{aligned}$$

Using the three-point identity (4), we obtain

$$\begin{aligned}\alpha_2\{D_f(M_1^2x, y) &+ D_f(y, M_1y) + \langle \nabla f(y) - \nabla f(M_1y), M_1^2x - y\rangle\} + \beta_2 D_f(M_1^2x, y)\\ &+ \alpha_1\{D_f(M_1x, y) + D_f(y, M_1y) + \langle \nabla f(y) - \nabla f(M_1y), M_1x - y\rangle\} + \beta_1 D_f(M_1x, y)\\ &+ \alpha_0\{D_f(x, y) + D_f(y, M_1y) + \langle \nabla f(y) - \nabla f(M_1y), x - y\rangle\} + \beta_0 D_f(x, y)\\ &\leq \gamma_2\{D_f(M_1y, M_1^2x) - D_f(M_1y, M_1x)\} + \gamma_1\{D_f(M_1y, M_1x) - D_f(M_1y, x)\}\\ &+ \gamma_0\{D_f(M_1y, x) - D_f(M_1y, M_1^2x)\} + \delta_2\{D_f(y, M_1^2x) - D_f(y, M_1x)\}\\ &+ \delta_1\{D_f(y, M_1x) - D_f(y, x)\} + \delta_0\{D_f(y, x) - D_f(y, M_1^2x)\}, \quad \forall x, y \in C.\end{aligned}$$

This implies

$$\begin{aligned}(\alpha_2 + \beta_2) D_f(M_1^2x, y) &+ (\alpha_1 + \beta_1) D_f(M_1x, y) + (\alpha_0 + \beta_0) D_f(x, y) + (\alpha_2 + \alpha_1 + \alpha_0) D_f(y, M_1y)\\ &+ \langle \nabla f(y) - \nabla f(M_1y), \alpha_2(M_1^2x - y) + \alpha_1(M_1x - y) + \alpha_0(x - y)\rangle\\ &\leq \gamma_2\{D_f(M_1y, M_1^2x) - D_f(M_1y, M_1x)\} + \gamma_1\{D_f(M_1y, M_1x) - D_f(M_1y, x)\}\\ &+ \gamma_0\{D_f(M_1y, x) - D_f(M_1y, M_1^2x)\} + \delta_2\{D_f(y, M_1^2x) - D_f(y, M_1x)\}\\ &+ \delta_1\{D_f(y, M_1x) - D_f(y, x)\} + \delta_0\{D_f(y, x) - D_f(y, M_1^2x)\}, \quad \forall x, y \in C.\end{aligned} \tag{5}$$

Since $-(\alpha_2 + \beta_2 + \alpha_1 + \beta_1) \leq \alpha_0 + \beta_0$, we obtain from Inequality (5) that

$$\begin{aligned}(\alpha_2 + \beta_2)\big(D_f(M_1^2x, y) &- D_f(x, y)\big) + (\alpha_1 + \beta_1)\big(D_f(M_1x, y) - D_f(x, y)\big) + (\alpha_2 + \alpha_1 + \alpha_0) D_f(y, M_1y)\\ &+ \langle \nabla f(y) - \nabla f(M_1y), \alpha_2(M_1^2x - x) + \alpha_1(M_1x - x) + (\alpha_2 + \alpha_1 + \alpha_0)(x - y)\rangle\\ &\leq \gamma_2\{D_f(M_1y, M_1^2x) - D_f(M_1y, M_1x)\} + \gamma_1\{D_f(M_1y, M_1x) - D_f(M_1y, x)\}\\ &+ \gamma_0\{D_f(M_1y, x) - D_f(M_1y, M_1^2x)\} + \delta_2\{D_f(y, M_1^2x) - D_f(y, M_1x)\}\\ &+ \delta_1\{D_f(y, M_1x) - D_f(y, x)\} + \delta_0\{D_f(y, x) - D_f(y, M_1^2x)\}, \quad \forall x, y \in C.\end{aligned} \tag{6}$$

