mathematics

MDPI

Article

On a System of ψ-Caputo Hybrid Fractional Differential Equations with Dirichlet Boundary Conditions

Muath Awadalla [1,*], Kinda Abuasbeh [1], Muthaiah Subramanian [2] and Murugesan Manigandan [3]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia; kabuasbeh@kfu.edu.sa
[2] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641020, India; subramanian.m@kpriet.ac.in
[3] Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore 641020, India; yogimani22@rmv.ac.in
* Correspondence: mawadalla@kfu.edu.sa

Abstract: In this article, we investigate sufficient conditions for the existence and stability of solutions to a coupled system of ψ-Caputo hybrid fractional derivatives of order $1 < \upsilon \leq 2$ subjected to Dirichlet boundary conditions. We discuss the existence and uniqueness of solutions with the assistance of the Leray–Schauder alternative theorem and Banach's contraction principle. In addition, by using some mathematical techniques, we examine the stability results of Ulam–Hyers. Finally, we provide one example in order to show the validity of our results.

Keywords: ψ-Caputo fractional derivative; existence; fixed point theorems; Ulam–Hyers stability

MSC: 26A33; 34K37; 34A08

check for updates

Citation: Awadalla, M.; Abuasbeh, K.; Subramanian, M.; Manigandan, M. On a System of ψ-Caputo Hybrid Fractional Differential Equations with Dirichlet Boundary Conditions. *Mathematics* **2022**, *10*, 1681. https://doi.org/10.3390/math10101681

Academic Editor: Vilém Novák

Received: 31 March 2022
Accepted: 5 May 2022
Published: 13 May 2022

1. Introduction

Fractional calculus has a long history, going all the way back to Leibniz's 17th-century explanation of the derivative order in 1965. Mathematicians use fractional calculus to study how derivatives and integrals of noninteger order work and how they change over time. Since then, the new theory has proven to be very appealing to mathematicians, biologists, chemists, economists, engineers, and physicists. Subsequently, the subject attracted the interest of numerous famous mathematicians, including Fourier, Laplace, Abel, Liouville, Riemann, and Letnikov. For current and wide-ranging analyses of fractional derivatives and their applications, we recommend the monographs [1–4]. In [5], the authors investigated new results of the existence and uniqueness of systems of nonlinear coupled DEs and inclusions involving Caputo-type sequential derivatives of fractional order and new kinds of boundary conditions. In [6], the authors investigated a new type of SFDE and inclusions involving ψ-Hilfer fractional derivatives, associated with integral multi-point BCs.

Fractional derivatives have played a very important role in mathematical modeling in many diverse applied sciences, see [7]. In [8], the authors applied a new technique called "local fractional Laplace homotopy perturbation method" (LFLHPM) on Helmholtz and coupled Helmholtz equations to obtain analytical approximate solutions. In [9], the authors present a new analytical method called the "local fractional Laplace variational iteration method" (LFLVIM) for solving the two-dimensional Helmholtz and coupled Helmholtz equations. In [10], the authors find the solution of the LFFPE on the Cantor set. They make a comparison between the RDTM and LFSEM used in LFFPE. For example, the authors in [11] employed the LFLVIM and LFLDM to obtain approximate solutions for solving the damped wave equation and dissipative wave equation within LFDOs. The authors in [12] employed the fractional derivative of the ψ-Caputo type in modeling the logistic population equation, through which they were able to show that the model with the fractional derivative led to a better approximation of the variables than the classical model. In addition, the authors in [13] employ the fractional derivative of the ψ-Caputo type, and use the kernel Rayleigh, to improve the model again in modeling the logistic population equation. Various research has studied the existence and uniqueness of solutions to initial and boundary value problems utilizing ψ-fractional derivatives, see [14–18].

Fractional differential equations have been used to describe a wide variety of occurrences in a number of different engineering and scientific areas. Differential equations of fractional order are suitable for critical aspects in finance, electromagnetics, acoustics, viscoelasticity, biochemistry, and material science, see [19–21].

Additionally, it is essential to examine coupled systems through the use of fractional differential equations, as these systems are found in a wide range of applications. A number of scholars have also investigated coupled fractional differential equation systems. Some theoretical work on coupled fractional differential equations is included in this article, see [22–24].

The fractional derivatives of an unknown function are included in hybrid differential equations, as is the nonlinearity that relies on them. This class of equations arises in a wide variety of applications and physical science areas, for example, in the redirection of a bent pillar with a constant or variable cross-area, a three-layer shaft, electromagnetic waves, or gravity-driven streams. In the literature, hybrid FDEs have been examined by employing a variety of different forms of fractional derivatives; see [23,25,26]. Some recent results on the existence and uniqueness of initial and boundary value problems and Ulam–Hyers stability can be found in [27–29] and the references therein. For recent results from the ψ-Caputo hybrid fractional derivatives (CHFDs), we refer to [22,23,30,31] and the references cited therein. Choukri Derbazi et al. recently investigated the existence of extremal solutions to the nonlinear coupled system in [32]. Using the so-called "monotone iterative technique" together with the method of upper and lower solutions, the authors investigate the existence of extremal solutions of the following BVP that involves the ψ-Caputo derivative with ICs.

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{v,\psi}\varphi(\omega) = \mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega)), & \omega\in\mathcal{J} \qquad [a,b]; \\ {}^{C}\mathcal{D}_{a^+}^{v,\psi}\zeta(\omega) = \mathfrak{g}(\omega,\varphi(\omega),\zeta(\omega)), & \omega\in\mathcal{J} \qquad [a,b]; \\ \varphi(a) = \varphi_a \qquad \zeta(b) = \zeta_b, \end{cases}$$

where ${}^{C}\mathcal{D}_{a^+}^{v;\psi}$ denote the ψ-Caputo fractional derivatives (CFDs) of order v and $\mathfrak{f},\mathfrak{g}:[a,b]\times\mathcal{R}_e^2\to\mathcal{R}_e$ are continuous functions and $\varphi_a,\zeta_b\in\mathcal{R}_e$ with $\varphi_a\le\zeta_b$.

Mohamed I Abbas [30] investigated the uniqueness of solutions for the following coupled system of fractional differential equations (CSFDEs). Based on the Leray–Schauder alternative and Banach's fixed point theorem, the authors investigated the existence and uniqueness of the following BVP associated with four-point BCs.

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{v,\psi}\varphi(\omega) = \mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega)), & \omega\in[0,1] \qquad 1<v<2; \\ {}^{C}\mathcal{D}_{0^+}^{\beta,\psi}\zeta(\omega) = \mathfrak{g}(\omega,\varphi(\omega),\zeta(\omega)), & \omega\in[0,1] \qquad 1<\beta<2; \\ \varphi(0) = \zeta(0) = 0, \\ \varphi(1) = \lambda\varphi(\eta), \qquad \zeta(1) = \mu\zeta(\zeta), & 0<\eta<\zeta<1, \qquad \lambda,\mu>0, \end{cases}$$

where ${}^{C}\mathcal{D}_{0^+}^{v;\psi}, {}^{C}\mathcal{D}_{0^+}^{\beta;\psi}$ denote the ψ-CFDs of order v,β and $\mathfrak{f},\mathfrak{g}:[0,1]\times\mathcal{R}_e^2\to\mathcal{R}_e$ are continuous functions.

In 2020, the authors of [33] studied the existence and uniqueness of the following BVP associated with multi-point BCs, with results obtained via topological degree theory and Banach's contraction principle:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha;\psi}_{a^+} z(\tau) + h(\tau, z(\tau)) = 0, \ 2 < \alpha \leq 3, \ a \leq \tau \leq b, \\ z(a) = z'(a) = 0, \ z(b) = \sum_{k=1}^{n} \delta_k z(\mu_k), \ a < \mu_k < b, \end{cases}$$

where ${}^{C}\mathcal{D}^{\alpha;\psi}_{a^+}$ denotes ψ-Caputo fractional derivatives, $h : [a, b] \times \mathcal{R}_e \to \mathcal{R}_e$ is assumed to be continous and $\delta_k \in \mathcal{R}_e, k = 1, 2, \ldots, n$.

