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Epistemic Configurations and Holistic Meaning of Binomial Distribution

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Abstract: The competencies that today's citizen must possess have led to a transformation of the teaching of probability, which has been repositioned on the school curriculum from an algorithmic view to a practical one based on the understanding of the concepts and their application in daily life. In this task, the understanding of the binomial distribution is essential as it allows the analysis of discrete data, the modeling of random situations, and the learning of other notions. However, weaknesses are identified in teachers and students with respect to the binomial distribution attributed to the lack of knowledge of its origin and meaning throughout history. For this reason, our work consists of the identification of its partial meanings and essential components as well as its relationships from a historical epistemological study and its analysis, based on the notions of the Ontosemiotic Approach (OSA) to Mathematical Knowledge and Instruction and the specialized literature on statistics and probability. As a result of our work, we present a reconstruction of the holistic meaning of the binomial distribution from the elements that compose it, which are essential for didactic purposes such as the identification and resolution of learning conflicts, the design or evaluation criteria, and teacher education.

Keywords: binomial distribution; primary objects; probabilistic culture; partial meanings; holistic meaning; epistemology; configurations

MSC: 97K60; 97K50; 60-03

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1. Introduction

Probability allows human beings to better understand the random events of everyday life by identifying possible outcomes, calculating the probabilities of occurrence of random events, and modeling random phenomena, among other uses. The above, together with the increase in probabilistic information available and the current development of technology, raises the need to train probabilistically educated students, considering the knowledge and dispositions they need to develop to be able to face a wide range of daily situations that involve the interpretation or generation of probabilistic messages as well as decision making [1]. Consequently, the teaching of probability has taken an important role in the mathematics curriculum, as well as in its didactics and research. Within the latter, weaknesses have been identified in the training process and the probabilistic reasoning of students and teachers, such as regarding the knowledge and notions of the different meanings of probability [2], conditional probabilistic reasoning [3], and the notions evidenced when facing a problem situation without previous formal instruction [4]. This makes probability a relevant discipline of study, with challenges such as the search and construction of didactic resources, the promotion of its learning at early ages, and the change of its teaching approach from one based on the application of formulas to one focused on experimentation, complemented by the theory and its understanding [5,6].

The binomial distribution, as one of the most important in probability and statistics by allowing the analysis of random phenomena [7], is part of the components of probabilistic literacy [8] and of the research on its didactics. For example, Taufiq et al. [9] conducted an analysis on the structuring of its learning in secondary textbooks and identified an incomplete discussion by not addressing essential prior concepts such as the discrete random variable or discrete probability distribution and not addressing probabilistic and statistical concepts related to the binomial, such as mean and variance. Other authors have focused on the study of the development of probabilistic reasoning associated with the binomial in teachers and students, identifying worrisome weaknesses in aspects such as modeling, the perception of randomness and variability, the illusion of linearity, case-counting tasks, and the application of constructs such as combinatorics [10–15]. These research lines evidence a link between the general study of probability didactics and the one focused on binomial distribution by seeking the understanding of its meaning from its essential components and the solution of learning conflicts by identifying its origins. However, despite being closely related to the mentioned aspects, the historical nature of the latter as a historical construct—that is, from the ideas and concepts that gave rise to it and how they have evolved throughout history—has not been effectively addressed. Consequently, the need to propose a holistic meaning of the binomial distribution that facilitates the study of its understanding and the way it is proposed in learning planification is evident and is stated with the following questions: What is the holistic meaning of the binomial distribution? What are the essential elements that compose the binomial distribution, and how are they related to the construction of its meaning? Accordingly, Batanero [16] indicated that to ensure meaningful learning of a mathematical object, it is necessary to know its fundamental components and those that give it meaning.

For identification of the essential elements of mathematical constructs, the Ontosemiotic Approach (OSA) of Mathematical Knowledge and Instruction [17] provides theoretical and methodological notions that address diverse aspects of the problematic in didactics in mathematics, such as the epistemological, (How does mathematics emerge and develop?) ontological, (What is a mathematical object, and what types intervene in the mathematical activity) semiotic-cognitive, (What does it mean to know a mathematical object given a time and circumstance?) and educational-instructional. (What are teaching and learning?) With this approach, the meanings of statistic, probabilistic, algebraic, arithmetic, and calculus objects have been addressed [18–22] and studied from different products of human mathematical practice, such as a specialized bibliography, the answers of students or experts to mathematical problem situations, and the history of the mathematical construct. To address the epistemic and historical aspect in this task, the Historical Epistemological Study (HES) [23] has been a prevalent tool in investigation based on the OSA, consisting of the study of the nature, genesis, and consolidation of, in this case, mathematical concepts and their development through history. By identifying the situation problems that gave rise to a mathematical concept, its solutions, and the dialectical nature of its evolutionary process, we would approach its holistic meaning, which is delivered partially and in different ways in the educational process.

