Some Fixed-Point Theorems in Proximity Spaces with Applications

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Abstract: Considering the ω-distance function defined by Kostić in proximity space, we prove the Matkowski and Boyd–Wong fixed-point theorems in proximity space using ω-distance, and provide some examples to explain the novelty of our work. Moreover, we characterize Edelstein-type fixed-point theorem in compact proximity space. Finally, we investigate an existence and uniqueness result for solution of a kind of second-order boundary value problem via obtained Matkowski-type fixed-point results under some suitable conditions.

Keywords: fixed point; proximity spaces; φ-contraction; ω-distance

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1. Introduction and Preliminaries

The basic concepts of proximity spaces were initially developed by Frigyes Riesz [1] in 1908, and later on this theory was revived and axiomatized by Efremovich [2] in 1934, which was published in 1951. Over the years, a lot of studies on proximity spaces have been carried out [3–6]. Smirnov [5] established the link between the proximity relation and topological spaces. In addition, he was the first to introduce the relationship between proximities and uniformities.

Let δ be a relation on a set 2Ω. Then, the pair (Ω, δ) is said to be a proximity space if the following hold: for all U, V, W ∈ 2Ω, where 2Ω the power set of Ω

1. UδV implies VδU;
2. UδV implies U δ V ≠ ∅;
3. Uδ(V ∪ W) iff UδV or UδW;
4. U δ V ≠ ∅ implies UδV;
5. For all Y ⊆ Ω, UδY or Vδ(Ω − Y) implies UδV;

For all ξ ∈ Ω and U ⊆ Ω, we will use the notation ξδU and Ud ξ instead of {ξ}δU and Ud {ξ}, respectively. A proximity space (Ω, δ) is called separated if ξδη implies ξ = η for all ξ, η ∈ Ω. The properties of proximity spaces are generalizations of uniform properties and continuity properties of metric and topology, respectively. Every proximity relation on a nonempty set Ω induces a topology τδ via the Kuratowski closure operator. The Kuratowski closure operator using proximity relation can be defined by cl(U) = {ξ ∈ Ω: ξδU} for all U ⊆ Ω. In this case, the topology τδ is always completely regular, further, it is Tychonoff if (Ω, δ) is separated. If (Ω, τ) is a topological space and δ is a proximity on Ω such that τδ = τ, then τ and δ are said to be compatible. Every completely regular topology on a nonempty set Ω has a compatible proximity. Additionally, if a sequence {ξn} converges to a point...
\(\zeta \in \Omega\) with respect to induced topology \(\tau_\delta\), then we obtain \(\zeta \delta \{\zeta_n\}\). Furthermore, every uniform space \((\Omega, U)\) has an associated proximity structure defined by for all \(U, V \subseteq \Omega\), \(U \delta V\) if \((U \times V) \cap W \neq \emptyset\) for all \(W \in U\). For further information, see [7,8].

Now we state some examples of proximity spaces.

**Example 1.** Let \((\Omega, \rho)\) be a metric space. Consider the relation \(\delta\) on \(2^\Omega\) as

\[
U \delta V \Leftrightarrow \rho(U, V) = 0,
\]

where \(\rho(U, V) = \inf\{\rho(u, v) : u \in U, v \in V\}\). Then \(\delta\) is a proximity on \(\Omega\). In addition, the metric topology \(\tau_\rho\) and \(\delta\) are compatible.

**Example 2.** Let \((\Omega, \tau)\) be a \(T_4\) (both normal and \(T_1\)) topological space. Consider the relation \(\delta\) on \(2^\Omega\) as

\[
U \delta V \leftrightarrow \overline{U} \cap \overline{V} \neq \emptyset.
\]

Then \(\delta\) is a proximity on \(\Omega\). In addition, \(\tau\) and \(\delta\) are compatible.

**Example 3.** Let \((\Omega, \tau)\) be a completely regular topological space. Consider the relation \(\delta\) on \(2^\Omega\) as

\[
U \delta V \Leftrightarrow \overline{U} \cap \overline{V} \neq \emptyset.
\]

Then \(\delta\) is a proximity on \(\Omega\). In addition, \(\tau\) and \(\delta\) are compatible.

Considering that fixed-point theory on proximity space will bring a new direction and interesting results, Kostić [9] defined the concept of \(\omega\)-distance and \(\omega_0\)-distance inspired by [10] (see [11] for more information about \(\omega\)-distance) in proximity space as follows and obtained the proximity space version of Banach fixed-point theorem.