In previous works, researchers investigated the existence and uniqueness of linear fractional differential equations involving ψ-Caputo.

This work is devoted to investigating the existence and uniqueness of the solutions for the following system of equations with Dirichlet BCs. Adding to this, we show that BVP is stable via the Ulam–Hyers technique.

$$\begin{cases} {}^{C}\mathcal{D}^{v_1;\psi}\left(\dfrac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) = \mathfrak{h}_1(\omega, \varphi(\omega), \zeta(\omega)), \qquad \omega \in [\xi, \mathcal{T}] \qquad 1 < v_1 \leq 2; \\ {}^{C}\mathcal{D}^{v_2;\psi}\left(\dfrac{\zeta(\omega)}{\mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))}\right) = \mathfrak{h}_2(\omega, \varphi(\omega), \zeta(\omega)), \qquad \omega \in [\xi, \mathcal{T}] \qquad 1 < v_2 \leq 2; \\ \varphi(\xi) = \varphi(\mathcal{T}) = 0, \\ \zeta(\xi) = \zeta(\mathcal{T}) = 0, \end{cases} \tag{1}$$

where ${}^{C}\mathcal{D}^{v_i,\psi}_{0^+}, i = 1, 2.$ is the ψ-CFDs of order v_i, and $\mathfrak{f}, \mathfrak{g} : [\xi, \mathcal{T}] \times \mathcal{R}_e^2 \to \mathcal{R}_e$ are continuous functions. To be valuable, the findings of this paper must be novel and generalize several earlier findings that are important to the research. To the best of our knowledge, there are no articles that discuss boundary value problems for systems of fractional differential equations with ψ-Caputo and no articles that investigate Ulam–Hyers stability for differential equations that contain ψ-Caputo derivatives. This paper is organized as follows. In Section 2, we will briefly recall some basic definitions and some preliminary concepts about fractional calculus and auxiliary results used in the following sections. In Section 3, we establish the existence of solutions to the ψ-Caputo fractional hybrid differential equation by using the Leray–Schauder alternative and Banach's fixed point theorem. In Section 4, the stability of Ulam–Hyers solutions is shown. In Section 5, we finally give an example to illustrate the application of the results obtained and we give our conclusion in Section 6.

2. Preliminaries

There are some basic definitions, lemmas and results of the ψ-CFDs with regard to another function ([1–4]).

Definition 1. *Let $v > 0$, $\mathfrak{f} \in L'([\xi, \mathcal{T}], \mathcal{R}_e)$ and $\psi : [\xi, \mathcal{T}] \to \mathcal{R}_e$ such that $\psi'(\omega) > 0$ $\forall\, \omega \in [\xi, \mathcal{T}]$. The ψ-Riemann–Liouville fractional integral of order v for the function $\mathfrak{f}$ is given by*

$$\mathcal{I}^{v;\psi}_{\xi}\mathfrak{f}(\omega) = \frac{1}{\Gamma(v)} \int_{\xi}^{\omega} (\psi(\omega) - \psi(\mathfrak{s}))^{v-1} \mathfrak{f}(\mathfrak{s}) \psi'(\mathfrak{s}) \mathfrak{d}\mathfrak{s}, \tag{2}$$

where Γ denotes the standard Euler gamma function.

Definition 2. *Let $v > 0$, $\mathfrak{f} \in \mathcal{C}^{m-1}([\xi, \mathcal{T}], \mathcal{R}_e)$ and $\psi \in \mathcal{C}^{m}([\xi, \mathcal{T}], \mathcal{R}_e)$ such that $\phi'(\omega) > 0$ $\forall \omega \in ([\xi, \mathcal{T}], \mathcal{R}_e)$. The ψ-Caputo fractional derivative (CFD) of order v for the function $\mathfrak{f}$ is given by*

$${}^{C}\mathcal{D}^{v;\psi}_{\xi}\mathfrak{f}(\omega) = \frac{1}{\Gamma(n - v)} \int_{\xi}^{\omega} \psi'(\mathfrak{s})(\psi(\omega) - \psi(\mathfrak{s}))^{n-v-1} \mathfrak{f}^{[n]}_{\psi}(\mathfrak{s}) \mathfrak{d}\mathfrak{s}, \tag{3}$$

where

$$\mathfrak{f}^{[n]}_{\psi}(\mathfrak{s}) = \left(\frac{1}{\psi'(\mathfrak{s})} \frac{d}{ds}\right)^{n} \mathfrak{f}(\mathfrak{s}) \ \text{and} \ n = [v] + 1,$$

and $[v]$ denotes the integer part of the real number v.

Remark 1. *If $v \in (0,1)$, then Equation (3) can be written as follows:*

$$^{C}\mathcal{D}_{\xi}^{v;\psi}\mathfrak{f}(\omega) = \frac{1}{\Gamma(v)}\int_{\xi}^{\omega}(\psi(\omega)-\psi(\mathfrak{s}))^{v-1}\mathfrak{f}'(\mathfrak{s})\mathfrak{d}\mathfrak{s}.$$

In another way, we have

$$^{C}\mathcal{D}_{\xi}^{v;\psi}\mathfrak{f}(\omega) = \mathcal{I}^{1-v,\psi}\left(\frac{\mathfrak{f}'(\omega)}{\psi'(\omega)}\right).$$

Remark 2. *Note that if $\psi(\omega) = \omega$ and $\psi(\omega) = \log(\omega)$, then Equation (2) is reduced to the Riemann–Liouville and Hadamard fractional integrals, respectively.*

Remark 3. *In particular, note that if $\psi(\omega) = \omega$ and $\psi(\omega) = \log(\omega)$, then Equation (3) is reduced to the CFDs and Caputo–Hadamard fractional integrals, respectively.*

Definition 3. *Let $v > 0$ and an increasing function $\psi : [\xi, \mathcal{T}] \to \mathcal{R}_e$ satisfy $\psi'(\omega)\ 0$ for all $\omega \in [\xi, \mathcal{T}]$. We define the left-side ψ-Riemann–Liouville integral of an integrable function $\mathfrak{f}$ on $[\xi, \mathcal{T}]$ in the fractional framework with regard to another differentiable function ψ as*

$$({}_{\xi}\mathcal{I}^{v;\psi}\mathfrak{f})(\omega) = \frac{1}{\Gamma(v)}\int_{\xi}^{\omega}(\psi(\omega)-\psi(\mathfrak{s}))^{v-1}\mathfrak{f}(\mathfrak{s})\psi'(\mathfrak{s})\mathfrak{d}\mathfrak{s},$$

where Γ denotes the standard Euler gamma function.

Definition 4. *Let $m \in \mathbb{N}$ with $m = [v] + 1$. The left-sided ψ-Riemann–Liouville fractional derivative of an existing function $\mathfrak{f} \in \mathcal{C}^m([\xi, \mathcal{T}], \mathcal{R}_e)$ with regard to a nondecreasing function ψ such that $\psi'(\omega) = 0$, for all $\omega \in [\xi, \mathcal{T}]$ in the functional framework, is represented as follows:*

$$\begin{aligned}\mathcal{D}_{\xi^+}^{v;\psi}\mathfrak{f}(\omega) =&\left(\frac{1}{\psi'(\omega)}\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}\right)^{m}(\mathcal{I}_{\xi}^{m-v;\psi}\mathfrak{f})(\omega),\\ =&\frac{1}{\Gamma(m-v)}\left(\frac{1}{\psi'(\omega)}\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}\right)^{m}\int_{\xi}^{\omega}(\psi(\omega)-\psi(\mathfrak{s}))^{m-v-1}\mathfrak{f}(\mathfrak{s})\psi'(\mathfrak{s})\mathfrak{d}\mathfrak{s}.\end{aligned}$$