As a consequence, to answer the questions proposed about the binomial distribution, in this paper, we present the reconstruction of its holistic meaning from the analysis of an HES and specialized bibliography in statistics and probability, based on the theoretical notions of the practice, object, and configuration of the OSA. Regarding the results that will be presented, we can preview that we identify five historical periods in which concepts and ideas related to binomial distribution are approached from different levels of complexity that gave origin to four partial meanings: (1) approximation for case-counting problems, (2) the study of particular binomial situations, (3) the study of general or complex binomial phenomena, and (4) approximation to other probabilistic mathematical ideas such as normal distribution, the law of large numbers, and multinomial distribution. In addition, for each of the first three meanings, essential elements are identified, such as the concepts of chance, numerical patterns, combinatorics, binomial, summation, mean, and hope; procedures such as direct counting of cases, representation by means of the tree diagram, creation and validation of mathematical models, and the use of the binomial distribution formula; the use of properties to link chance with mathematics, link mathematics with probability, and link binomial distribution with the rest of mathematics and probability; inductive, recursive, and deductive arguments; and different levels of formality and complexity in the language. Finally, we reflect on the relationship between these elements, identifying, for example, that the evolution of the binomial distribution is closely related to that of other notions, such as probability and the expectation of a random phenomenon.

2. Theoretical Framework

To reconstruct the holistic meaning of the binomial distribution, we rely on the notions of the Ontosemiotic Approach (OSA) to Mathematical Knowledge and Instruction [24–27]. This theoretical approach articulates various theoretical perspectives related to the learning and teaching of mathematics, with the aim of addressing problematics of dimensions such as the epistemological, (What is mathematics?) ontological, (What is a mathematical object, and how does it intervene in the mathematical activity?) semiotic-cognitive, (What does it mean to know a mathematical object?) and educational [28]. (What are teaching and learning mathematics?) In the search to facilitate its learning, by answering the cited questions, the OSA defines mathematics as a human activity centered on the resolution of problem situations whose global understanding requires understanding of the different elements that are generated in it and how these are presented in the content and the reasoning of teachers and students. Furthermore, to facilitate its analysis and promote its learning, it organizes and systematizes the identifiable elements in mathematical activity, introducing notions such as practice, object, and configuration, which we will explore in more detail below.

A mathematical practice corresponds to "any action or expression (verbal, graphic, etc.) carried out by someone to solve mathematical problems, communicate the solution obtained to others, validate it or generalize it to other contexts and problems" [29] (p. 334). Thus, when trying to solve a problem, a person puts into play various practices that generate sequences, either of a personal nature or that are shared within an institution, which correspond to the meaning attributed to the notion or knowledge that the problem addresses, known as partial meaning. Based on this, global or holistic meaning is considered to be the potential systems of practices that a subject can manifest which, due to the historical nature of mathematics, are closely related to those of the past. Therefore, if we wish to foster an adequate understanding of mathematics and its notions, it will be necessary to consider the historical and epistemic components that gave rise to it and allowed it to evolve into what it is known as today. Consequently, for promoting the learning of the binomial distribution from the OSA, it is first necessary to address its epistemic aspect (What is the binomial distribution?) from the different systems of practices generated to answer the problem situations that refer to it and the different meanings that these have generated throughout history, addressing its historical and epistemological aspects and serving as a basis for extending its study to the personal meanings of students or teachers and how it is promoted in the curriculum by answering questions such as the following: What are the institutional mathematical-probabilistic practices necessary to answer a certain type of task of a binomial nature? What are the principles that allow the construction of the binomial distribution and facilitate its application to the resolution of problem situations?

The notion of object is more specific than the practice and derives from it. Formally, an object is any identifiable entity in the resolution of a practice [25], fulfilling an instrumental or representational, regulative, explanatory, or justificatory role. In this study, we rely on the main elements (primary objects) identified and described by Batanero [16] in the study of the meaning of probability from a historical perspective:

- Problem situations: A problem field from which the object emerges, usually in an informal state that is formalized over time and generalized for application to similar situations.

- Processes: These correspond to the resolutive methods, such as algorithms and operations, associated with the resolution of the problem situation. These evolve over time, becoming refined and generating complementary or alternative approaches, such as demonstration or verification.
- Definitions-concepts: Descriptions with which the different probabilistic ideas are evoked. To each concept corresponds its definition, which may vary depending on the problem situation and in which a set of properties is presented.
- Properties-propositions. Properties correspond to the attributes possessed by the mentioned objects, expressed by means of propositions which allow linking or associating them with others. For example, the multiplicative principles of probability allow linking probability with calculus.
- Arguments: Statements used to validate refutable objects (propositions and procedures). In current mathematics, deductive arguments predominate. However, those of antiquity, with which the first problematic situations were modeled, were mostly inductive or recursive.
- Linguistic elements: These correspond to the terms, expressions, notations, graphs, and tables evidenced in different registers through which data, solutions, and ideas are expressed and represented. Vásquez and Alsina [2] point out the following essential linguistic elements for probability: common language, probabilistic language, representation in tables and graphs, numerical representation, and the tree diagram. In the answers studied in our work, to facilitate their analysis, we will consider the categories of common (words), symbolic (purely mathematical expressions, such as equations), tabular, and graphical language.