**Definition 1.** Let \((\Omega, \delta)\) be proximity space and \(\omega : \Omega \times \Omega \rightarrow [0, \infty)\) be a function. If \(\omega\) satisfies

1. \(\omega(\zeta, U) = 0\) and \(\omega(\zeta, V) = 0\) implies \(U \delta V\) for all \(\zeta \in \Omega\) and \(U, V \subseteq \Omega\),

then \(\omega\) is said to be a \(\omega\)-distance on \(\Omega\), where

\[
\omega(\zeta, U) = \inf\{\omega(\zeta, \xi) : \xi \in U\}.
\]

In addition, a \(\omega\)-distance on a proximity space \((\Omega, \delta)\) is called \(\omega_0\)-distance if the following axioms hold:

1. \(\omega(\xi, \eta) \leq \omega(\xi, \zeta) + \omega(\zeta, \eta)\) for all \(\xi, \eta, \zeta \in \Omega\),

2. \(\omega\) is lower semicontinuous in both variables with respect to \(\tau_\delta\), i.e., for all \(\xi, \eta \in \Omega\), we have

\[
\omega(\xi, \eta) = \liminf_{\zeta \rightarrow \xi} \omega(\zeta, \eta) = \sup_{V \in U_\xi} \inf_{\xi' \in V} \omega(\xi', \eta)
\]

and

\[
\omega(\eta, \zeta) = \liminf_{\zeta' \rightarrow \zeta} \omega(\eta, \zeta') = \sup_{V \in U_\xi} \inf_{\xi' \in V} \omega(\eta, \xi'),
\]

where \(U_\xi\) is a base of neighborhoods of the point \(\xi \in \Omega\).

**Remark 1.** It is clear that if \(\omega\) is lower semicontinuous in both variables with respect to \(\tau_\delta\), then we have \(\omega(\xi, \eta) \leq \liminf_{n \rightarrow \infty} \omega(\xi_n, \eta)\) and \(\omega(\eta, \zeta) \leq \liminf_{n \rightarrow \infty} \omega(\eta, \zeta_n)\) for every sequence \(\{\xi_n\}\) converging to \(\xi\) with respect to \(\tau_\delta\).

**Example 4.** Let \(\Omega = \mathbb{R}\) be endowed with the usual metric and the proximity \(\delta\) defined in Example 1. Define \(\omega_1, \omega_2 : \Omega \times \Omega \rightarrow [0, \infty)\) by

\[
\omega_1(\xi, \eta) = \max\{|\xi|, |\eta|\}
\]
and

\[ \omega_2(\xi, \eta) = \frac{|\xi| + |\eta|}{2}, \]

then both \( \omega_1 \) and \( \omega_2 \) are \( \omega_0 \)-distance on \( \Omega \).

**Example 5.** Let \( \Omega = [0, \infty) \) be endowed with the lower limit topology \( \tau_l \) and the proximity \( \delta \) defined in Example 2. It is well-known that \( (\mathbb{R}, \tau_l) \) is not metrizable but normal topological space. Define \( \omega : \Omega \times \Omega \to [0, \infty) \) by

\[ \omega(\xi, \eta) = \eta, \]

then \( \omega \) is \( \omega_0 \)-distance on \( \Omega \).

**Example 6.** Let \( \Omega = C[0,2] \) be endowed with the metric \( \rho \) defined by

\[ \rho(f, g) = \int_0^2 |f(\xi) - g(\xi)|d\xi \]

and the proximity \( \delta \) defined in Example 1. Define \( \omega : \Omega \times \Omega \to [0, \infty) \) by

\[ \omega(f, g) = \int_0^2 |g(\xi)|d\xi, \]

then \( \omega \) is \( \omega_0 \)-distance on \( \Omega \).

**Proof.** Let \( U, V \subseteq C[0,2] \), \( h \in C[0,2] \) and

\[ \omega(h, U) = 0 = \omega(h, V). \]  \hspace{1cm} (1)

Let \( \varepsilon > 0 \). Then by (1) there exist \( f \in U \) and \( g \in V \) such that

\[ \omega(h, f) = \int_0^2 |f(\xi)|d\xi < \frac{\varepsilon}{2} \] and \( \omega(h, g) = \int_0^2 |g(\xi)|d\xi < \frac{\varepsilon}{2} \).

Hence we have

\[ \rho(f, g) \leq \omega(h, f) + \omega(h, g) < \varepsilon. \]

Thus we have

\[ \rho(U, V) = \inf\{\rho(f, g) : f \in U, g \in V\} = 0 \]

and hence \( U \delta V \). Therefore, (\( w1 \)) holds, and so \( \omega \) is a \( \omega \)-distance on \( \Omega \). It is clear that (\( w2 \)) holds. Now, let \( f, g \in \Omega \), then we have

\[ \omega(f, g) = \int_0^2 |g(\xi)|d\xi = \lim_{f \to f'} \inf_{g \in V} \omega(f', g) = \sup_{V \in \mathcal{U}_f} \inf_{f' \in V} \omega(f', g), \]

that is, \( \omega \) is lower semicontinuous in the first variable. On the other hand, let \( V \subseteq \mathcal{U}_f \) and \( f' \in V \), then there exists \( \varepsilon > 0 \) such that \( \rho(f, f') < \varepsilon \), that is \( \sup_{V \in \mathcal{U}_f} \inf_{f' \in V} \rho(f, f') = 0 \). Hence, we have

\[ \omega(g, f) = \int_0^2 |f(\xi)|d\xi \]

\[ \leq \int_0^2 |f(\xi) - f'(\xi)|d\xi + \int_0^2 |f'(\xi)|d\xi \]

\[ \leq \varepsilon + \omega(g, f'). \]
and so we obtain
\[ \omega(g, f) \leq \sup_{V \in \Omega} \inf_{f' \in V} \omega(g, f'), \]
that is, \( \omega \) is lower semicontinuous in second variable. \( \square \)