Definition 5. *Let $m \in \mathbb{N}$ with $m = [v] + 1$. The left-sided ψ-Caputo fractional derivative of an existing function $\mathfrak{f} \in \mathcal{C}^m([\xi, \mathcal{T}], \mathcal{R}_e)$ with regard to a nondecreasing function ψ such that $\psi'(\omega) = 0$, for all $\omega \in [\xi, \mathcal{T}]$ in the functional framework, is represented as follows:*

$$\begin{aligned}{}^{c}\mathcal{D}_{\xi^+}^{v;\psi}\mathfrak{f}(\omega) =&\mathcal{I}_{\xi^+}^{m-v;\psi}\left(\frac{1}{\psi'(\omega)}\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}\right)^{m}\mathfrak{f}(\omega),\\ =&\frac{1}{\Gamma(m-v)}\left(\frac{1}{\psi'(\omega)}\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}\right)^{m}\int_{\xi}^{\omega}(\psi(\omega)-\psi(\mathfrak{s}))^{m-v-1}\mathfrak{f}(\mathfrak{s})\psi'(\mathfrak{s})\mathfrak{d}\mathfrak{s}.\end{aligned}$$

Definition 6. *Let $\psi \in \mathcal{C}^n([\xi, \mathcal{T}])$ be such that $\psi'(\omega) > 0$ on $[\xi, \mathcal{T}]$. Then,*

$$\mathcal{AC}^{m;\psi}([\xi, \mathcal{T}]) = \left\{\mathfrak{f} : [\xi, \mathcal{T}] \to \mathbb{C} \ \text{and}\ \mathfrak{f}^{[m-1]} = \left(\frac{1}{\psi'(\omega)}\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}\right)^{m-1}\mathfrak{f}\right\}.$$

Proposition 1. *Let $v > 0$ and $\beta > 0$, then*

(1) $\mathcal{I}_{\xi^+}^{v;\psi}(\psi(\omega)-\psi(\xi))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(v+\beta)}(\psi(\omega)-\psi(\xi))^{v+\beta-1}$,

(2) $^{C}\mathcal{D}_{\xi^+}^{v;\psi}(\psi(\omega)-\psi(\xi))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-v)}(\psi(\omega)-\psi(\xi))^{\beta-v-1}$,

(3) $^{C}\mathcal{D}_{\xi^+}^{v;\psi}(\psi(\omega)-\psi(\xi))^{k} = 0$, *for any $k = 0, \ldots, m-1; m \in \mathbb{N}$.*

Proposition 2. *Let $v > 0$, if $\mathfrak{f} \in \mathcal{C}^{m-1}([\xi, \mathcal{T}], \mathcal{R}_e)$, then we have*

(1) ${}^{C}\mathcal{D}^{v,\psi}_{\xi^+}\mathcal{I}^{v,\psi}_{\xi^+}\mathfrak{f}(\omega) = \mathfrak{f}(\omega)$,

(2) $\mathcal{I}^{v,\psi}_{\xi^+}{}^{C}\mathcal{D}^{v,\psi}_{\xi^+}\mathfrak{f}(\omega) = \mathfrak{f}(\omega) - \sum_{J=0}^{n-1} \frac{\mathfrak{f}^{[k]}{}_{\psi}(0)}{k!}(\psi(\omega) - \psi(0))^k$.

(3) $\mathcal{I}^{v,\psi}_{\xi^+}$ *is linear and bounded from* $\mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e)$ *to* $\mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e)$.

Lemma 1 (Hybrid Fixed Point Theorem). *Let $\mathcal{X}$ be a convex, bounded and closed set contained in the Banach algebra $\mathcal{Y}$ and the operators $\mathcal{P}, \mathcal{S} : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ be such that:*

(1) *$\mathcal{P}$ and $\mathcal{S}$ are Lipschitz maps with Lipschitz constant $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{S}}$, respectively;*

(2) *$\mathcal{Q}$ is continuous and compact;*

(3) *$\varphi = \mathcal{P}\varphi\mathcal{Q}\zeta + \mathcal{S}\varphi \ \forall \ \zeta \in \mathcal{X} \implies \varphi \in \mathcal{X}$; and*

(4) *$\mathcal{L}_{\mathcal{P}}\mathcal{M}_{\mathcal{Q}} + \mathcal{L}_{\mathcal{S}} < 1$, where $\mathcal{M}_{\mathcal{Q}} = ||\mathcal{Q}(\mathcal{X})|| = \sup\{||\mathcal{Q}\varphi|| : \varphi \in \mathcal{X}\}$, then the operator equation $\varphi = \mathcal{P}\varphi + \mathcal{S}\varphi$ possesses a solution in $\mathcal{X}$.*

Theorem 1. *A contraction mapping $\mathcal{T} : \Omega \to \Omega$ possesses a unique fixed point where Ω is a nonempty closed set contained in a Banach space $\mathcal{Y}$.*

Theorem 2 (Banach Contraction Mapping Principle). *A contraction mapping on a complete metric space has exactly one fixed point.*

Theorem 3 (Arzelà–Ascoli Theorem). *A set of functions in $\mathcal{C}([a, b])$ with supremum norm is relatively compact if, and only if, it is uniformly bounded and equicontinuous on $[a, b]$.*

Before presenting our main results, the following auxiliary lemma is presented.

Lemma 2. *The solution of the following boundary value problem (BVP):*

$$\begin{cases} {}^{C}\mathcal{D}^{v_1;\psi}\left(\dfrac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) = \mathfrak{H}_1(\omega), \qquad \omega \in [\xi, \mathcal{T}] \qquad 1 < v_1 \le 2; \\ {}^{C}\mathcal{D}^{v_2;\psi}\left(\dfrac{\zeta(\omega)}{\mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))}\right) = \mathfrak{H}_2(\omega), \qquad \omega \in [\xi, \mathcal{T}] \qquad 1 < v_2 \le 2; \\ \varphi(\xi) = \varphi(\mathcal{T}) = 0, \\ \zeta(\xi) = \zeta(\mathcal{T}) = 0, \end{cases} \tag{4}$$

is given by

$$\begin{aligned} \varphi(\omega) =& \mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\varsigma)\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\varsigma)\mathfrak{d}\varsigma, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \zeta(\omega) =& \mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_2)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_2-1}\mathfrak{H}_2(\varsigma)\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_2)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_2-1}\mathfrak{H}_2(\varsigma)\mathfrak{d}\varsigma. \end{aligned} \tag{6}$$

Proof. First, we apply the fractional integral ${}^{\psi}\mathcal{I}^{v_1}_{\xi^+}$ to the equation

$${}^{C}\mathcal{D}^{v_1;\psi}\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) = \mathfrak{H}_1(\omega),$$

and we obtain

$$\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) = {}^{\psi}\mathcal{I}^{v_1}_{\xi^+}\mathfrak{H}_1(\omega) + \mathfrak{b}_0 + \mathfrak{b}_1(\psi(\omega) - \psi(\xi)), \tag{7}$$

the first boundary condition $\varphi(\xi) = 0$, which yields

$$\mathfrak{b}_0 = 0,$$

and the second boundary condition $\varphi(\mathcal{T})$, which implies

$$\mathfrak{b}_1 = \frac{-{}^{\psi}\mathcal{I}_{\xi^+}^{v_1}\mathfrak{H}_1(\mathcal{T})}{(\psi(\omega) - \psi(\xi))}.$$

Substituting the obtained values of $\mathfrak{b}_0$ and $\mathfrak{b}_1$ in Equation (7), we have

$$\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) = {}^{\psi}\mathcal{I}_{\xi^+}^{v_1}\mathfrak{H}_1(\omega) - \frac{1}{(\psi(\omega) - \psi(\xi))}{}^{\psi}\mathcal{I}_{\xi^+}^{v_1}\mathfrak{H}_1(\mathcal{T}),$$

$$\begin{aligned}\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) =& \left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega}\psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\varsigma)\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\varsigma)\mathfrak{d}\varsigma,\end{aligned}$$

which completes the proof. □

3. Main Result

Defining the space $\mathcal{B} = \{(\varphi(\omega), \zeta(\omega)) : (\varphi, \zeta) \in \mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e) \times \mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e)\}$, it is obvious that $\mathcal{B}$ is a Banach space. Furthermore, this space is endowed with the norm

$$||(\varphi, \zeta)||_{\mathcal{B}} = ||\varphi|| + ||\zeta|| \;\; \forall(\varphi, \zeta) \in \mathcal{B}.$$