The dimensions and elements proposed in the OSA have been used in the investigation of mathematics and the promotion of its learning. For example, Vásquez and Alsina [2] elaborated upon a model for the analysis of probabilistic objects in Chilean textbooks of primary education, focused on some of the elements mentioned (problematic situations, linguistic elements, and concepts) and identifying essential aspects such as the notions of chance and probability, their application in daily life, the experimentation, and distinction between theoretical and empirical probability. Beltrán-Pellicer et al. [30], using the OSA notion of suitability, elaborated upon the indicators of didactic suitability in probability teaching in secondary school, considering as elements to improve the consideration of a greater number of registers and representations, allowing students to generate their own modeling and compare them, and approaching the content from an intuitive approach over a transmissive one. Gómez et al. [19] analyzed the meaning of probability evidenced in textbooks for primary education in Andalusia (Spain), identifying that although the four meanings of probability suggested by the curriculum are partially developed, most of the concepts and their properties are not explicitly defined, and this, added to the fact that simulation, frequential meaning, and subjective meaning are not sufficiently mentioned, gives space to the generation of semiotic conflicts, biases, or incorrect heuristics when trying to apply them to situations in real life.

From the notions and investigations presented, we consider the concept of binomial distribution as a probabilistic mathematical object, originating from the solving practices of problem situations of a binomial nature and, with that, being approachable by OSA notions to facilitate the design of its teaching and the resolving of learning conflicts. However, to meaningfully analyze its understanding or competence in its use in different practices [26] and identify its essential elements (primary objects), it is necessary to analyze the relationships that exist between them. To this end, OSA introduces the notion of configuration, which represents the relationship between the elements of a practice (Figure 1) and is dependent on different factors, such as the institution's vision of the objects in question and the subject's level of theoretical training. Knowledge of this configuration, which can be reconstructed from different perspectives such as the historical one, is a facilitator of decision making focused on its teaching and learning.

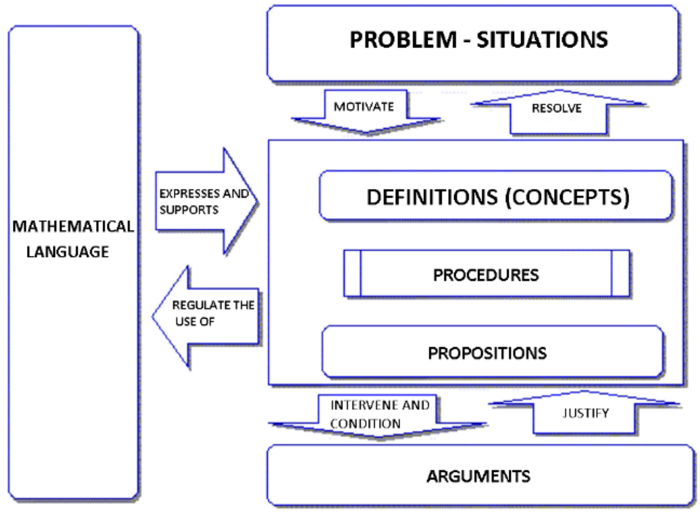

Figure 1. Components and relationships in an epistemic configuration. The mathematical object is considered a human product to resolve different problem situations, generating elements such as definitions, procedures, and propositions, justified by arguments and expressed by the language ("Componentes y relaciones en una configuración epistémica" by [31], licenced under CC BY-NC-ND 4.0 / translated from original).

In relation with the OSA and our object of study, Alvarado and Batanero [32] showed how engineering students approach a normal distribution by meanings of binomial distribution, evidencing that it is a complex task mostly because, although the group of students recognized essential the aspects of the binomial distribution, they presented numerous conflicts such as the confusion of its variance, mean, and proportion and its convergence with the general statement of the central limit theorem. This incomplete comprehension of the meaning of the binomial distribution can be associated with the disregard of its epistemic aspect and how the mathematical objects mentioned are related through its history.

The HES, as part of the OSA, allows for addressing this aspect of mathematical objects by allowing the identification of diverse conceptions of a mathematical object, its elements, and their development throughout history, providing the members of the educational world a system of knowledge about the mathematical object that considers its nature as a social construct and the result of a human need and intense theory. This is evidenced in investigations such as the ones presented by Batanero [16] and Gordillo and Pino-Fan [17], consisting of the reconstruction of the holistic meaning of probability and antiderivative, respectively. The first one articulates the different meanings of the probability identified and, using OSA notions, analyzes how the elements identified generate the holistic meaning of probability, also facilitating the analysis of how these are presented in an educative institution, the student learning, and the students' errors when solving probability problems. In a similar way, Gordillo and Pino-Fan [17] identified the practices approached by mathematicians in different historical periods, which gave way to the emergence and evolution of the notion of antiderivative, evidencing its relationship with two essential mathematical objects—the derivative and the integral—and four partial meanings: tangent quadratures, fluxion fluents, summation differences, and elementary functions.