Example 7. Let \( \Omega = C[0, 1] \) be endowed with the metric \( \rho \) defined by
\[ \rho(f, g) = \sup\{|f(\xi) - g(\xi)| : \xi \in [0, 1]\} \]
and the proximity \( \delta \) defined in Example 1. Define \( \omega_1, \omega_2 : \Omega \times \Omega \rightarrow [0, \infty) \) by
\[ \omega_1(f, g) = \sup\{|g(\xi)| : \xi \in [0, 1]\} \]
and
\[ \omega_2(f, g) = \sup\{|f(\xi)| + |g(\xi)| : \xi \in [0, 1]\}, \]
then \( \omega_1 \) and \( \omega_2 \) are \( \omega_0 \)-distance on \( \Omega \).

Lemma 1 ([9,10]). Let \( (\Omega, \delta) \) be a proximity space with \( \omega \)-distance \( \omega \). Then, the following properties hold:

(i) If \( (\Omega, \delta) \) is separated, then \( \omega(\xi, \xi) = 0 \) and \( \omega(\xi, \eta) = 0 \) implies \( \xi = \eta \).
(ii) If \( \omega(\xi, \xi) = 0 \) and \( \omega(\xi, \xi_n) \rightarrow 0 \) as \( n \rightarrow \infty \), then \( \{\xi_n\} \) subsequently converges to \( \xi \) with respect to \( \tau_\delta \).

In [9], Kostić introduced the Banach contraction principle in proximity space using \( \omega_0 \)-distance \( \omega \) as follows:

Theorem 1. Let \( (\Omega, \delta) \) be a separated proximity space with \( \omega_0 \)-distance \( \omega \). Suppose that the mapping \( T : \Omega \rightarrow \Omega \) satisfies the following:

(i) There exists \( k \in (0, 1) \) such that
\[ \omega(T\xi, T\eta) \leq k \omega(\xi, \eta) \]
for all \( \xi, \eta \in \Omega \);
(ii) For all \( \xi \in \Omega \), any iterative sequence \( \{T^n\xi\} \) has a convergent subsequence with respect to \( \tau_\delta \).

Then, there exists \( \xi \in \Omega \), such that \( \xi = T\xi \) and \( \omega(\xi, \xi) = 0 \).

In this study, we will use some auxiliary functions in the contraction inequality to generalize Kostić’s theorem. Thus, we will obtain equivalents in proximity space of the famous Boyd–Wong [12] and Matkowski [13] fixed-point theorems known in metric fixed-point theory. In addition, for the equivalent of Edelstein’s theorem [14], in this space, we will consider that the space is compact according to the topology induced by proximity.

Let \( \phi : [0, \infty) \rightarrow [0, \infty) \) be a function. Next, we will consider the below properties for \( \phi \):

(\( \phi_1 \)) \( \phi \) is non-decreasing;
(\( \phi_2 \)) For all \( t \geq 0 \), \( \lim_{n \rightarrow \infty} \phi^n(t) = 0 \);
(\( \phi_3 \)) For all \( t > 0 \), \( \phi(t) < t \);
(\( \phi_4 \)) \( \phi \) is upper semicontinuous from the right.

We will symbolize the set of all functions with properties (\( \phi_1 - \phi_2 \)) and (\( \phi_3 - \phi_4 \)), by \( \Phi \) and \( \Psi \) respectively. It is easy to see that both \( \Phi \setminus \Psi \) and \( \Psi \setminus \Phi \) are nonempty and also every function in \( \Phi \) satisfies the property (\( \phi_3 \)). Some examples of the functions belonging both \( \Phi \) and \( \Psi \) are \( \phi_1(t) = Lt \), where \( 0 \leq L < 1 \) and \( \phi_2(t) = \frac{t}{1+L} \).
2. Main Result

Here, we present our main theorems.

**Theorem 2.** Let \((\Omega, \delta)\) be a separated proximity space with \(\omega_0\)-distance \(\omega\) and \(T : \Omega \rightarrow \Omega\) be a mapping satisfying the followings:

(i) There exists \(\phi \in \Phi\), satisfying

\[
\omega(T\xi, T\eta) \leq \phi(\omega(\xi, \eta))
\]

for all \(\xi, \eta \in \Omega\);

(ii) For all \(\xi \in \Omega\), any iterative sequence \(\{T^n\xi\}\) has a convergent subsequence with respect to \(\tau_\delta\). Then there exists \(\zeta \in \Omega\), such that \(\zeta = T\zeta\) and \(\omega(\zeta, \zeta) = 0\).