By Lemma 2, we define an operator $\Phi : \mathcal{B} \to \mathcal{B}$ as

$$\Phi(\varphi, \zeta)(\omega) = \begin{cases}\Phi_1(\varphi, \zeta)(\omega), \\ \Phi_2(\varphi, \zeta)(\omega),\end{cases} \tag{8}$$

where

$$\begin{aligned}&\Phi_1((\varphi, \zeta)(\omega)) \\ &= \mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega}\psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\Phi_2((\varphi, \zeta)(\omega)) \\ &= \mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_2)}\int_{\xi}^{\omega}\psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_2-1}\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_2)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_2-1}\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma.\end{aligned} \tag{10}$$

Now, let us assume that the following assumptions hold true:

$(\mathcal{A}_1)$ φ, ζ are assumed to be continuous and bounded, and there exist $\partial_{\mathfrak{f}}, \partial_{\mathfrak{g}} > 0$ such that

$$|\mathfrak{f}(\omega, \varphi, \zeta)| \leq \partial_{\mathfrak{f}}, \;\text{ and }\; |\mathfrak{g}(\omega, \varphi, \zeta)| \leq \partial_{\mathfrak{g}}, \forall(\omega, \varphi, \zeta) \in [\xi, \mathcal{T}] \times \mathcal{R}_e^2.$$

($\mathcal{A}_2$) Both $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are assumed to be continuous and there exist $\delta_i, \varepsilon_i > 0, i = 1,2$ such that

$$
\begin{aligned}
|\mathfrak{H}_1(\omega, \varphi_1, \zeta_1) - \mathfrak{H}_1(\omega, \varphi_2, \zeta_2)| \le& \delta_1|\varphi_1 - \varphi_2| + \delta_2|\zeta_1 - \zeta_2|,\\
|\mathfrak{H}_2(\omega, \varphi_1, \zeta_1) - \mathfrak{H}_2(\omega, \varphi_2, \zeta_2)| \le& \varepsilon_1|\varphi_1 - \varphi_2| + \varepsilon_2|\zeta_1 - \zeta_2|,
\end{aligned}
$$

$\forall\ \omega \in [\xi, \mathcal{T}], \varphi_i, \zeta_i \in \mathcal{R}_e, i = 1,2.$

($\mathcal{A}_3$) There exist $\lambda_0, \mu_0 > 0$, and $\lambda_i, \mu_i \le 0, i = 1,2$ such that

$$
\begin{aligned}
|\mathfrak{H}_1(\omega, \varphi, \zeta)| \le& \lambda_0 + \lambda_1|\varphi| + \lambda_2|\zeta|,\\
|\mathfrak{H}_2(\omega, \varphi, \zeta)| \le& \mu_0 + \mu_1|\varphi| + \mu_2|\zeta|,\ \ \forall \omega \in [\xi, \mathcal{T}], \varphi, \zeta \in \mathcal{R}_e.
\end{aligned}
$$

($\mathcal{A}_4$) Let $\mathcal{S} \subset \mathcal{B}$ be a bounded set, then there exist $\mathscr{K}_i > 0, i = 1,2$ such that

$$
\begin{aligned}
|\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))| \le& \mathscr{K}_1,\\
|\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))| \le& \mathscr{K}_2,\ \ \forall\ \omega \in [\xi, \mathcal{T}],\ \ \forall \varphi, \zeta \in \mathcal{S}.
\end{aligned}
$$

Using $\mathcal{A}_4$, observe that $\forall\ \ i = 1,2.$

$$
\begin{aligned}
&\Bigg|\left(\frac{1}{\Gamma(v_i)} \int_\xi^\omega \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_i - 1}\mathfrak{H}_i(\varsigma, \varphi(\varsigma), \zeta(\varsigma))\mathfrak{d}\varsigma\right)\\
&-\frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_i)(\psi(\mathcal{T}) - \psi(\xi))} \int_\xi^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_i - 1}\mathfrak{H}_1(\varsigma, \varphi(\varsigma), \zeta(\varsigma))\mathfrak{d}\varsigma\Bigg|,\\
\le& \frac{2\mathscr{K}_i(\psi(\mathcal{T}) - \psi(\xi))^{v_i}}{\Gamma(v_i + 1)}.
\end{aligned}
$$

For computational convenience, we let

$$
\mathcal{L}_i = \frac{(\psi(\mathcal{T}) - \psi(\xi))^{v_i}}{\Gamma(v_i + 1)}. \tag{11}
$$

Next, we introduce our main result by setting two theorems with their proofs.

Theorem 4. *If the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ hold, and*

$$
\mathcal{P} = 2(\partial_{\mathfrak{f}}\mathcal{L}_1(\delta_1 + \delta_2) + \partial_{\mathfrak{g}}\mathcal{L}_2(\varepsilon_1 + \varepsilon_2)) < 1, \tag{12}
$$

then the BVP in (1) has a unique solution on $[\xi, \mathcal{T}]$.

Proof. Considering the operator given by (1), let

$$
\hat{\mathcal{B}}_{\mathfrak{r}} = \{(\varphi, \zeta) \in \mathcal{B} : ||(\varphi, \zeta)|| \le \mathfrak{r}\}
$$

be closed ball in $\mathcal{B}$ with

$$
\mathfrak{r} \ge \frac{2\big(\partial_{\mathfrak{f}}\mathcal{L}_1(\mathcal{N}_{\mathfrak{H}_1}) + \partial_{\mathfrak{g}}\mathcal{L}_2(\mathcal{N}_{\mathfrak{H}_2})\big)}{1 - [2(\partial_{\mathfrak{f}}\mathcal{L}_1(\delta_1 + \delta_2) + \partial_{\mathfrak{g}}\mathcal{L}_2(\varepsilon_1 + \varepsilon_2))]},
$$

where

$$
\mathcal{N}_{\mathfrak{H}_1} = \sup_{\xi \le \omega \le \mathcal{T}} |\mathfrak{H}_1(\omega, 0, 0)| \text{ and } \mathcal{N}_{\mathfrak{H}_2} = \sup_{\xi \le \omega \le \mathcal{T}} |\mathfrak{H}_2(\omega, 0, 0)|.
$$

Observe that

$$
\begin{aligned}
|\mathfrak{H}_1(\omega, \varphi, \zeta)| =& |\mathfrak{H}_1(\omega, \varphi, \zeta) - \mathfrak{H}_1(\omega, 0, 0) + \mathfrak{H}_1(\omega, 0, 0)|,\\
\le& \delta_1\|\varphi\| + \delta_2\|\zeta\| + \mathcal{N}_{\mathfrak{H}_1},
\end{aligned}
$$

$$\leq (\delta_1+\delta_2)\mathfrak{r}+\mathcal{N}_{\mathfrak{H}_1}.$$

Now, we demonstrate that $\Phi\hat{\mathcal{B}}_{\mathfrak{r}} \subset \hat{\mathcal{B}}_{\mathfrak{r}}, \forall\,(\varphi,\zeta)\in\hat{\mathcal{B}}_{\mathfrak{r}}, \omega\in[\xi,\mathcal{T}]$, then

$$\begin{aligned}
&|\Phi_1((\varphi,\zeta)(\omega))|\\
=&\bigg|\mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega))\bigg(\frac{1}{\Gamma(v_1)}\int_\xi^\omega \psi'(\varsigma)(\psi(\omega)-\psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))\mathfrak{d}\varsigma\bigg)\\
&-\frac{(\psi(\omega)-\psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_\xi^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))\mathfrak{d}\varsigma\bigg|,\\
\leq&\partial_{\mathfrak{f}}\sup_{\xi\leq\omega\leq\mathcal{T}}\bigg\{\bigg(\frac{1}{\Gamma(v_1)}\int_\xi^\omega \psi'(\varsigma)(\psi(\omega)-\psi(\varsigma))^{v_1-1}|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\mathfrak{d}\varsigma\bigg)\\
&+\frac{(\psi(\omega)-\psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_\xi^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\mathfrak{d}\varsigma\bigg\},\\
\leq&\partial_{\mathfrak{f}}\big((\delta_1+\delta_2)\mathfrak{r}+\mathcal{N}_{\mathfrak{H}_1}\big)\sup_{\xi\leq\omega\leq\mathcal{T}}\bigg\{\bigg(\frac{1}{\Gamma(v_1)}\int_\xi^\omega \psi'(\varsigma)(\psi(\omega)-\psi(\varsigma))^{v_1-1}\mathfrak{d}\varsigma\bigg)\\
&+\frac{(\psi(\omega)-\psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_\xi^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}\mathfrak{d}\varsigma\bigg\},
\end{aligned}$$

$$|\Phi_1((\varphi,\zeta)(\omega))| \leq\partial_{\mathfrak{f}}(2\mathcal{L}_1)\big[(\delta_1+\delta_2)\mathfrak{r}+\mathcal{N}_{\mathfrak{H}_1}\big],$$

and

$$\|\Phi_1((\varphi,\zeta)(\omega))\| \leq \partial_{\mathfrak{f}}(2\mathcal{L}_1)\big[(\delta_1+\delta_2)\mathfrak{r}+\mathcal{N}_{\mathfrak{H}_1}\big], \tag{13}$$

similarly,

$$\|\Phi_2((\varphi,\zeta)(\omega))\| \leq \partial_{\mathfrak{g}}(2\mathcal{L}_1)\big[(\varepsilon_1+\varepsilon_2)\mathfrak{r}+\mathcal{N}_{\mathfrak{H}_2}\big]. \tag{14}$$