In conclusion, the HES is evidenced to be an effective study in the didactics of mathematics investigation, especially for objects that are usually taught purely in a technical or operative way, since it facilitates resolving learning conflicts attributed to its epistemological aspects while also allowing its integration with the rest of the mathematics objects. As part of the theorical-methodological framework of the OSA, the HES would allow us to identify the essential elements in the origin and development of the binomial distribution throughout history and how they are related, which has not been addressed yet, and to reconstruct its holistic meaning and address its epistemic aspect, facilitating its investigation in today's education with the diverse perspectives of the OSA.

3. Methodology

Our study is of the qualitative [33] and documentary [34] types with an exploratory-descriptive design [35], dealing with the identification, analysis, and description of the essential elements of the binomial distribution and its relationships, collected and organized from a HES and the literature specialized in statistics and probability with the objective of thoroughly understanding the binomial distribution and the principles behind it in order to promote its adequate learning. For this, in the context of our work, the descriptive-exploratory design implies a series of tasks: (1) a previous work, in which the elements to be studied regarding the understanding of the binomial distribution are diagnosed and the modality of analysis of the components of the meaning of the binomial distribution are selected, (2) the recollection and synthesis of information about the binomial distribution, and (3) the identification, synthesis, and representation of new ideas regarding the meaning of the binomial distribution. These tasks are considered and resolved in the three phases of our research, which are further elaborated upon below.

For the first phase, we decided on the type of elements to be identified for a mathematical or probabilistic object to address its epistemological aspect with educational purposes. For this, in the investigation of didactics in the mathematics cited, we identified a consideration of problem situations and how people resolve them as essential, since the elements of a historic origin can be associated with the student learning that happens in the classroom. However, these are generic terms and would not, by themselves, allow the analysis of the epistemic aspect of the binomial distribution with educational purposes (i.e., proposing essential elements (such as combinatorics) that need to be considered in the planning of its teaching). For this reason, we selected the OSA as the theorical-methodological background of our analysis since, thanks to its notions and the HES, it facilitates the identification of essential components (objects) related to the resolution of problem situations (practices) and how they are articulated (configuration) while also considering their historical nature.

In the second phase, we compiled the historical elements and the problems that gave origin and development to the binomial distribution from the literature that addresses their history or teaching as a part of the field of statistics and probability [36–39]. The results of this documentary work were subsequently organized into periods, depending on the complexity of the problems they addressed and the way in which the binomial distribution was presented, in order to facilitate the identification of families or fields of problems that gave rise to the rest of the primary objects and were key in the process of the theoretical construction of the binomial distribution.

The third phase was approached on two levels. On the first one, we propose partial meanings from the identified problem families and their resolution practices based on the OSA notions of practice and object, thus making a first approach to the holistic meaning of the binomial distribution and its essential components without yet delving into how these are related. On a second level, using the literature designed for statistics and probability learning [40–45], we deepen the relationships between the primary objects of each of the partial meanings and the way in which they are configured to give way to the solution practices referred. These analyses were performed individually by every author and then contrasted, yielding to a triangulation of experts and the reaching of a consensus. By relying on OSA notions and the configuration from a historical perspective, in addition to complementing the elements identified thanks to cited bibliography, we propose a deeper level of analysis of the mathematical activity associated with the binomial distribution that addresses its history and epistemology and should be considered in the design of its learning and its epistemic configuration.

The phases presented resemble the methodology used by other authors for the reconstruction of the holistic meaning of other mathematical notions such as the derivative, antiderivative, and random variable from history [28,46,47]. These, once formalized, were extended, enriched, and strengthened by the authors in their use for the analysis of answers given by experts or students, the construction of evaluative instruments, the analysis of curricular documents, and the development of training strategies.

4. Historical Periods of Binomial Distribution Development Identified by the HES

The historical epistemological study allowed us to identify five periods, in which a specific family of problem situations associated with the binomial distribution were addressed with practices that increased in complexity and the number of applications.

4.1. 600 B.C.–14th Century: Case Counting and Informal Numerical Patterns

In Porphyry's *Isagoge* (234–305), one of the first questions of a combinatorial nature is identified: In how many ways can two things be selected from *n* different things? This was solved by analyzing the specific case n = 5 through direct case counting and recursive reasoning [38]. Similarly, problems related to case counting and combinations applied in other areas such as geometry are identified and solved by case counting or graphical representations.

This type of methodology and reasoning is applicable to the study of the number of possible, favorable, and unfavorable cases of phenomena of a binomial nature (e.g., coin tossing) and is also associated with primitive notions of probability [37]. In Dante's *Divine Comedy*, the first hints of probability, in the form of ratios, are identified [36].