**Proof.** Let \(\xi_0 \in \Omega\) be arbitrary. Consider the corresponding Picard sequence \(\{\xi_n\}\) constructed by

\[
\xi_n = T^n\xi_0 = T\xi_{n-1}.
\]

Since \(\phi\) is non-decreasing, and by (i), we have

\[
\omega(\xi_n, \xi_{n+1}) = \omega(T\xi_{n-1}, T\xi_n) \\
\leq \phi(\omega(\xi_{n-1}, \xi_n)) \\
\vdots \\
\leq \phi^n(\omega(\xi_0, \xi_1)).
\]

Similarly, we can obtain

\[
\omega(\xi_{n+1}, \xi_n) \leq \phi^n(\omega(\xi_1, \xi_0)).
\]

By (\(\phi_2\)), we have

\[
\lim_{n \rightarrow \infty} \omega(\xi_n, \xi_{n+1}) = 0 \tag{2}
\]

and

\[
\lim_{n \rightarrow \infty} \omega(\xi_{n+1}, \xi_n) = 0. \tag{3}
\]

On the other hand, since \(\phi \in \Phi\), we have \(\phi(\epsilon) < \epsilon\) for all \(\epsilon > 0\). Hence by (2), there exists a natural number \(N\) satisfying

\[
\omega(\xi_N, \xi_{N+1}) < \epsilon - \phi(\epsilon)
\]

for all \(\epsilon > 0\).

Since \(\omega\) is a \(\omega_0\)-distance and \(\phi\) is non-decreasing, then we have

\[
\omega(\xi_N, \xi_{N+2}) \leq \omega(\xi_N, \xi_{N+1}) + \omega(\xi_{N+1}, \xi_{N+2}) \\
< \epsilon - \phi(\epsilon) + \omega(T\xi_N, T\xi_{N+1}) \\
\leq \epsilon - \phi(\epsilon) + \phi(\omega(\xi_N, \xi_{N+1})) \\
< \epsilon - \phi(\epsilon) + \phi(\epsilon - \phi(\epsilon)) \\
\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\
= \epsilon.
\]
Similarly,

\[ \omega(\xi_N, \xi_{N+3}) \leq \omega(\xi_N, \xi_{N+1}) + \omega(\xi_{N+1}, \xi_{N+3}) < \epsilon - \phi(\epsilon) + \omega(T\xi_N, T\xi_{N+2}) \]

\[ \leq \epsilon - \phi(\epsilon) + \phi(\omega(\xi_N, \xi_{N+2})) \]

\[ < \epsilon - \phi(\epsilon) + \phi(\epsilon) \]

\[ = \epsilon. \]

Continuing this process, we obtain \( \omega(\xi_N, \xi_{N+k}) < \epsilon \) for all \( k = 1, 2, \cdots \) and similarly by (3) we obtain \( \omega(\xi_{N+k}, \xi_N) < \epsilon \) for all \( k = 1, 2, \cdots \). Thus, for all \( m, n > N \), we have

\[ \omega(\xi_n, \xi_m) \leq \omega(\xi_n, \xi_N) + \omega(\xi_N, \xi_m) < 2\epsilon. \]  

By (ii), there exists a subsequence \( \{\xi_{n_k}\} \) of the sequence \( \{\xi_n\} \) which is convergent with respect to \( \tau_2 \) to some \( \zeta \in \Omega \), and by (w3), Remark 1 and Inequality (4), we have

\[ \omega(\zeta, \xi_{n_k}) \leq \liminf_{l \to \infty} \omega(\xi_{n_l}, \xi_{n_k}) \leq 2\epsilon, \]

and symmetrically, we obtain

\[ \omega(\xi_{n_k}, \zeta) \leq 2\epsilon \]

for all \( n_k > N \). By Inequality (6), we have

\[ \omega(T\xi_{n_k}, T\xi) \leq \phi(\omega(\xi_{n_k}, \zeta)) \leq \phi(2\epsilon) < 2\epsilon \]

for all \( n_k > N \). Finally, by Inequalities (5) and (7), we have

\[ \omega(\zeta, T\xi) \leq \omega(\zeta, \xi_{n_k}) + \omega(\xi_{n_k}, T\xi) + \omega(T\xi_n, T\zeta) \]

\[ < 2\epsilon + \phi(k_1)(\omega(\xi_0, \xi_1)) + 2\epsilon \]

\[ = 4\epsilon + \phi(k_1)(\omega(\xi_0, \xi_1)) \]

for all \( n_k > N \). Since, by (phi), \( \phi(k_1)(\omega(\xi_0, \xi_1)) \to 0 \) as \( k \to \infty \), it follows that \( \omega(\zeta, T\xi) = 0 \).