Equations (13) and (14) yield

$$||\Phi(\varphi,\zeta)|| \leq \mathfrak{r}.$$

Next, we show that Φ is a contraction. Let $(\varphi_1,\zeta_1),(\varphi_2,\zeta_2)\in\mathcal{B}$, then

$$\begin{aligned}
&|\Phi_1(\varphi_1,\zeta_1)(\omega)-\Phi_1(\varphi_2,\zeta_2)(\omega)|\\
\leq&\partial_{\mathfrak{f}}\sup_{\xi\leq\omega\leq\mathcal{T}}\bigg\{\bigg(\frac{1}{\Gamma(v_1)}\int_\xi^\omega \psi'(\varsigma)(\psi(\omega)-\psi(\varsigma))^{v_1-1}\\
&|\mathfrak{H}_1(\omega,\varphi_1(\omega),\zeta_1(\omega))-\mathfrak{H}_1(\omega,\varphi_2(\omega),\zeta_2(\omega))|\mathfrak{d}\varsigma\bigg)\\
&+\frac{(\psi(\omega)-\psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_\xi^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}\\
&|\mathfrak{H}_1(\omega,\varphi_1(\omega),\zeta_1(\omega))-\mathfrak{H}_1(\omega,\varphi_2(\omega),\zeta_2(\omega))|\mathfrak{d}\varsigma,\},\\
\leq&\partial_{\mathfrak{f}}(\delta_1||\varphi_1-\varphi_2||+\delta_2||\zeta_1-\zeta_2||)(2\mathcal{L}_1),
\end{aligned}$$

$$\begin{aligned}
&||\Phi_1(\varphi_1,\zeta_1)-\Phi_1(\varphi_2,\zeta_2)||\\
&\leq \partial_{\mathfrak{f}}(\delta_1+\delta_2)(||\varphi_1-\varphi_2||+||\zeta_1-\zeta_2||)(2\mathcal{L}_1).
\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}
&||\Phi_2(\varphi_1,\zeta_1)-\Phi_2(\varphi_2,\zeta_2)||\\
&\leq \partial_{\mathfrak{g}}(\varepsilon_1+\varepsilon_2)(||\varphi_1-\varphi_2||+||\zeta_1-\zeta_2||)(2\mathcal{L}_2).
\end{aligned} \tag{16}$$

Equations (15) and (16) give

$$\begin{aligned}
&||\Phi(\varphi_1,\zeta_1)-\Phi(\varphi_2,\zeta_2)|| \\
&\leq [\partial_{\mathfrak{f}}(2\mathcal{L}_1)(\delta_1+\delta_2)+\partial_{\mathfrak{g}}(2\mathcal{L}_2)(\varepsilon_1+\varepsilon_2)](||\varphi_1-\varphi_2||+||\zeta_1-\zeta_2||) \\
&\leq ||\varphi_1-\varphi_2||+||\zeta_1-\zeta_2||.
\end{aligned} \tag{17}$$

Operator Φ is a contraction, and the Banach contraction mapping principle applies, that is, on $[\xi,\mathcal{T}]$, the BVP (1) has a unique solution. □

Theorem 5. *If $(\mathcal{A}_1),(\mathcal{A}_3)$ and $(\mathcal{A}_4)$ are satisfied, and if*

$$2(\partial_{\mathfrak{f}}\mathcal{L}_1\lambda_1+\partial_{\mathfrak{g}}\mathcal{L}_2\mu_1)<1$$

and

$$2(\partial_{\mathfrak{f}}\mathcal{L}_1\lambda_2+\partial_{\mathfrak{g}}\mathcal{L}_2\mu_2)<1,$$

then the proposed problem given by (1) has at least one solution on $[\xi,\mathcal{T}]$.

Proof. To begin, we show that Φ is (c.c), if $\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{f}$ and $\mathfrak{g}$ are both continuous, which implies that Φ is continuous.

By $\mathcal{A}_4$, for any $(\mathfrak{f},\mathfrak{g})\in\mathcal{S}$, we have

$$\begin{aligned}
&|\Phi_1((\varphi,\zeta)(\omega))| \\
&\leq \partial_{\mathfrak{f}}\sup_{\xi\leq\omega\leq\mathcal{T}}\left\{\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega}\psi'(\varsigma)(\psi(\omega)-\psi(\varsigma))^{v_1-1}|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\partial\varsigma\right)\right. \\
&\left.+\frac{(\psi(\omega)-\psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\partial\varsigma\right\},
\end{aligned}$$

that is

$$\|\Phi_1(\varphi,\zeta)\|\leq\partial_{\mathfrak{f}}(2\mathcal{L}_1)\mathscr{K}_1, \tag{18}$$

similarly,

$$\|\Phi_2(\varphi,\zeta)\|\leq\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mathscr{K}_2, \tag{19}$$

and, from (18) and (19), we obtain

$$\|\Phi(\varphi,\zeta)\|\leq\partial_{\mathfrak{f}}(2\mathcal{L}_1)\mathscr{K}_1+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mathscr{K}_2, \tag{20}$$

which implies that our operator Φ is uniformly bounded.

Next, we investigate the equicontinuity of our operator to see this, $\forall\omega_1,\omega_2\in[\xi,\mathcal{T}]$ with $\omega_1<\omega_2, i=1,2$. We have

$$\begin{aligned}
&|\Phi_1((\varphi,\zeta)(\omega_2))-\Phi_1((\varphi,\zeta)(\omega_1))| \\
&\leq\partial_{\mathfrak{f}}\left\{\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega_1}\psi'(\varsigma)\left[(\psi(\omega_1)-\psi(\varsigma))^{v_1-1}-(\psi(\omega_2)-\psi(\varsigma))^{v_1-1}\right]|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\partial\varsigma\right)\right. \\
&+\left(\frac{1}{\Gamma(v_1)}\int_{\omega_1}^{\omega_2}\psi'(\varsigma)\left[(\psi(\omega_2)-\psi(\varsigma))^{v_1-1}\right]|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\partial\varsigma\right) \\
&\left.+\frac{(\psi(\omega_2)-\psi(\omega_1))}{\Gamma(v_1)(\psi(\mathcal{T})-\psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}|\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|\partial\varsigma\right\}, \\
&\leq\frac{\partial_{\mathfrak{f}}\mathscr{K}_1}{\Gamma(v_1)}\left\{\left(\int_{\xi}^{\omega_1}\psi'(\varsigma)\left[(\psi(\omega_1)-\psi(\varsigma))^{v_1-1}-(\psi(\omega_2)-\psi(\varsigma))^{v_1-1}\right]\partial\varsigma\right)\right. \\
&+\left(\int_{\omega_1}^{\omega_2}\psi'(\varsigma)\left[(\psi(\omega_2)-\psi(\varsigma))^{v_1-1}\right]\partial\varsigma\right) \\
&\left.+\frac{(\psi(\omega_2)-\psi(\omega_1))}{(\psi(\mathcal{T})-\psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_1-1}\partial\varsigma\right\},
\end{aligned}$$