Following the hypothesis, we can take $x \in C$ such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded. Now, we replace x with $M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$ so that from Inequality (6) we obtain

$$\begin{aligned}&(\alpha_2 + \alpha_1 + \alpha_0) D_f(y, M_1y) + (\alpha_2 + \beta_2)\big(D_f(M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y) - D_f(M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y)\big)\\ &+ (\alpha_1 + \beta_1)\big(D_f(M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y) - D_f(M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y)\big)\\ &+ \langle \nabla f(y) - \nabla f(M_1y), \alpha_2(M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x - M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\ &+ \alpha_1(M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x - M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) + (\alpha_2 + \alpha_1 + \alpha_0)(M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x - y)\rangle\\ &\leq \gamma_2\{D_f(M_1y, M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1y, M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\ &+ \gamma_1\{D_f(M_1y, M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1y, M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\ &+ \gamma_0\{D_f(M_1y, M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1y, M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\ &+ \delta_2\{D_f(y, M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\ &+ \delta_1\{D_f(y, M_1^{\mu_1+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\ &+ \delta_0\{D_f(y, M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1^{\mu_1+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}, \quad \forall y \in C.\end{aligned} \tag{7}$$

Summing Inequality (7) with respect to $\mu_1 = 0, 1, \cdots, n$, we obtain

$$
\begin{aligned}
&(\alpha_2+\alpha_1+\alpha_0)(n+1)D_f(y, M_1 y) \\
+\quad & (\alpha_2+\beta_2)\big(D_f(M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) + D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}u, y) - D_f(M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) \\
-\quad & D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x, y)\big) + (\alpha_1+\beta_1)\big(D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) - D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x, y)\big) \\
+\quad & \langle \nabla f(y) - \nabla f(M_1 y), \alpha_2\big((M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x + M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
-\quad & (M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x + M_2^{\mu_2}\cdots M_N^{\mu_N}x)\big) \\
+\quad & \alpha_1(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x - M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
+\quad & (\alpha_2+\alpha_1+\alpha_0)(\sum_{\mu_1=0}^{n} M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x - (n+1)y)\rangle \\
\leq\quad & \gamma_2\{D_f(M_1 y, M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(M_1 y, M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \\
+\quad & \gamma_1\{D_f(M_1 y, M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(M_1 y, M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \\
+\quad & \gamma_0\{D_f(M_1 y, M_2^{\mu_2}\cdots M_N^{\mu_N}x) + D_f(M_1 y, M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(M_1 y, M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
-\quad & D_f(M_1 y, M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} + \delta_2\{D_f(y, M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(y, M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \\
+\quad & \delta_1\{D_f(y, M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(y, M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \\
+\quad & \delta_0\{D_f(y, M_2^{\mu_2}\cdots M_N^{\mu_N}x) + D_f(y, M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x) - D_f(y, M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
-\quad & D_f(y, M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)\}, \quad \forall y \in C.
\end{aligned} \tag{8}
$$

Again, summing Inequality (8) with respect to $\mu_2 = 0, 1, \cdots, n$, we obtain

$$
\begin{aligned}
&(\alpha_2+\alpha_1+\alpha_0)(n+1)^2 D_f(y, M_1 y) \\
+\quad & (\alpha_2+\beta_2)\sum_{\mu_2=0}^{n}\big(D_f(M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) + D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) - D_f(M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) \\
-\quad & D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x, y)\big) + (\alpha_1+\beta_1)\sum_{\mu_2=0}^{n}\big(D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x, y) - D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x, y)\big) \\
+\quad & \langle \nabla f(y) - \nabla f(M_1 y), \alpha_2\sum_{\mu_2=0}^{n}\big((M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x + M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
-\quad & (M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x - M_2^{\mu_2}\cdots M_N^{\mu_N}x)\big) \\
+\quad & \alpha_1\sum_{\mu_2=0}^{n}(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x - M_1 M_2^{\mu_2}\cdots M_N^{\mu_N}x) \\
+\quad & (\alpha_2+\alpha_1+\alpha_0)(\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n} M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x - (n+1)^2 y)\rangle
\end{aligned}
$$

$$\begin{aligned}
\leq\quad & \gamma_2 \sum_{\mu_2=0}^{n} \{D_f(M_1 y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x)\} \qquad (9)\\
+\quad & \gamma_1 \sum_{\mu_2=0}^{n} \{D_f(M_1 y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \gamma_0 \sum_{\mu_2=0}^{n} \{D_f(M_1 y, M_2^{\mu_2} \cdots M_N^{\mu_N} x) + D_f(M_1 y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
-\quad & D_f(M_1 y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\} + \delta_2 \sum_{\mu_2=0}^{n} \{D_f(y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \delta_1 \sum_{\mu_2=0}^{n} \{D_f(y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \delta_0 \sum_{\mu_2=0}^{n} \{D_f(y, M_2^{\mu_2} \cdots M_N^{\mu_N} x) + D_f(y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
-\quad & D_f(y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}, \quad \forall y \in C.
\end{aligned}$$