On the other hand, by (w3) and Remark 1, we have

\[ \omega(\zeta, \xi) \leq \liminf_{l \to \infty} \omega(\zeta, \xi_{n_l}) \leq 2\epsilon, \]

and consequently, \( \omega(\zeta, \xi) = 0 \). Thus, by Lemma 1 (i), we have \( \zeta = T\xi \). To check the uniqueness, let \( \zeta \in \Omega \) be a different fixed point of \( T \). Then, \( \omega(\zeta, \xi) > 0 \), because \( \zeta \neq \xi \) and \( \omega(\zeta, \xi) = 0 \). By (i), we obtain

\[ 0 < \omega(\xi, \zeta) = \omega(T\xi, T\xi) \leq \phi(\omega(\zeta, \xi)) < \omega(\xi, \zeta), \]

a contradiction. Hence, the fixed point of \( T \) is unique. \( \square \)

**Remark 2.** Note that for \( \phi(t) = kt, 0 \leq k < 1 \), Theorem 2 reduces to Theorem 1.

**Example 8.** Let \( (\Omega, \Delta) \) be the proximity space and \( \omega \) be the \( \omega_N \)-distance given in Example 5. Clearly \( (\Omega, \Delta) \) is separated proximity space. Define a mapping \( T : \Omega \to \Omega \) by \( T\xi = \frac{\xi}{1+\xi} \) and consider the function \( \phi \in \Phi \) as \( \phi(t) = \frac{t}{1+t} \). In this case, we have

\[ \omega(T\xi, T\eta) = T\eta = \frac{\eta}{1+\eta} = \phi(\eta) = \phi(\omega(\xi, \eta)) \]

\[ = \phi(\omega(\xi, \eta)), \]
for all $\xi, \eta \in \Omega$. In addition, for all $\xi \in \Omega$, $T^n \xi = \frac{\xi}{1+n\xi}$ is convergent with respect to $\tau_\varphi$. Hence, all conditions of Theorem 2 hold. Therefore, $T$ has a unique fixed point. Note that, since

$$\sup_{\eta \in \Omega} \frac{\omega(T\xi, T\eta)}{\omega(\xi, \eta)} = 1,$$

we can not find a constant $k \in (0, 1)$ satisfying

$$\omega(T\xi, T\eta) \leq k\omega(\xi, \eta).$$

Hence, Theorem 1 cannot be applied.

**Theorem 3.** Let $(\Omega, \delta)$ be a separated proximity space with $\omega_0$-distance $\omega$ and $T : \Omega \longrightarrow \Omega$ be a mapping satisfying the followings:

(i) There exists $\phi \in \Psi$ such that $\phi(0) = 0$ and $\phi$ is right continuous at 0 satisfying $\omega(T\xi, T\eta) \leq \phi(\omega(\xi, \eta))$ for all $\xi, \eta \in \Omega$;

(ii) For all $\xi \in \Omega$, any iterative sequence $\{T^n\xi\}$ has a convergent subsequence with respect to $\tau_\varphi$.

Then, there exists $\xi \in \Omega$ such that $\xi = T\xi$ and $\omega(\xi, \xi) = 0$.

**Proof.** Let $\xi_0 \in \Omega$ be arbitrary. Consider the corresponding Picard sequence $\{\xi_n\}$ constructed by

$$\xi_n = T^n\xi_0 = T^n\xi_{n-1}.$$  

for all $n \in \mathbb{N}$. For simplicity, call $\omega_n = \omega(\xi_n, \xi_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $\omega_{n_0} = \omega_{n_0+1} = 0$, then by triangular inequality we have

$$\omega(\xi_{n_0}, \xi_{n_0+2}) \leq \omega(\xi_{n_0}, \xi_{n_0+1}) + \omega(\xi_{n_0+1}, \xi_{n_0+2}) = \omega_{n_0} + \omega_{n_0+1} = 0$$

and so by Lemma 1 (i) we have $\xi_{n_0+1} = \xi_{n_0+2}$. This shows that $\xi_{n_0+1}$ is a fixed point of $T$. Now assume that any of consecutive terms of the $\{\omega_n\}$ are not zero. In this case, there exists a subsequence $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $\omega_{n_k} > 0$ for all $k \in \mathbb{N}$. By (i), we have

$$\omega_{n_{k+1}} = \omega(\xi_{n_{k+1}}, \xi_{n_{k+2}}) = \omega(T\xi_{n_k}, T\xi_{n_{k+1}}) \leq \phi(\omega(\xi_{n_k}, \xi_{n_{k+1}})) = \phi(\omega_{n_k}) < \omega_{n_k}.  \quad (8)$$