and

$$\begin{aligned}
&|\Phi_2((\varphi,\zeta)(\omega_2)) - \Phi_2((\varphi,\zeta)(\omega_1))| \\
\leq&\partial_{\mathfrak{g}}\Bigg\{\left(\frac{1}{\Gamma(v_2)}\int_{\xi}^{\omega_1}\psi'(\varsigma)\left[(\psi(\omega_1)-\psi(\varsigma))^{v_2-1}-(\psi(\omega_2)-\psi(\varsigma))^{v_2-1}\right]|\mathfrak{H}_2(\omega,\varphi(\omega),\zeta(\omega))|\mathfrak{d}\varsigma\right) \\
&+\left(\frac{1}{\Gamma(v_2)}\int_{\omega_1}^{\omega_2}\psi'(\varsigma)\left[(\psi(\omega_2)-\psi(\varsigma))^{v_2-1}\right]|\mathfrak{H}_2(\omega,\varphi(\omega),\zeta(\omega))|\mathfrak{d}\varsigma\right) \\
&+\frac{(\psi(\omega_2)-\psi(\omega_1))}{\Gamma(v_2)(\psi(\mathcal{T})-\psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_2-1}|\mathfrak{H}_2(\omega,\varphi(\omega),\zeta(\omega))|\mathfrak{d}\varsigma\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\leq&\frac{\partial_{\mathfrak{g}}\mathscr{K}_2}{\Gamma(v_2)}\Bigg\{\left(\int_{\xi}^{\omega_1}\psi'(\varsigma)\left[(\psi(\omega_1)-\psi(\varsigma))^{v_2-1}-(\psi(\omega_2)-\psi(\varsigma))^{v_1-1}\right]\mathfrak{d}\varsigma\right) \\
&+\left(\int_{\omega_1}^{\omega_2}\psi'(\varsigma)\left[(\psi(\omega_2)-\psi(\varsigma))^{v_2-1}\right]\mathfrak{d}\varsigma\right) \\
&+\frac{(\psi(\omega_2)-\psi(\omega_1))}{(\psi(\mathcal{T})-\psi(\xi))}\int_{\xi}^{\mathcal{T}}\psi'(\varsigma)(\psi(\mathcal{T})-\psi(\varsigma))^{v_2-1}\mathfrak{d}\varsigma\Bigg\}.
\end{aligned}$$

Note that the above inequality approaches zero and is independent of $(\mathfrak{f},\mathfrak{g})$, that is, Φ is equicontinuous. Finally, we let $\Delta = \{(\varphi,\zeta)\in\mathcal{B} : (\varphi,\zeta) = \mathfrak{r}\Phi(\varphi,\zeta), \mathfrak{r}\in[0,1]\}\forall\,\omega\in[0,1]$ and we obtain $\varphi(\omega) = \mathfrak{r}\Phi_1(\varphi,\zeta)(\omega)$ and $\zeta(\omega) = \mathfrak{r}\Phi_2(\varphi,\zeta)(\omega)$. By $(\mathcal{A}_3)$, we obtain

$$||\varphi|| \leq \partial_{\mathfrak{f}}(2\mathcal{L}_1)(\lambda_0+\lambda_1||\varphi||+\lambda_2||\zeta||) \tag{21}$$
$$||\zeta|| \leq \partial_{\mathfrak{g}}(2\mathcal{L}_2)(\mu_0+\mu_1||\varphi||+\mu_2||\zeta||), \tag{22}$$

and adding (21) and (22), we obtain

$$\begin{aligned}
||\varphi||+||\zeta|| \leq&(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_0+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_0) \\
&+(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_1+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_1)||\varphi|| \\
&+(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_2+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_2)||\zeta||.
\end{aligned} \tag{23}$$

Equation (23) can be rewritten as

$$||(\varphi,\zeta)|| \leq \frac{(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_0+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_0)}{\min\{1-(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_1+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_1),1-(\partial_{\mathfrak{f}}(2\mathcal{L}_1)\lambda_2+\partial_{\mathfrak{g}}(2\mathcal{L}_2)\mu_2)\}}, \tag{24}$$

which shows that the defined subset Δ is bounded. Now, applying the Leray–Schauder alternative, the problem (1) has at least one solution on $[\xi,\mathcal{T}]$. □

4. Ulam–Hyers Stability

This section is devoted to the investigation of Hyers–Ulam stability for our system. Consider the following equations:

$$\begin{aligned}
\varphi(\omega) =&\Phi_1(\varphi,\zeta)(\omega), \\
\zeta(\omega) =&\Phi_2(\varphi,\zeta)(\omega),
\end{aligned} \tag{25}$$

where Φ_1 and Φ_2 are given by (9) and (10), respectively. Consider the following definitions of nonlinear operators $\mathfrak{h}_1,\mathfrak{h}_2\in\mathcal{C}([\xi,\mathcal{T}],\mathcal{R}_e)\times\mathcal{C}([\xi,\mathcal{T}],\mathcal{R}_e)\to\mathcal{C}([\xi,\mathcal{T}],\mathcal{R}_e)$:

$$\begin{aligned}
&{}^{C}\mathcal{D}^{v_1;\psi}\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega))}\right)-\mathfrak{h}_1(\omega,\varphi(\omega),\zeta(\omega)) = \mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega)), \qquad \omega\in[a,\mathcal{T}], \\
&{}^{C}\mathcal{D}^{v_2;\psi}\left(\frac{\zeta(\omega)}{\mathfrak{g}(\omega,\varphi(\omega),\zeta(\omega))}\right)-\mathfrak{h}_2(\omega,\varphi(\omega),\zeta(\omega)) = \mathfrak{H}_2(\omega,\varphi(\omega),\zeta(\omega)), \qquad \omega\in[a,\mathcal{T}].
\end{aligned}$$

Considering the following inequalities for some $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$,

$$||\mathfrak{H}_1(\omega,\varphi(\omega),\zeta(\omega))|| \leq\hat{\Lambda}_1, \tag{26}$$

$$||\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))|| \leq \hat{\Lambda}_2.$$

Definition 7. *The coupled system 1 is said to have Hyers–Ulam stability, if there exist $\mathcal{M}_1, \mathcal{M}_2 > 0$, showing that, for every solution $(\varphi', \zeta') \in \mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e) \times \mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e)$ of the inequalities (26),*

$$\begin{aligned} ||\varphi(\omega) - \varphi'(\omega)|| &\leq \mathcal{M}_1 \hat{\Lambda}_1, \\ ||\zeta(\omega) - \zeta'(\omega)|| &\leq \mathcal{M}_2 \hat{\Lambda}_2, \text{ and } \omega \in [\xi, \mathcal{T}]. \end{aligned}$$

Theorem 6. *If all conditions of Theorem 4 are satisfied, the CSFDEs given by (1) are U-H stable.*

Proof. Let $\mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e) \times \mathcal{C}([\xi, \mathcal{T}], \mathcal{R}_e)$ be the solution to (1).