We continue summing Inequality (9) until with respect to $\mu_N = 0, 1, \cdots, n$, and we obtain

$$\begin{aligned}
& (\alpha_2 + \alpha_1 + \alpha_0)(n+1)^N D_f(y, M_1 y)\\
+\quad & (\alpha_2 + \beta_2) \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \big(D_f(M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y) + D_f(M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y)\\
-\quad & D_f(M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x, y) - D_f(M_2^{\mu_2} \cdots M_N^{\mu_N} x, y)\big)\\
+\quad & (\alpha_1 + \beta_1) \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \big(D_f(M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, y) - D_f(M_2^{\mu_2} \cdots M_N^{\mu_N} x, y)\big)\\
+\quad & \langle \nabla f(y) - \nabla f(M_1 y), \alpha_2 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \big((M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x + M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
-\quad & (M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x + M_2^{\mu_2} \cdots M_N^{\mu_N} x)\big) + \alpha_1 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} (M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x - M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
+\quad & (\alpha_2 + \alpha_1 + \alpha_0)\Big(\sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x - (n+1)^N y\Big)\rangle\\
\leq\quad & \gamma_2 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(M_1 y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x)\} \qquad (10)\\
+\quad & \gamma_1 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(M_1 y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \gamma_0 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(M_1 y, M_2^{\mu_2} \cdots M_N^{\mu_N} x) + D_f(M_1 y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(M_1 y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
-\quad & D_f(M_1 y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\} + \delta_2 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \delta_1 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}\\
+\quad & \delta_0 \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \{D_f(y, M_2^{\mu_2} \cdots M_N^{\mu_N} x) + D_f(y, M_1 M_2^{\mu_2} \cdots M_N^{\mu_N} x) - D_f(y, M_1^{n+1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\\
-\quad & D_f(y, M_1^{n+2} M_2^{\mu_2} \cdots M_N^{\mu_N} x)\}, \quad \forall y \in C.
\end{aligned}$$

Dividing both sides of Inequality (10) by $(n+1)^N$ and letting $S_n x = \frac{1}{(n+1)^N}\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n} M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x$, we obtain

$$
\begin{aligned}
&(\alpha_2+\alpha_1+\alpha_0)D_f(y,M_1y)\\
&+\frac{(\alpha_2+\beta_2)}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\big(D_f(M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)+D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)\\
&-D_f(M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)-D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)\big)\\
&+\frac{(\alpha_1+\beta_1)}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\big(D_f(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)-D_f(M_2^{\mu_2}\cdots M_N^{\mu_N}x,y)\big)\\
&+\langle\nabla f(y)-\nabla f(M_1y),\frac{\alpha_2}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}(M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x+M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x\\
&-M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x-M_2^{\mu_2}\cdots M_N^{\mu_N}x)\\
&+\frac{\alpha_1}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}(M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x)+(\alpha_2+\alpha_1+\alpha_0)(S_nx-y)\rangle\\
&\le\frac{\gamma_2}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(M_1y,M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(M_1y,M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \qquad (11)\\
&+\frac{\gamma_1}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(M_1y,M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(M_1y,M_2^{\mu_2}\cdots M_N^{\mu_N}x)\}\\
&+\frac{\gamma_0}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(M_1y,M_2^{\mu_2}\cdots M_N^{\mu_N}x)+D_f(M_1y,M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x)\\
&-D_f(M_1y,M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(M_1y,M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)\}\\
&+\frac{\delta_2}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(y,M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(y,M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x)\} \qquad (12)\\
&+\frac{\delta_1}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(y,M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(y,M_2^{\mu_2}\cdots M_N^{\mu_N}x)\}\\
&+\frac{\delta_0}{(n+1)^N}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\{D_f(y,M_2^{\mu_2}\cdots M_N^{\mu_N}x)+D_f(y,M_1M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(y,M_1^{n+1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)\\
&-D_f(y,M_1^{n+2}M_2^{\mu_2}\cdots M_N^{\mu_N}x)\},\quad\forall y\in C.
\end{aligned}
$$

Since E is reflexive and $\{S_nx\}$ is bounded, then there exists a subsequence $\{S_{n_j}x\}$ of $\{S_nx\}$ such that $\{S_{n_j}x\}$ converges weakly to some point $p\in E$. Now, replacing n with n_j in Inequality (11) and allowing $j\to\infty$, we obtain

$$(\alpha_2+\alpha_1+\alpha_0)\big(D_f(y,M_1y)+\langle\nabla f(y)-\nabla f(M_1y),p-y\rangle\big)\le 0. \qquad (13)$$