4.2. 15th–17th Century: Formalization of Numerical Patterns and Probability as a Numerical Concept

During this period, mathematical patterns that could be associated with the binomial phenomenon were formalized, giving way to constructs such as Pascal's Triangle (Table 1), and game of chance phenomena were studied by means of probability in the form of ratios [35]. In the referred construction, figurative numbers, binomial coefficients, and combinatorics are related in a simple way.

Table 1. Pascal's triangle values (adapted from [38]).

r/c	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	4	10	20	35	56	84	120
4	1	5	15	35	70	126	210	330
5	1	6	21	56	126	252	462	792
6	1	7	28	84	210	462	924	1716
7	1	8	36	120	330	792	1716	3432

In this table adaptation, the binomial coefficients, terms of the Pascal triangle, and combinatorics are articulated, with "c" being the column and "r" being the row. We can identify the number of combinations of "n " things taken "k" at a time by looking for the term in the $(n - k)$th row and kth column.

Arnauld and Nicole in 1662 introduced probability as a numerical value applicable to other situations (aside from games of chance) by means of mathematical principles, promoting the passage from the calculus of chance to the theory of probability calculus.

4.3. 17th Century: The Beginning of Probability Theory and the Problem of Points

This period is characterized by the problem of points, to which the origin of probability theory is attributed and whose resolution consists of associating, by means of a binomial model, a number of points to each winner and, based on these, distributing the money [36,37].

A series of manuscripts by Cardano published in 1663 addressed topics that reflected the usefulness of mathematical analysis for profit in games of chance. In addition to introducing chance as the ratio of the number of favorable cases to the number of possible cases, he formulated the principles of addition and multiplication of probability [37].

Blaise Pascal and Pierre de Fermat introduced the concept of value. Specifically, if in a dice game the participant bets to roll a 6 in 8 throws, the player should receive 1/6 of the total for not making the first throw, 1/6 of the remainder for not making the second throw, and so on [36].

From this period, one can identify the evident arithmetization of probabilistic phenomena, making it no longer necessary to approach them from the counting of cases but rather through more abstract ideas that can be the basis of new arguments or deductions.

4.4. 17th–18th Century: The Informal Binomial Distribution for Modeling of the p = 1/2 Case and Modeling by Binomial Expansion

In this period, the generalization of expressions to model binomial situations was identified. Using combinatorics and Pascal's triangle as a base, Pascal developed the binomial for p = 1/2 to analyze the notion of what is currently known as hope and to provide a solution to the problem of points. Table 2 summarizes Pascal's reasoning for calculating e(a,b), the money that would correspond to A if the game stopped and player A was short of winning "a" games while player B was short of winning "b". It can be read, for example, in the case that the number of remaining games is 3 and A needs to win 1 of these, while B needs to win 2. If A wins the next game (3a), he or she should receive the total of the bet. However, if A loses the next game (3b), as each one lacks a win, then he or she would be entitled to half. By averaging these two possible cases, he obtained that A should receive 3/4 of the total bet [37].

Table 2. Recursive calculus of e(a,b) by Pascal (retrieved from [37]).

Description Event	a	b	e(a−1,b) e(a,b−1)	e(a,b)
1 game max	0	n > 0	-	1
2 games max	n	n		1/2
3 games max	1	2		
A wins	0	2	1	3/4
B wins	1	1	1/2	
4 games max	1	3		
A wins	0	3	1	7/8
B wins	1	2	3/4	

An association with the general formula is made in the letters between Fermat and Pascal: if the players have a and b games left, the game should end in a maximum of "a + b-1" games. Then, considering that all are played, there are 2^{a+b-1} possible cases, and the number of favorable cases for A relative to 2^{a+b-1} will give the fraction corresponding to A. As an example, for (a,b) = (2,3), there are 2^4 cases, of which 11 are favorable for A, whereas with Fermat's formula, e(2,3)= 11/16. This would be rearranged by Fermat, giving the general answer to the following point problem:

$$e(a,b) = \sum ({}_{a-1+i}C_{a-1})(1/2)^{a+i},\ i = 0, \ldots, b-1 \quad (1)$$

On the other hand, John Arburthnot presented an approach to the binomial phenomenon of coin tossing with the expression "$(S + F)^n$", with "n" equal to the number of tosses and with "S" and "F" equal to the probabilities of success and failure (nowadays designed with the letters "p" and "q"), respectively, which are not necessarily defined. Table 3 shows the study of the binomial phenomenon with p = 1/2 and n = 6 [36,39].

Table 3. Aburthnot's binomial phenomenon reasoning (adapted from from [39]).