Let (φ, ζ) be any solution that meets the condition (26):

$$\begin{aligned} {}^{C}\mathcal{D}^{v_1;\psi}\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))}\right) &= \mathfrak{h}_1(\omega, \varphi(\omega), \zeta(\omega)) + \mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega)), \qquad \omega \in [\xi, \mathcal{T}], \\ {}^{C}\mathcal{D}^{v_2;\psi}\left(\frac{\zeta(\omega)}{\mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))}\right) &= \mathfrak{h}_2(\omega, \varphi(\omega), \zeta(\omega)) + \mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega)), \qquad \omega \in [\xi, \mathcal{T}], \end{aligned}$$

so,

$$\begin{aligned} &\varphi(\omega) \\ &= \varphi'(\omega) + \mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_1 - 1}\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_1 - 1}\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma, \end{aligned}$$

$$\begin{aligned} &|\varphi(\omega) - \varphi'(\omega)| \\ &= \mathfrak{f}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_1)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_1 - 1}|\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))|\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_1)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_1 - 1}|\mathfrak{H}_1(\omega, \varphi(\omega), \zeta(\omega))|\mathfrak{d}\varsigma, \\ &\leq \frac{(\psi(\mathcal{T}) - \psi(\xi))^{v_1}}{\Gamma(v_1 + 1)}\hat{\Lambda}_1 \\ &\leq \mathcal{L}_1 \hat{\Lambda}_1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\zeta(\omega) \\ &= \zeta'(\omega) + \mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_2)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_2 - 1}\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_2)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_2 - 1}\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma, \end{aligned}$$

$$\begin{aligned} &|\zeta(\omega) - \zeta'(\omega)| \\ &= \mathfrak{g}(\omega, \varphi(\omega), \zeta(\omega))\left(\frac{1}{\Gamma(v_2)}\int_{\xi}^{\omega} \psi'(\varsigma)(\psi(\omega) - \psi(\varsigma))^{v_2 - 1}\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))\mathfrak{d}\varsigma\right) \\ &- \frac{(\psi(\omega) - \psi(\xi))}{\Gamma(v_2)(\psi(\mathcal{T}) - \psi(\xi))}\int_{\xi}^{\mathcal{T}} \psi'(\varsigma)(\psi(\mathcal{T}) - \psi(\varsigma))^{v_2 - 1}|\mathfrak{H}_2(\omega, \varphi(\omega), \zeta(\omega))|\mathfrak{d}\varsigma, \\ &\leq \frac{(\psi(\mathcal{T}) - \psi(\xi))^{v_2}}{\Gamma(v_2 + 1)}\hat{\Lambda}_2 \\ &\leq \mathcal{L}_2 \hat{\Lambda}_2, \end{aligned} \tag{28}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (11). Hence, Definition (7) is verified, with the help of (27) and (28). Hence, the problem (1) is Ulam–Hyers stable. □

5. Example

Example 1. *Let us consider the following CSFDEs:*

$$\begin{cases} {}^{C}\mathcal{D}^{v_1;\psi}\left(\frac{\varphi(\omega)}{\mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega))}\right) = \mathfrak{h}_1(\omega,\varphi(\omega),\zeta(\omega)), & \omega \in [\xi,\mathcal{T}] \quad 1 < v_1 \leq 2; \\ {}^{C}\mathcal{D}^{v_2;\psi}\left(\frac{\zeta(\omega)}{\mathfrak{g}(\omega,\varphi(\omega),\zeta(\omega))}\right) = \mathfrak{h}_2(\omega,\varphi(\omega),\zeta(\omega)), & \omega \in [\xi,\mathcal{T}] \quad 1 < v_2 \leq 2; \\ \varphi(\xi) = \varphi(\mathcal{T}) = 0; \\ \zeta(\xi) = \zeta(\mathcal{T}) = 0. \end{cases} \tag{29}$$

The problem (29) has a coupled system of hybrid FDEs (1), where $v_1 = \frac{1}{2}, v_2 = \frac{1}{3}$, $\mathcal{T} = 1, \psi(\omega) = \omega, \xi = 0$. To prove Theorem 4 , let $\omega \in [\xi,\mathcal{T}]$ and $\varphi,\zeta \in \mathcal{R}_e$, then we have

$$\begin{aligned} \mathfrak{h}_1(\omega,\varphi(\omega),\zeta(\omega)) &= \frac{1}{99}\left(\frac{\omega\zeta(\omega)}{2+\zeta(\omega)} - \frac{\zeta(\omega)}{2+\zeta(\omega)}\right), \\ \mathfrak{h}_2(\omega,\varphi(\omega),\zeta(\omega)) &= \frac{e^{-\omega}}{87}\left(\frac{\omega^2 - \varphi(\omega)\zeta(\omega)}{2+\zeta(\omega)\varphi(\omega)}\right), \\ \mathfrak{f}(\omega,\varphi(\omega),\zeta(\omega)) &= \frac{1}{99}\left(\frac{\omega\zeta(\omega)}{3} + \frac{\omega\varphi(\omega)}{2} + \frac{5}{6}\right), \\ \mathfrak{g}(\omega,\varphi(\omega),\zeta(\omega)) &= \frac{1}{98}\left(\frac{\zeta(\omega)}{5} + \omega\varphi(\omega) + 6\right), \end{aligned}$$

$$\begin{aligned} |\mathfrak{f}(\omega,\varphi,\zeta)| \leq \frac{2}{97}, |\mathfrak{g}(\omega,\varphi,\zeta)| &\leq \frac{1}{87}, \\ |\mathfrak{h}_1(\omega,\varphi,\zeta) - \mathfrak{h}_1(\omega,\hat{\varphi},\hat{\zeta}| &\leq \frac{1}{99}\{|\varphi-\hat{\varphi}| + |\zeta-\hat{\zeta}|\}, \\ |\mathfrak{h}_2(\omega,\varphi,\zeta) - \mathfrak{h}_2(\omega,\hat{\varphi},\hat{\zeta}| &\leq \frac{1}{98}\{|\varphi-\hat{\varphi}| + |\zeta-\hat{\zeta}|\}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathcal{L}_1 = 1.183791995, \mathcal{L}_2 &= 1.1193470177, \\ \partial_{\mathfrak{f}} = 0.02061855676, \partial_{\mathfrak{g}} &= 0.0114942528, \\ \delta_i = 0.01010101, \varepsilon_i &= 0.010200816, \end{aligned} \tag{30}$$

as $i = 1,2$. We substitute values in Equation (12), and we obtain

$$2(\partial_{\mathfrak{f}}\mathcal{L}_1(\delta_1+\delta_2) + \partial_{\mathfrak{g}}\mathcal{L}_2(\varepsilon_1+\varepsilon_2)) \approx 0.0014651668 < 1.$$

Based on the computations mentioned above, all conditions of Theorem 4 are satisfied. Therefore, the BVP given by (29) guaranteed a unique solution on $[\xi,\mathcal{T}]$ (Table 1 and Figure 1) .

Table 1. The impact of fractional order (v) on the condition $\mathcal{P}$ given by (12).

$\mathcal{T}$	$v = 0.15$	$v = 0.30$	$v = 0.45$	$v = 0.60$	$v = 0.75$	$v = 0.90$
			$\mathcal{P}$			
0.3	0.00169289	0.00164105	0.00156982	0.00148344	0.00138603	0.00128146
1.3	0.00338579	0.00364173	0.00386534	0.00405286	0.00420167	0.00431029
2.3	0.00507868	0.00580513	0.00654797	0.00729616	0.00803837	0.00876327
4.3	0.00677158	0.0080815	0.00951759	0.0110728	0.0127371	0.014498
5.3	0.00677158	0.0104457	0.0127207	0.0153029	0.0182023	0.0214241
6.3	0.0101574	0.0128824	0.016123	0.0199337	0.0243678	0.029476
7.3	0.0118503	0.015381	0.0197005	0.0249265	0.0311839	0.0386034

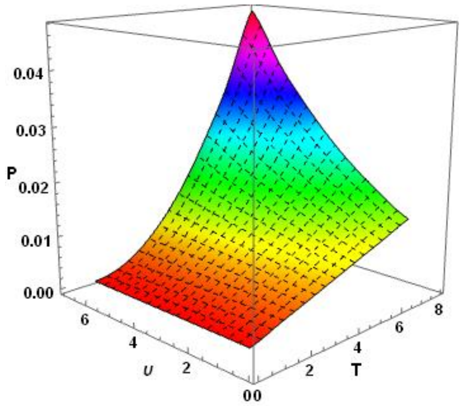

Figure 1. The impact of fractional order (υ) on the condition $\mathcal{P}$ given by (12) is represented graphically. Based on the $\mathcal{P}$ value given by (12) and the conditions $\mathcal{A}_1$ and $\mathcal{A}_2$, the graph shown above describes the behavior of the solution of problem (29) for different values of $\upsilon \in (0,1)$. It is noted that as $\mathcal{T}$ increases, the value of $\mathcal{P}$ increases as well and, with an increase in time, the condition $\mathcal{P}$ increases gradually for all values of $\upsilon \in (0,1)$, and the $\mathcal{P}$ is clearly less than 1, satisfying the condition obtained in Theorem 4. An important observation to be made is that when order (υ) is small, the value of $\mathcal{P}$ decreases with increasing time. As the order (υ) increases, this trend changes with the value of $\mathcal{P}$ increasing with time. The figure describes the behavior of the solution.