Using Equation (4) and the fact that $\alpha_2+\alpha_1+\alpha_0\ge 0$, we obtain

$$D_f(y,M_1y)+D_f(p,M_1y)-D_f(y,M_1y)-D_f(p,y)\le 0,\forall y\in C.$$

Thus,

$$D_f(p,M_1y)\le D_f(p,y),\forall y\in C. \qquad (14)$$

By the commutative nature of $M_1, M_2, \cdots, M_N$, we can replace M_1 in (14) with any of the $M_2 \cdots, M_N$ so that for all $y \in C$ we obtain

$$D_f(p, M_2 y) \leq D_f(p, y) \tag{15}$$
$$\vdots$$
$$D_f(p, M_N y) \leq D_f(p, y). \tag{16}$$

Therefore, from (14)–(16), we have $v \in \cap_{l=1}^{N} A_f(M_l)$. Hence, every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then we put $y = v$ in (14)–(16) and we see that by Lemma 7, $v \in \cap_{l=1}^{N} F(M_l)$. Hence, every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} F(M_l)$. This completes the proof. □

Following a similar argument as in the proof of Lemma 8, the following new results with respect to finite generic 2-generalized nonspreading mappings and normally 2-generalized hybrid mappings can be established.

Lemma 9. *Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative generic 2-generalized nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} A(M_l)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} F(M_l)$.

Proof. Let E be a smooth Banach space and $f(x) = ||x||^2$. Then by Remark 2, the mapping reduces to generic 2-generalized nonspreading in the sense of Takahashi [23]. Following a similar argument as in the proof of Lemma 8 with the use of $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ in the place where Equation (4) is applied, we obtain the desired results. This completes the proof. □

Lemma 10. *Let E be a real Hilbert space and C be a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative normally 2-generalized hybrid mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} A(M_l)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{N} F(M_l)$.

Proof. Since E is a real Hilbert space, then the mapping reduces to normally 2-generalized hybrid [21], i.e., there exist $\bar{\alpha_2}, \bar{\alpha_1}, \bar{\alpha_0}, \bar{\beta_2}, \bar{\beta_1}, \bar{\beta_0} \in \mathbb{R}$ such that

$$\begin{aligned} \bar{\alpha_2}||M^2x - My||^2 &+ \bar{\alpha_1}||Mx - My||^2 + \bar{\alpha_0}||x - My||^2 \\ &\leq \bar{\beta_2}||M^2x - y||^2 + \bar{\beta_1}||Mx - y||^2 + \bar{\beta_0}||x - y||^2, \quad \forall\, x, y \in C, \end{aligned}$$

where $\bar{\alpha_2} = \alpha_2 - \gamma_2 + \gamma_0$, $\bar{\alpha_1} = \alpha_1 + \gamma_2 - \gamma_1$, $\bar{\alpha_0} = \alpha_0 + \gamma_1 - \gamma_0$, $\bar{\beta_2} = \beta_2 - \delta_2 + \delta_0$, $\bar{\beta_1} = \beta_1 + \delta_2 - \delta_1$ and $\bar{\beta_0} = \beta_0 + \delta_1 - \delta_0$ satisfying $\bar{\alpha_2} + \bar{\alpha_1} + \bar{\alpha_0} + \bar{\beta_2} + \bar{\beta_1} + \bar{\beta_0} \geq 0$ and $\bar{\alpha_2} + \bar{\alpha_1} + \bar{\alpha_0} > 0$. Following similar argument as in the proof of Lemma 8 with the use of $||x-y||^2 = ||x-z||^2 + ||z-y||^2 + 2\langle x-z, z-y\rangle$ in the place where Equation (4) is applied, we obtain the desired results. This completes the proof. □

As direct consequences of Lemmas 8–10, the following results corresponding to the ones in Ali and Haruna [15], Hojo and Takahashi [17] and Hojo et al. [9] can be obtained as corollaries.

Corollary 1 ([15], Theorem 3.3)**.** *Let $f : E \to \mathbb{R}$ be a convex and uniformly Fréchet-differentiable function which is bounded on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $M_1, M_2 : C \to C$ be two commutative generic generalized Bregman nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} x : \mu_1, \mu_2 \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} F(M_l)$.