Since the index sequence $n_k$ is strictly increasing, we have $n_{k+1} > n_k$ and so $n_{k+1} \geq n_k + 1$. Therefore, from (8), we have

$$\omega_{n_{k+1}} \leq \omega_{n_k+1} < \omega_{n_k}.  \quad (9)$$

It follows that $\{\omega_{n_k}\}$ is a decreasing sequence which is also bounded below. Thus, there exists $\gamma \geq 0$ such that $\lim_{k \to \infty} \omega_{n_k} = \gamma$. If $\gamma > 0$, then by (9), $(\phi_3)$ and $(\phi_4)$, we have

$$0 < \gamma = \lim_{k \to \infty} \omega_{n_k+1} \leq \lim_{k \to \infty} \omega_{n_k+1} \leq \lim_{k \to \infty} \phi(\omega_{n_k}) \leq \lim_{k \to \infty} \sup_{k \to \infty} \phi(\omega_{n_k}) \leq \phi(\gamma) < \gamma,$$

a contradiction. Therefore, $\gamma = 0$, and consequently,

$$\lim_{k \to \infty} \omega_{n_k} = 0.  \quad (10)$$
Similarly, it can be obtained that the limit of all subsequences of \( \{\omega_n\} \) whose all terms are greater than zero, is zero. Hence we have
\[
\lim_{n \to \infty} \omega_n = \lim_{n \to \infty} \omega(\xi_n, \xi_{n+1}) = 0.
\]

Now, if we call \( \omega_n = \omega(\xi_{n+1}, \xi_n) \), similarly we can obtain
\[
\lim_{n \to \infty} \omega_n = \lim_{n \to \infty} \omega(\xi_{n+1}, \xi_n) = 0. \tag{11}
\]

Now, we claim that for every \( \epsilon > 0 \), there exists a natural number \( N \) satisfying
\[
\omega(\xi_m, \xi_n) < \epsilon
\]
for all \( m > n > N \). Assume the contrary, then there exist \( \epsilon > 0 \), \( m_k, n_k \in \mathbb{N} \) with \( m_k > n_k > k, k \in \mathbb{N} \) such that \( \omega(\xi_{m_k}, \xi_{n_k}) \geq \epsilon \). Here, we can choose \( m_k \) as the smallest positive integer that satisfies \( \omega(\xi_{m_k}, \xi_{n_k}) \geq \epsilon \). Therefore, we obtain \( \omega(\xi_{m_k-1}, \xi_{n_k}) < \epsilon \), and since \( \omega \) is a \( \omega \)-distance, we obtain
\[
\begin{align*}
\epsilon & \leq \omega(\xi_{m_k}, \xi_{n_k}) \\
& \leq \omega(\xi_{m_k}, \xi_{m_k-1}) + \omega(\xi_{m_k-1}, \xi_{n_k}) \\
& \leq \omega(\xi_{m_k}, \xi_{m_k-1}) + \epsilon \\
& = \omega_{m_k} + \epsilon.
\end{align*}
\]

By Equation (11), it follows that \( \lim_{k \to \infty} \omega(\xi_{m_k}, \xi_{n_k}) = \epsilon \). On other hand,
\[
\begin{align*}
\omega(\xi_{m_k}, \xi_{n_k}) & \leq \omega(\xi_{m_k}, \xi_{m_k+1}) + \omega(\xi_{m_k+1}, \xi_{n_k+1}) + \omega(\xi_{n_k+1}, \xi_{n_k}) \\
& = \omega_{m_k} + \omega(T\xi_{m_k}, T\xi_{n_k}) + \omega_{n_k} \\
& \leq \omega_{m_k} + \phi(\omega(\xi_{m_k}, \xi_{n_k})) + \omega_{n_k}.
\end{align*}
\]

Taking limit \( k \to \infty \) and by \((\phi_4)\), we have \( \epsilon \leq \phi(\epsilon) \), a contradiction. Thus, for every \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \), such that
\[
\omega(\xi_m, \xi_n) < \epsilon \tag{12}
\]
for all \( m > n > N \). Similarly, we obtain
\[
\omega(\xi_n, \xi_m) < \epsilon \tag{13}
\]
for all \( m > n > N \).

By \((ii)\), there exists a subsequence \( \{\xi_{n_k}\} \) of sequence \( \{\xi_n\} \) which is convergent with respect to \( \tau_j \) to some \( \xi \in \Omega \), and by \((\omega_3)\) and Inequality (12), we have
\[
\omega(\xi, \xi_{n_k}) \leq \lim_{l \to \infty} \inf \omega(\xi_{n_l}, \xi_{n_k}) \leq \epsilon, \tag{14}
\]
for all \( n_k \in \mathbb{N} \) and symmetrically by (13), we obtain
\[
\omega(\xi_{n_k}, \xi) \leq \epsilon \tag{15}
\]
for all \( n_k \in \mathbb{N} \). Then, we have
\[
\omega(\xi, \xi) \leq \lim_{k \to \infty} \inf \omega(\xi, \xi_{n_k}) \leq \epsilon,
\]
and consequently, \( \omega(\xi, \xi) = 0 \). On the other hand, we have
Let \( T \) be a mapping. Since \( \omega(T, 0) = \omega(0, T) = 0 \), we have \( \omega(T, T) = 0 \). Thus, by Lemma 1, we have \( \xi = T \cup T \).