Number of Successes	0	1	2	3	4	5	6
Possibilities	$(1/2)^6$	$6(1/2)^6$	$15(1/2)^6$	$20(1/2)^6$	$15(1/2)^6$	$6(1/2)^6$	$(1/2)^6$

In summary, in this period, we can identify that a probabilistic phenomenon is modeled using probabilistic principles, and although the binomial distribution is not presented in a formal way as we know it, the calculation of the currently known expectation and other values involving binomial reasoning are introduced.

4.5. 18th Century: Formalization of the Binomial Distribution

Jacob Bernoulli established the general expression of the binomial distribution for the analysis of what today is known as hope in the problems embodied in his work *Ars Conjectandi* (1613). Since the probability of "m" successes and "n−m" failures is "$p^m q^{m-n}$", then by combinatorial methods, it is deduced that there are "${}_nC_m$" different orders of "m" successes and "n−m" failures, which originates the formula of the binomial distribution [36]:

$$B\,(k,n,p) = {}_nC_k\; p^k q^{n-k}, 0 < p < 1 \tag{2}$$

In one of the letters exchanged with Pierre Redmond, Bernoulli remarked that the solution of the point problem for any value of "p" is obtained by expanding "$(p + q)^{a+b-1}$", giving the solution that corresponds to the generalization of Pascal's proposal:

$$e(a,b) = \sum\nolimits_{a+b-1} C_i p^i q^{a+b-1-i} \text{ with } i = a, a + 1, \ldots , a + b - 1 \tag{3}$$

Another application of this type of expression is for finding the number "n" of attempts needed to have good probabilities of at least "c" successes:

$$P = \{x \leq c - 1\} = \sum\nolimits_n C_x p^x q^{n-x} = 1/2, x = 0, 1, \ldots , c - 1 \tag{4}$$

Laudański [39] presented other examples of the use of the binomial, such as comparison with theoretical values, calculation of the mean and variance, analysis of the tails of the probability distribution, demonstration of the law of large numbers, approximation to the normal distribution by increasing cases and the notions of limit, multinomial distribution, and negative binomial distribution.

5. Reconstruction of the Partial Meanings of the Binomial Distribution

From the Historical Epistemological Study (HES), we can identify four families of problem situations, which increase in generality and can be categorized as follows: (1) problems related to the counting of cases, approached from the direct method, the graphical method, and the generation of numerical patterns and their generalization; (2) problems related to the calculation of the probability of particular binomial phenomena, approached from probabilistic and combinatorial principles (allowing support in graphical representations) and resulting in values and expressions that can be extended to an infinite number of trials; (3) problems related to general or complex binomial phenomena, in which the focus is on the study of the behavior of the distribution with different values and whose effective solution is achieved through the direct use of the formula or the study of the graph of the distribution; and (4) problems in which the binomial distribution is extended to other areas of mathematics and probability. In the following sections, we will elaborate on these families of problems and the systems of practices generated for their resolution (epistemic configurations).

5.1. Partial Meaning 1: Practice System for Binomial Case-Counting Problems

As shown in Table 4, the situations or family of problems belonging to this meaning are those that refer to the way in which elements can be ordered and their use for counting cases and their study, such as how many ways two things can be selected from "n" different things. These can be approached from two levels of complexity: an informal one (in which the answer is constructed from the direct counting of cases by means of numbers or graphic representations) and a formal one (in which expressions such as "$1/2n(n-1)$" are used for more complex situations).

Table 4. Practice system for binomial problems involving case counting.

Problem Situation (PS)	Mathematical Practices (EC)	Partial Meaning (PM)
PS1: Study the cases that a binomial phenomenon can present	**EC1: By means of direct counting method, use of graphical representations, or mathematical patterns, study the possible cases of a binomial phenomenon**	**PM1: The binomial phenomenon for case counting**
PS1.1: How many ways can two things be selected from "X" different things?	EC1.1: Direct counting or recursive techniques (fixed "X") and use of numerical patterns and inductive reasoning (indefinite or variable "X")	PM1.1: Selection of two things from "X" things
PS1.2: How many cases (favorable and unfavorable) do we have when rolling a die or coin if we want "X"?	EC1.2: Direct counting, graphical representation, and use of combinatorics	PM1.2: Analysis of possible cases: favorable or unfavorable

Main problem-situation, mathematical practices and partial meaning are shown in bold. The problem situations given consist strictly of the study of the number of cases of a binomial phenomenon or its characteristics, thus addressing its possibilities. Because it was not developed as part of the formal probability theory, the maximum complexity of this practice system would be reached in a task such as comparing proportions of the cases studied and identified, obtained by direct counting or the use of patterns identified or the formula, concluding, for example, that it is more possible to obtain an even number when throwing a 6-faced die than to obtain a multiple of three.

5.2. Partial Meaning 2: Practice System for Binomial Case-Counting Problems

The second system of practices (Table 5) goes hand in hand with understanding probability as being applicable to other situations beyond games of chance from mathematical principles, approaching the binomial phenomenon with the best known value (p = 1/2) and situations, such as the problem of points (money distribution).

Table 5. Practice system for binomial situations with one variable or for specific cases.