Corollary 2 ([17], Lemma 3.1)**.** *Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let $M_1, M_2 : C \to C$ be two commutative generic 2-generalized nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} x : \mu_1, \mu_2 \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} A(M_l)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} F(M_l)$.

Corollary 3 ([9], Lemma 3.1)**.** *Let E be a real Hilbert space and C be a nonempty subset of E. Let $M_1, M_2 : C \to C$ be two commutative normally 2-generalized hybrid mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} x : \mu_1, \mu_2 \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} A(M_l)$. Additionally, if C is closed and convex then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^{2} F(M_l)$.

In view of the fact that the generic 2-generalized Bregman nonspreading (simply nonspreading) mapping unifies the generic generalized Bregman nonspreading (simply nonspreading) mapping and the 2-generalized Bregman nonspreading (simply nonspreading) mapping, the following results can be obtained from Lemmas 8 and 9 as corollaries. These results correspond to the ones in Ali and Haruna [15], Alsulami et al. [14] Takahashi et al. [12], Takahashi et al. [13] and Lin et al. [11] when one or two mappings are considered.

Corollary 4 ([15])**.** *Let $f : E \to \mathbb{R}$ be a convex and uniformly Fréchet-differentiable function which is bounded on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative generic generalized Bregman nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

Corollary 5 ([15])**.** *Let $f : E \to \mathbb{R}$ be a convex and uniformly Fréchet-differentiable function which is bounded on bounded subsets of X. Let C be a nonempty subset of $int(dom(f))$ and $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative 2-generalized Bregman nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

Corollary 6 ([11,12])**.** *Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative generic generalized nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

Corollary 7 ([13,14])**.** *Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative 2-generalized nonspreading mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} x : \mu_1, \mu_2 \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A_f(M_l)$. Additionally, if f is strictly convex and C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

In addition, in view of the fact that the normally 2-generalized hybrid mapping unifies the normally generalized hybrid mapping and 2-generalized hybrid mapping, we obtain the following results from Lemma 10 as corollaries. These results correspond to the ones in Hojo et al. [8], Takahashi et al. [7] and Takahashi et al. [33] when only one or two mappings are considered.

Corollary 8 ([7])**.** *Let E be a real Hilbert space and C be a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative normally generalized hybrid mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A(M_l)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

Corollary 9 ([8,33])**.** *Let E be a real Hilbert space and C be a nonempty subset of E. Let $M_1, M_2, \cdots, M_N : C \to C$ be finite commutative 2-generalized hybrid mappings such that the set $\{M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$ is bounded for some $x \in C$. Define a sequence $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $n \in \mathbb{N} \cup \{0\}$. Then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N A(M_l)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $\cap_{l=1}^N F(M_l)$.

We now prove a nonlinear mean convergence theorem for finite commutative generic 2-generalized Bregman nonspreading mappings in a reflexive Banach space E. Let $D = \{(\mu_1, \mu_2, \cdots, \mu_N) : \mu_1, \mu_2, \cdots, \mu_N \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(\mu_1, \mu_2, \cdots, \mu_N) \leq (\nu_1, \nu_2, \cdots, \nu_N) \quad \text{if} \quad \mu_1 \leq \nu_1, \mu_2 \leq \nu_2, \cdots, \mu_N \leq \nu_N.$$

Theorem 1. *Let E be a smooth, strictly convex and reflexive Banach space and $f : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded sets. Let $M_1, M_2, \cdots, M_N$ be finite commutative generic 2-generalized Bregman nonspreading mappings of a nonempty subset C of $int(dom(f))$ into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2, \cdots, N$. Let $\mathcal{R}$ be the sunny Bregman generalized nonexpansive retraction of E onto $\cap_{l=1}^N B_f(M_l)$, and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^N A_f(M_l)$, where $q = \lim_{(\mu_1, \mu_2, \cdots, \mu_N) \in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^N F(M_l)$.