6. Conclusions

In previous works, researchers investigated the existence and uniqueness of linear fractional differential equations involving ψ-Caputo. The legacy of this work lies in verifying the existence and uniqueness of solutions to a coupled system of ψ-Caputo hybrid fractional differential equations with Dirichlet boundary conditions. Our major findings are demonstrated using the Banach fixed point theorem and the alternative of Leray–Schauder. The stability of the solutions involved in the Hyers–Ulam type was investigated. We provide an example to demonstrate the study results. ψ-fractional calculus has its own prominence. For example, some researchers showed that by considering different ψs, a particular natural phenomenon can be remodeled with more accuracy. For replacing the fractional calculus by ordinary calculus, see [34]. In future studies, researchers can verify the existence, uniqueness and stability of the solutions for the system of equations given by Equation (1) using the ψ-Hilfer fractional derivative or any other derivatives such as the fractional Katugambula derivative. In addition, this system can be used in practical applications of the subject by taking our results as proven facts.

Author Contributions: Funding acquisition, M.A.; Investigation, K.A.; Methodology, M.A. and M.M.; Project administration, M.A.; Software, M.M.; Supervision, M.S.; Writing—original draft, M.A. and M.M.; Writing—review & editing, K.A., M.S. and M.M. The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Project No. GRANT293).

Institutional Review Board Statement: Not serviceable.

Informed Consent Statement: Not serviceable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors, therefore, acknowledge technical and financial support of DSR at KFU.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998.
3. Burton, T.A. A Fixed-Point Theorem of Krasnoselskii. *Appl. Math. Lett.* **1998**, *11*, 85–88. [CrossRef]
4. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: Cambridge, UK, 1980; Volume 66.
5. Manigandan, M.; Muthaiah, S.; Nandhagopal, T.; Vadivel, R.; Unyong, B.; Gunasekaran, N. Existence results for coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order. *AIMS Math.* **2022**, *7*, 723755. [CrossRef]
6. Sitho, S.; Ntouyas, S.K.; Samadi, A.; Tariboon, J. Boundary value problems for ψ-Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions. *Mathematics* **2021**, *9*, 1001. [CrossRef]
7. Baleanu, D.; Jassim, H.K. Approximate solutions of the damped wave equation and dissipative wave equation in fractal strings. *Fractal Fract.* **2019**, *3*, 26. [CrossRef]
8. Baleanu, D.; Jassim, H.K. A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on Cantor sets. *Fractal Fract.* **2019**, *3*, 30. [CrossRef]
9. Baleanu, D.; Jassim, H.K.; Al Qurashi, M. Solving Helmholtz equation with local fractional derivative operators. *Fractal Fract.* **2019**, *3*, 43. [CrossRef]
10. Singh, J.; Jassim, H.K.; Kumar, D. An efficient computational technique for local fractional Fokker Planck equation. *Phys. A Stat. Mech. Appl.* **2020**, *555*, 124525. [CrossRef]
11. Baleanu, D.; Jassim, H.K. Exact solution of two-dimensional fractional partial differential equations. *Fractal Fract.* **2020**, *4*, 21. [CrossRef]
12. Awadalla, M.; Yameni Noupoue, Y.Y.; Asbeh, K.A. ψ-Caputo Logistic Population Growth Model. *J. Math.* **2021**, *2021*, 8634280.
13. Awadalla, M.; Noupoue, Y.Y.Y.; Abuasbeh, K. Population Growth Modeling via Rayleigh-Caputo Fractional Derivative. *J. Stat. Appl. Probab.* **2021**, *10*, 11–16.
14. Almeida, R. Variational problems involving a Caputo-type fractional derivative. *J. Optim. Theory Appl.* **2017**, *174*, 276–294. [CrossRef]
15. Awadalla, M.; Yameni, Y.Y. Modeling exponential growth and exponential decay real phenomena by ψ-Caputo fractional derivative. *J. Adv. Math. Comput. Sci.* **2018**, *28*, 1–13. [CrossRef]
16. Boutiara, A.; Abdo, M.S.; Benbachir, M. Existence results for ψ-Caputo fractional neutral functional integro-differential equations with finite delay. *Turk. J. Math.* **2020**,*44*, 2380–2401. [CrossRef]
17. Abdo, M.S.; Panchal, S.K.; Hussien, H.S. Fractional integro-differential equations with nonlocal conditions and *psi*–Hilfer fractional derivative. *Math. Model. Anal.* **2020**, *24*, 564–584. [CrossRef]
18. Almalahi, M.A.; Panchal, S.K. Some existence and stability results for ψ-Hilfer fractional implicit differential equation with periodic conditions. *J. Math. Anal. Model.* **2020**, *1*, 15. [CrossRef]
19. Subramanian, M.; Manigandan, M.; Tunç, C.; Gopal, T.N.; Alzabut, J. On system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. *J. Taibah Univ. Sci.* **2022**, *16*, 1–23. [CrossRef]
20. Muthaiah, S.; Murugesan, M.; Thangaraj, N.G. Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations. *Adv. Theory Nonlinear Anal. Appl.* **2019**, *3*, 162–173. [CrossRef]
21. Awadalla, M. Some Existence Results for a System of Nonlinear Sequential Fractional Differential Equations with Coupled Nonseparated Boundary Conditions. *Complexity* **2022**, *2022*, 8992894. [CrossRef]
22. Jiang, D.; Bai, C. On coupled Gronwall inequalities involving a ψ-fractional integral operator with its applications. *AIMS Math.* **2020**, *7*, 7728–7741. [CrossRef]
23. Boutiara, A.; Etemad, S.; Hussain, A.; Rezapour, S. The generalized U-H and U-H stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving ψ-Caputo fractional operators. *Adv. Differ. Equ.* **2020**, *2021*, 95. [CrossRef]
24. Samadi, A.; Ntouyas, S.K.; Tariboon, J. Nonlocal coupled hybrid fractional system of mixed fractional derivatives via an extension of Darbo's theorem. *AIMS Math.* **2021**, *6*, 3915–3926. [CrossRef]
25. Ji, D.; Ge, W. A nonlocal boundary value problems for hybrid ψ-Caputo fractional integro-differential equations. *AIMS Math.* **2020**, *5*, 7175–7190. [CrossRef]
26. Shammakh, W.; Selvam, A.G.M.; Dhakshinamoorthy, V.; Alzabut, J. A Study of Generalized Hybrid Discrete Pantograph Equation via Hilfer Fractional Operator. *Fractal Fract.* **2022**, *6*, 152. [CrossRef]
27. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpath. J. Math.* **2010**, *26*, 103–107.
28. Almalahi, M.A.; Abdo, M.S.; Panchal, S.K. Existence and Ulam–Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations. *Results Appl. Math.* **2021**, *10*, 100142. [CrossRef]
29. Abdo, M.S.; Thabet, S.; Ahmad, B. The existence and Ulam–Hyers stability results for ψ-Hilfer fractional integrodifferential equations. *J. Pseudo-Differ. Oper. Appl.* **2020**, *11*, 1757–1780. [CrossRef]
30. Abbas, M.I. Four-point boundary value problems for a coupled system of fractional differential equations with ψ-Caputo fractional derivatives. *arXiv* **2020**, arXiv:2007.10325.

31. Aydin, M.; Mahmudov, N.I.; Aktuğlu, H.; Baytunç, E.; Atamert, M.S. On a study of the representation of solutions of a ψ-Caputo fractional differential equations with a single delay. *Electron. Res. Arch.* **2022**, *30*, 1016–1034. [CrossRef]
32. Derbazi, C.; Baitiche, Z.; Benchohra, M.; Graef, J.R. Extremal solutions to a coupled system of nonlinear fractional differential equations with Caputo fractional derivatives. *J. Math. Appl.* **2021**, *44*, 19–34. [CrossRef]
33. BAİTİCHE, Z.; DERBAZİ, C.; BENCHOHRA, M. ψ-Caputo fractional differential equations with multi-point boundary conditions by Topological Degree Theory. *Results Nonlinear Anal.* **2020**, *3*, 167–178.
34. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]