To check the uniqueness, let \( v \in \Omega \) with \( v \neq \xi \) be another fixed point of mapping \( T \). Since \( \omega(v, \xi) = 0 \) it must be \( \omega(v, 0) > 0 \). Hence by (i), we obtain

\[
\omega(v, 0) = \omega(T, T) \leq \phi(\omega(v, 0)) < \omega(v, 0),
\]

a contradiction. Thus, the fixed point of \( T \) is unique. \( \square \)

Finally, in the theoretical part, we present a fixed-point result, taking into account the compactness of the space and a strict contraction inequality. Here, we will use the following well-known lemma.

**Lemma 2** ([15]). Let \( f : \Omega \to R \) a lower semicontinuous function, where \( \Omega \) is a compact topological space. Then, there exists an element \( \xi_0 \in \Omega \) such that

\[
f(\xi_0) = \inf\{f(\xi) : \xi \in \Omega\}.
\]

**Definition 2.** Let \((\Omega, \delta)\) be a proximity space with \( \omega \)-distance \( \omega \) and \( T : \Omega \to \Omega \) be a mapping. Then \( T \) is said to have \( 0 \)-property if \( \omega(T, T) = 0 \) whenever \( \omega(\xi, \eta) = 0 \) for all \( \xi, \eta \in \Omega \).

**Example 9.** Let \( \Omega = [0, \infty) \) be endowed with the lower limit topology \( \tau_\delta \) and the proximity \( \delta \) defined in Example 2. Consider the \( \omega \)-distance on \( \Omega \) defined by \( \omega(\xi, \eta) = 0 \). Then every self-mapping \( T \) of \( \Omega \) has \( 0 \)-property, provided that \( T0 = 0 \).

**Theorem 4.** Let \((\Omega, \delta)\) be a separated proximity space with \( \omega_0 \)-distance \( \omega \) such that \( \Omega \) is compact with respect to \( \tau_\delta \), and \( T : \Omega \to \Omega \) be a continuous mapping with \( 0 \)-property. Assume that

\[
\omega(T, T) < \omega(\xi, \eta)
\]

for all \( \xi, \eta \in \Omega \) with \( \omega(\xi, \eta) > 0 \). Then, there exists \( \zeta \in \Omega \) such that \( \zeta = T \zeta \).

**Proof.** Define \( f, g : \Omega \to R \) by \( f(\xi) = \omega(\xi, T\xi) \) and \( g(\xi) = \omega(T\xi, \xi) \). Since \( \omega \) is lower semicontinuous in both variables and \( T \) is continuous, then \( f \) and \( g \) are lower semicontinuous. Since \( \Omega \) is compact with respect to \( \tau_\delta \), then from Lemma 2 there exist \( \xi, \zeta \in \Omega \), such that

\[
f(\xi) = \inf\{f(\xi) : \xi \in \Omega\}
\]

and

\[
g(\xi) = \inf\{g(\xi) : \xi \in \Omega\}.
\]

In this case, if \( \omega(\xi, T\xi) > 0 \), then we obtain

\[
f(T\xi) = \omega(T\xi, TT\xi) < \omega(\xi, T\xi) = f(\xi),
\]

which is a contradiction. Hence, we obtain \( f(\xi) = \omega(\xi, T\xi) = 0 \). Similarly, we obtain \( \omega(T\xi, \zeta) = 0 \).
Now, if $\omega(\zeta, \varsigma) > 0$, then we obtain

$$\omega(\zeta, \varsigma) \leq \omega(\zeta, T\zeta) + \omega(T\zeta, T\varsigma) + \omega(T\varsigma, \varsigma)$$

which is a contradiction. Hence, we have $\omega(\zeta, \varsigma) = 0$. Therefore, from Lemma 1 (i), we have $\varsigma = T\zeta$ and so $\omega(T\zeta, T\varsigma) = 0$. Now,

$$\omega(\zeta, \varsigma) = \omega(T\zeta, T\varsigma)$$

and so from Lemma 1 (i), we have $\varsigma = T\zeta$. Additionally,

$$\omega(\varsigma, \varsigma) = \omega(T\zeta, T\varsigma) = 0.$$

Now let $v$ be another fixed point of $T$. Then, we have $v = Tv$ and $\omega(\varsigma, v) > 0$. Therefore, we obtain

$$\omega(\varsigma, v) = \omega(T\zeta, Tv) < \omega(\varsigma, v),$$

which is a contradiction. □

3. Application

Now, by considering Theorem 2, we give existence and uniqueness results about the solution of the second-order boundary value problem (BVP) as follows:

$$\begin{cases} \frac{d^2}{dt^2} \xi(t) = F(t, \xi(t)), & t \in [0, 1] \\ \xi(0) = \xi(1) = 0 \end{cases} \quad (16)$$

where $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous function. By taking into account some certain conditions $F$, some existence theorems were recently presented for problem (16) in the literature (see [16–18]). Here, we will consider some different conditions on $F$, and we provide a new theorem. We can see that the problem (16) is equivalent to the integral equation

$$\xi(t) = \int_{0}^{1} G(t, s) F(s, \xi(s)) ds, \quad t \in [0, 1], \quad (17)$$

where $G(t, s)$ is associated Green’s function defined as

$$G(t, s) = \begin{cases} t(1-s) & , 0 \leq t \leq s \leq 1 \\ s(1-t) & , 0 \leq s \leq t \leq 1 \end{cases}.$$