Proof. Using Lemmas 1 and 2, we see that $\cap_{l=1}^N B_f(S_l)$ and $\nabla f(\cap_{l=1}^N A_f(S_l))$ are closed and convex, respectively. Thus, by Lemma 4, there exists a sunny Bregman generalized nonexpansive retraction $\mathcal{R}$ of E on $\cap_{l=1}^N B_f(M_l)$ which is characterized (see Lemma 5) by

$$0 \leq \langle v - \mathcal{R}v, \nabla f(\mathcal{R}v) - \nabla f(u) \rangle, \quad \forall u \in \cap_{l=1}^N B_f(S_l), v \in C. \tag{17}$$

Adding $D_f(\mathcal{R}v, u)$ on both sides of Inequality (17), we obtain

$$\begin{aligned} D_f(\mathcal{R}v, u) &\leq D_f(\mathcal{R}v, u) + \langle v - \mathcal{R}v, \nabla f(\mathcal{R}v) - \nabla f(u) \rangle \\ &= D_f(\mathcal{R}v, u) + D_f(v, u) - D_f(v, \mathcal{R}v) - D_f(\mathcal{R}v, u) \\ &= D_f(v, u) - D_f(v, \mathcal{R}v). \end{aligned} \tag{18}$$

Since $\cap_{l=1}^N B_f(M_l) \neq \emptyset$, then for any $u \in \cap_{l=1}^N B_f(M_l)$, $D_f(M_1 v, u) \leq D_f(v, u), D_f(M_2 v, u) \leq D_f(v, u) \cdots D_f(M_N v, u) \leq D_f(v, u)$. It follows that for any $(\mu_1, \mu_2 \cdots, \mu_N)$, $(\nu_1, \nu_2 \cdots, \nu_N) \in D$ with $(\mu_1, \mu_2 \cdots, \mu_N) \leq (\nu_1, \nu_2 \cdots, \nu_N)$, we have

$$\begin{aligned} D_f(M_1^{\nu_1} M_2^{\nu_2} \cdots M_N^{\nu_N} x, \mathcal{R} M_1^{\nu_1} M_2^{\nu_2} \cdots M_N^{\nu_N} x) &\leq D_f(M_1^{\nu_1} M_2^{\nu_2} \cdots M_N^{\nu_N} x, \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x) \\ &\leq D_f(M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x). \end{aligned}$$

Therefore, the net $D_f(M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x, \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x)$ is nonincreasing. Putting $u = \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$ and $v = M_1^{\nu_1} M_2^{\nu_2} \cdots M_N^{\nu_N} x$ in (18), we obtain from (2) of Lemma 6 that

$$\rho_r(||\mathcal{R}M_1^{\nu_1}M_2^{\nu_2}\cdots M_N^{\nu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots,M_N^{\mu_N}x||)\leq D_f(\mathcal{R}M_1^{\nu_1}M_2^{\nu_2}\cdots M_N^{\nu_N}x,\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)$$
$$\leq D_f(M_1^{\nu_1}M_2^{\nu_2}\cdots M_N^{\nu_N}x,\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-D_f(M_1^{\nu_1}M_2^{\nu_2}\cdots M_N^{\nu_N}x,\mathcal{R}M_1^{\nu_1}M_2^{\nu_2}\cdots M_N^{\nu_N}x),$$

where ρ_r is a gauge function of uniform convexity. From the properties of ρ_r, $\{\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}\}$ is a Cauchy net, see [34]. Hence, $\{\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}\}$ converges strongly to $q\in\cap_{l=1}^N B_f(M_l)$ since $\cap_{l=1}^N B_f(M_l)$ is closed by Lemma 1.

Next, we consider a fixed $x\in C$ and an arbitrary subsequence $\{S_{n_j}x\}$ of $\{S_nx\}$ that converges weakly to v. We know from Lemma 8 that $v\in\cap_{l=1}^N A_f(M_l)$. Rewriting the characterization of the retraction, we have that for any $u\in\cap_{l=1}^N B_f(M_l)$,

$$0\leq\langle M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,\nabla f(\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-\nabla f(u)\rangle.$$

Thus,

$$\begin{aligned}
&\langle M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,\nabla f(u)-\nabla f(q)\rangle\\
&\leq \langle M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,\nabla f(\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-\nabla f(q)\rangle\\
&\leq ||M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x||\cdot||\nabla f(\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-\nabla f(q)||\\
&\leq M||\nabla f(\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-\nabla f(q)||, \qquad (19)
\end{aligned}$$

where M is an upper bound for $||M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x-\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x||$. Summing Inequality (19) with respect to $\mu_1=0,1,2,\cdots,n$, $\mu_2=0,1,2,\cdots,n$ up to $\mu_N=0,1,2,\cdots,n$, and dividing through by $(n+1)^N$, we obtain