Problem Situation (PS)	Mathematical Practices (EC)	Partial Meaning (PM)
PS2: Calculate the probability in binomial situations with specific data	**EC2: Mathematical modeling and analysis (with or without support of graphical representations such as flowchart or distribution), generation of formulas, and calculation with mathematical rules**	**PM2: Probabilities of specific binomial phenomena**
PS2.1: What is the value of a binomial trial with "p" probability of success and "q" probability of failure?	EC2.1: Multiplication of probabilities by their respective values	PM2.1: Assigning value to a trial
PS2.2: Does X behave randomly?	EC2.2: Calculation of objective probabilities and comparison with empirical results	PM2.2: Validation of randomness of phenomena
PS2.3: What is the probability of obtaining the same result "n" times in a binomial trial?	EC2.3: Raise "p" or "q" by the power "n"	PM2.3: Probability of edge cases
PS2.4: What is the probability of obtaining "X" successes in "n" trials?	EC2.4: Calculation of the probability of one of the cases and its combinatorics	PM2.4: Probability of a specific number of successes in a binomial phenomenon
PS2.5: Probability of obtaining at least "X" hits in "n" throws (defined probability)	EC2.5: Additive principles, Pascal's triangle, or combinatorics	PM2.5: Probability of an interval of the values of the random variable of the binomial distribution
PS2.6: Study of A's expectation (against B) if a series of trials stops, where A needs "a" successful trials and B needs "b" failed trials (with equal probability)	EC2.6: Calculation with recursive, combinatorial, and multiplicative principles, similar to the point problem	PM2.6: Calculate the expectation of a series of incomplete binomial phenomena

Main problem-situation, mathematical practices and partial meaning are shown in bold. The problem situations given address the calculus of probability by using the basic principles introduced as the basis of probability theory and its development. As an object not fully developed, the binomial probability is addressed as a construction made with punctual application of the multiplicative principle of probability and combinatorics. From a simple task, such as calculating a punctual probability of a binomial phenomenon, to more complex task, such as the study of the problem of points with specified data, this practice system can be addressed as one that does not require the binomial distribution formula for an effective or possible approach.

5.3. Partial Meaning 3: Practice System for Complex or Varying Binomial Situations

The third system (Table 6) involves the binomial distribution for the study of phenomena, in which a high or undefined number of trials is performed and gives rise to the general formula of the binomial distribution, applied directly to the modeling of probabilistic phenomena.

Table 6. Practice system for problems involving more complex or varying binomial situations.

Problem Situation (PS)	Mathematical Practices (EC)	Partial Meaning (PM)
PS3: Study probability in variable or difficult binomial situations	**EC3: Modeling and mathematical analysis with the general formula or its variations and deductive analysis and calculation of properties such as mean and variance**	**PM3: Probability in general or complex binomial phenomena**
PS3.1: What is the probability of having "m" successes in "n" attempts?	EC3.1: Use of the binomial formula	PM3.1: The probability of a random variable value of a binomial phenomenon
PS3.2: What is the expectation of a complex binomial phenomenon?	EC3.2: Calculation of the mean	PM3.2: Expectation of a complex binomial phenomenon
PS3.3: What is the most probable value of the random variable "n" of successes?	EC3.3: Calculation of the mode	PM3.3: Most probable number of hits
PS3.4: What is the degree of dispersion of the number of successes obtained in a binomial phenomenon?	EC3.4: Calculation of variance	PM3.4: Dispersion number of successes
PS3.5: Probability of obtaining at least "X" hits in "n" throws (indefinite or variable probability)	EC3.5: Summation of probabilities (formula)	PM3.5: Probability of an interval of the values of the random variable of a complex binomial phenomena
PS3.6: With "x" being the number of successes and "y" being the number of failures ($x + y = n$), and letting A win if $x \geq m$, what is the expectation that B will win?	EC3.6: Calculus with recursive, combinatorial, and multiplicative principles, similar to the general problem of points	PM3.6: Expectation of incomplete binomial phenomenon
PS3.7: How many trials are necessary to have at least "c" successes?	EC3.7: Equalizing the sum of probabilities to 1/2	PM3.7: Favorable number of trials

Main problem-situation, mathematical practices and partial meaning are shown in bold. This practice system consists of resolving problem situations that require the use of the binomial distribution formula to be solved effectively, which also means an efficient use of time. By using the fully developed and proven formula, students can address diverse types of binomial situations where multiple binomial phenomena or varied probabilities are involved and study how their graphics change in different situations. This new approachable field of situation problems, which considers the binomial distribution formula as an instrument, also requires the comprehension of more complex probability ideas such as the variance, mean, and mode and reaches its maximum complexity when analyzing generic solutions to problems such as the problem of points or the degree of dispersion of the number of successes.