Therefore, $\xi \in C^2[0, 1]$ is a solution of (16) if and only if it is a solution of (17). It is clear that

$$\int_{0}^{1} G(t, s) ds = \frac{t(1-t)}{2}.$$

Let $(\Omega, \delta)$ be the proximity space, where $\Omega = C[0, 1]$ and $\delta$ is induced by the uniform metric $\rho_{\infty}$

$$\rho_{\infty}(\xi, \eta) = \sup \{|\xi(t) - \eta(t)| : t \in [0, 1]|.$$

In this case, $(\Omega, \delta)$ is separated proximity space. Consider the following $\omega_0$-distances $\omega_0$ on $\Omega$ defined by

$$\omega_0(\xi, \eta) = \sup \{|e^{-at}|\xi(t) - \eta(t)| : t \in [0, 1]|,$$

where $a > 0$ is a constant.
Theorem 5. The second-order BVP given by (16) has a unique solution under the following assumptions:

(a) There exists a non-decreasing function \( \varphi : [0, \infty) \to [0, \infty) \) such that

\[
|F(s, \xi) - F(s, \eta)| \leq \varphi(e^{-as}|\xi - \eta|)
\]

for all \( s \in [0, 1] \).

(b) \( \lim_{n \to \infty} \varphi^n(t) = 0 \) for all \( t \geq 0 \), where \( \varphi = K_a \varphi \) and

\[
K_a = \sup \left\{ e^{-at} \frac{t(1-t)}{2} : t \in [0, 1] \right\}.
\]

Proof. Consider the operator \( T : C[0, 1] \times C[0, 1] \to C[0, 1] \) defined by

\[
T \xi(t) = \int_0^1 G(t, s)F(s, \xi(s))ds
\]

Then, for any \( \xi, \eta \in C[0, 1] \) and \( t \in [0, 1] \), we have

\[
|T \xi(t) - T \eta(t)| = \left| \int_0^1 G(t, s)F(s, \xi(s))ds - \int_0^1 G(t, s)F(s, \eta(s))ds \right|
\]

\[
\leq \int_0^1 G(t, s)|F(s, \xi(s)) - F(s, \eta(s))|ds
\]

\[
\leq \int_0^1 G(t, s)\varphi(e^{-as}|\xi(s) - \eta(s)|)ds
\]

\[
\leq \varphi(\omega_a(\xi, \eta)) \int_0^1 G(t, s)ds
\]

\[
= \varphi(\omega_a(\xi, \eta)) \frac{t(1-t)}{2}
\]

and then we obtain

\[
e^{-at}\left| T \xi(t) - T \eta(t) \right| \leq \varphi(\omega_a(\xi, \eta))e^{-at} \frac{t(1-t)}{2}.
\]

By taking supremum over \( t \in [0, 1] \), we have

\[
\omega_a(T \xi, T \eta) \leq K_a \varphi(\omega_a(\xi, \eta)) = \varphi(\omega_a(\xi, \eta)).
\]  

(18)

Therefore, condition (i) of Theorem 2 is satisfied. Now let \( \xi \in C[0, 1] \) be an arbitrary function. Define a sequence of functions \( \{ \xi_n \} \) as \( \xi_n = T^n \xi \). As in the proof of Theorem 2 and by (18), we have

\[
\omega_a(\xi_n, \xi_m) \to 0
\]

as \( m, n \to \infty \). On the other hand, since

\[
e^{-a} \rho_{\omega}(\xi, \eta) \leq \omega_a(\xi, \eta) \leq \rho_{\omega}(\xi, \eta)
\]

for all \( \xi, \eta \in C[0, 1] \), we have

\[
\rho_{\omega}(\xi_n, \xi_m) \to 0
\]

as \( m, n \to \infty \). That is, the sequence \( \{ \xi_n \} \) is Cauchy and so has a convergent subsequence with respect to \( \rho_{\omega} \) since \( (\Omega, \rho_{\omega}) \) is complete. Therefore, condition (ii) of Theorem 2 is satisfied. Consequently, there exists unique \( \xi \in C[0, 1] \) which is a fixed point of the operator \( T \), moreover \( \omega_a(\xi, \xi) = 0 \). Hence, the (16) has a unique solution. \( \square \)
4. Conclusions

In this paper, following Kostić’s idea we present some fixed-point theorems for single-valued mappings on proximity space by using ω-distance and some auxiliary functions. Then, we support our theoretical results with some examples and an existence and uniqueness theorem.

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