$$\begin{aligned}
&\langle S_nx-\frac{1}{(n+1)^N}\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,\nabla f(u)-\nabla f(q)\rangle\\
&\leq M\frac{1}{(n+1)^N}\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}||\nabla f(\mathcal{R}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x)-\nabla f(q)||, \qquad (20)
\end{aligned}$$

where $S_nx=\frac{1}{(n+1)^N}\sum_{\mu_1=0}^{n}\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x$. Replacing n with n_j in (20) and allowing $j\to\infty$, keeping in mind that ∇f is continuous, we obtain

$$\langle v-q,\nabla f(u)-\nabla f(q)\rangle\leq 0.$$

This inequality holds for any $u\in\cap_{l=1}^N B_f(M_l)$. Thus, $\mathcal{R}v=q$. Since $v\in\cap_{l=1}^N B_f(M_l)$, then $v=q$. Therefore, the sequence $\{S_nx\}$ converges weakly to the point q. If, in addition, C is closed and convex, then $q\in C$. Hence, $\{S_nx\}$ converges weakly to a point of $\cap_{l=1}^N F(M_l)$. This completes the proof. □

Following a similar argument as in Theorem 1, we can establish the following new results for finite generic 2-generalized nonspreading mappings and normally 2-generalized hybrid mappings.

Corollary 10. *Let E be a uniformly convex Banach space with a Fréchet-differentiable norm and C be a nonempty subset of E. Let $M_1,M_2,\cdots,M_N$ be finite commutative generic 2-generalized nonspreading mappings of the nonempty subset C of E into itself such that $A_f(M_l)=B_f(M_l)\neq\emptyset$, for $l=1,2,\cdots,N$. Let $\mathcal{R}$ be the sunny generalized nonexpansive retraction of E onto $\cap_{l=1}^N B_f(M_l)$ and define $\{S_nx\}$ by*

$$S_nx=\frac{1}{(n+1)^N}\sum_{\mu_1=1}^{n}\sum_{\mu_2=1}^{n}\cdots\sum_{\mu_N=1}^{n}M_1^{\mu_1}M_2^{\mu_2}\cdots M_N^{\mu_N}x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2,\cdots,\mu_N)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

Corollary 11. *Let E be a real Hilbert space and $M_1, M_2, \cdots, M_N$ be finite commutative normally 2-generalized hybrid mappings of a nonempty subset C of E into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2, \cdots, N$. Let $\mathcal{R}$ be the metric projection of E onto $\cap_{l=1}^{N} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2,\cdots,\mu_N)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

As direct consequences of Theorem 1, Corollary 10 and Theorem 11, the following results can be obtained as corollaries. These results correspond to the ones in Ali and Haruna [15], Hojo and Takahashi [17], Hojo et al. [9] and Kondo and Takahashi [21].

Corollary 12 ([15])**.** *Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded sets. Let M_1, M_2 be two commutative generic 2-generalized Bregman nonspreading mappings of a nonempty subset C of $int(dom(f))$ into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2$. Let $\mathcal{R}$ be the sunny Bregman generalized nonexpansive retraction of E onto $\cap_{l=1}^{2} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{2} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

Corollary 13 ([17], Theorem 4.4)**.** *Let E be a uniformly convex Banach space with a Fréchet-differentiable norm and C be a nonempty subset of E. Let M_1, M_2 be two commutative generic 2-generalized nonspreading mappings of the nonempty subset C of E into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2$. Let $\mathcal{R}$ be the sunny generalized nonexpansive retraction of E onto $\cap_{l=1}^{2} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{2} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} x$.

Corollary 14 ([9], Theorem 3.2)**.** *Let E be a real Hilbert space and M_1, M_2 be two commutative normally 2-generalized hybrid mappings of a nonempty subset C of E into itself such that $A_f(M_l) \neq \emptyset$, for $l = 1, 2$. Let $\mathcal{R}$ be the metric projection of E onto $\cap_{l=1}^{2} A_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} M_1^{\mu_1} M_2^{\mu_2} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{2} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

In view of the fact that the generic 2-generalized Bregman nonspreading (simply nonspreading) mapping unifies the generic generalized Bregman nonspreading (simply nonspreading) mapping and the 2-generalized Bregman nonspreading (simply nonspreading) mapping, we can prove the following results as corollaries which correspond to the ones in Alsulami et al. [14], Lin et al. [11] and Takahashi et al. [12,13] when only one or two mappings are considered.

Corollary 15. *Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded sets. Let $M_1, M_2, \cdots, M_N$ be finite commutative generic generalized Bregman nonspreading mappings of a nonempty subset C of $int(dom(f))$ into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2, \cdots, N$. Let $\mathcal{R}$ be the sunny Bregman generalized nonexpansive retraction of E onto $\cap_{l=1}^{N} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2,\cdots,\mu_N)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

Corollary 16. *Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded sets. Let $M_1, M_2, \cdots, M_N$ be finite commutative 2-generalized Bregman nonspreading mappings of a nonempty subset C of $int(dom(f))$ into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2, \cdots, N$. Let $\mathcal{R}$ be the sunny Bregman generalized nonexpansive retraction of E onto $\cap_{l=1}^{N} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2,\cdots,\mu_N)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

Corollary 17 ([11,12])**.** *Let E be a uniformly convex Banach space with a Fréchet-differentiable norm and C be a nonempty subset of E. Let $M_1, M_2, \cdots, M_N$ be finite commutative generic generalized nonspreading mappings of the nonempty subset C of E into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$, for $l = 1, 2, \cdots, N$. Let $\mathcal{R}$ be the sunny generalized nonexpansive retraction of E onto $\cap_{l=1}^{N} B_f(M_l)$ and define $\{S_n x\}$ by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} A_f(M_l)$, where $q = \lim_{(\mu_1,\mu_2,\cdots,\mu_N)\in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to an element $q \in \cap_{l=1}^{N} F(M_l)$.

Corollary 18 ([13,14])**.** *Let E be a uniformly convex Banach space with a Fréchet-differentiable norm and C be a nonempty subset of E. Let $M_1, M_2, \cdots, M_N$ be finite commutative 2-generalized nonspreading mappings of the nonempty subset C of E into itself such that $A_f(M_l) = B_f(M_l) \neq \emptyset$,* *for* $l = 1, 2, \cdots, N$*. Let* $\mathcal{R}$ *be the sunny generalized nonexpansive retraction of* E *onto* $\cap_{l=1}^{N} B_f(M_l)$ *and define* $\{S_n x\}$ *by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ *and* $n \in \mathbb{N} \cup \{0\}$*. Then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} A_f(M_l)$*, where* $q = \lim_{(\mu_1, \mu_2, \cdots, \mu_N) \in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$*. Additionally, if* C *is closed and convex, then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} F(M_l)$*.*

In view of the fact that the class of normally 2-generalized hybrid mappings unifies those of normally generalized hybrid and 2-generalized hybrid mappings, these results correspond to the ones in [7,8] when only one or two mappings are considered.

Corollary 19 ([7])**.** *Let* E *be a real Hilbert space and* $M_1, M_2, \cdots, M_N$ *be finite commutative normally generalized hybrid mappings of a nonempty subset* C *of* E *into itself such that* $A_f(M_l) = B_f(M_l) \neq \emptyset$*, for* $l = 1, 2, \cdots, N$*. Let* $\mathcal{R}$ *be the metric projection of* E *onto* $\cap_{l=1}^{N} B_f(M_l)$ *and define* $\{S_n x\}$ *by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ *and* $n \in \mathbb{N} \cup \{0\}$*. Then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} A_f(M_l)$*, where* $q = \lim_{(\mu_1, \mu_2, \cdots, \mu_N) \in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$*. Additionally, if* C *is closed and convex, then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} F(M_l)$*.*

Corollary 20 ([8])**.** *Let* E *be a real Hilbert space and* $M_1, M_2, \cdots, M_N$ *be finite commutative 2-generalized hybrid mappings of a nonempty subset* C *of* E *into itself such that* $A_f(M_l) = B_f(M_l) \neq \emptyset$*, for* $l = 1, 2, \cdots, N$*. Let* $\mathcal{R}$ *be the metric projection of* E *onto* $\cap_{l=1}^{N} B_f(M_l)$ *and define* $\{S_n x\}$ *by*

$$S_n x = \frac{1}{(n+1)^N} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_N=1}^{n} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x,$$

for all $x \in C$ *and* $n \in \mathbb{N} \cup \{0\}$*. Then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} A_f(M_l)$*, where* $q = \lim_{(\mu_1, \mu_2, \cdots, \mu_N) \in D} \mathcal{R} M_1^{\mu_1} M_2^{\mu_2} \cdots M_N^{\mu_N} x$*. Additionally, if* C *is closed and convex, then* $\{S_n x\}$ *converges weakly to an element* $q \in \cap_{l=1}^{N} F(M_l)$*.*

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