5.4. Partial Meaning 4: Practice System for Situations That Go beyond the Simple Application of Binomial Distribution

Finally, the fourth meaning is composed of the system of practices in which the binomial distribution is used to analyze other concepts of probability and mathematics, such as the law of large numbers and the normal distribution. Therefore, we consider this meaning as the extended meaning of the binomial distribution, a meaning that is not exclusive to it and that links it with other probabilistic-mathematical objects (Table 7).

Our results suggest that it is important to start with the construction of the combinatorial and probabilistic principles applied to the counting of cases, followed by the analysis of specific binomial situations, the obtaining of probabilities associated with these, and the construction of expressions that represent their behavior, finally ending with the construction of the binomial and its application to the direct calculation of probabilities and answering other questions. This linear development of the binomial distribution also shows that the mathematical practices of one meaning can be applied to the family of problems of the previous meaning.

Table 7. Practice system for problems involving the binomial distribution in other areas of probability or mathematics.

Problem Situation (PS)	Mathematical Practices (EC)	Partial Meaning (PM)
SP4: Study of other probabilistic or mathematical phenomena	**EC4: Mathematical deductive and inductive modeling and analysis with the general formula or its variations**	**PM4: Extended meaning of the binomial distribution**
SP4.1: How close to the experimental values are the theoretical values? How many trials do we need to ensure that we are getting close to them?	EC4.1: Search for the value "n" such that $P_n = P\{\lvert h_n - p \rvert \leq \varepsilon\} > c > 0$	PM4.1: Law of large numbers (probability)
SP4.2: With what degree of certainty can we rely on specific values of the binomial distribution?	EC4.2: Calculation of intervals	PM4.2: Confidence intervals
SP4.3: What happens to the distribution as we increase the number of trials?	EC4.3: Graphical method and boundary analysis	PM4.3: Extension of the binomial distribution to the normal distribution
SP4.4: What happens when the results fall into different categories and trials or selections occur without replacement? Considering "n" independent trials with "f" equally likely outcomes, what is the number of ways to obtain "n_i" outcomes for one type?	EC4.4: Generate, with combinatorial and other binomial principles, the probability of the multinomial phenomenon	PM4.4: Multinomial distribution
SP4.5: What is the expectation, when an event has occurred "p" times and failed "q" times, that the original ratio of occurrence or non-occurrence of an event is different from the ratio between "p" and "q"?	EC4.5: Relate combinatorics, integral, probabilistic principles, and conditional probability	PM4.5: Inductive inference
SP4.6: Given a binomial phenomenon with a given "p", what is the probability that in the fifth trial there were "n" successes?	EC4.6: Construction of the probability function from binomial principles	PM4.6: Negative binomial distribution

Main problem-situation, mathematical practices and partial meaning are shown in bold. This practice system is only accessible if the previous one has been completely comprehended and consists of the use of the binomial distribution and its formula to approach other mathematical or probabilistic objects. By fully developing its meaning, the binomial distribution and its behavior can be used to study or complement other aspects, thus giving it an active role as part of the probability and mathematical theory while addressing questions that extend what is already known of the binomial distribution, such as limit theory or multinomial phenomena.

6. Primary Objects of Partial Meanings

Based on the problem situations identified in the HES, and relying on a complementary specialized bibliography on probability, we deepen our understanding of the primary objects associated with each of the exclusive partial meanings of the binomial distribution (EC1, EC2, and EC3), considering their main characteristics and the relationship they have with other mathematical or probabilistic notions.

6.1. Primary Objects of Partial Meaning 1 (Case Counts)

As mentioned above, the problem situations that are part of this meaning correspond to those that refer to the counting of cases and their study. This type of problem situation can be approached informally with the counting of cases or, in a more formal way, using the principles of combinatorics and constructing numerical patterns. This requires the subject to become familiar with inductive inference, breaking with determinism and extending empiricism into the unknown by means of induction while also considering the possible error involved, which is associated with variability and the existence of a degree of security when generating deductions.

For this reason, the definitions and concepts of this meaning can be linked to those of the primitive meaning of probability that existed long before it was defined [16]. For the knowledge of chance or random phenomenon, the important elements are chance, the case or event, numerical pattern, power, inequality, variable, unknown, combinatorics (permutation and variation), value of the variable, order, probability in the form of ratios, figurative numbers, and Pascal's triangle, while for the knowledge of the event, the notions of sample space, events (excluding and non-excluding), and discrete random variables are important [40]. For the understanding and construction of combinatorics, in a similar way to the basic ideas of probability calculation, the understanding of different situations depending on the number of elements selected, the existence or non-existence of repetition (variations, permutations, and combinatorics), and constructs such as combinatorial numbers and the factorial are considered essential [41].

Regarding resolution procedures, the direct counting of cases, the construction of the sample space, the construction and validation of algorithms originating from numerical patterns, and the use of graphical representations to record and analyze the phenomena studied are notable. This agrees with the study of combinatorics by Espinoza and Roa [48], who identified as important tasks for combinatorics its recognition and application of the operation and the enumeration and demonstration or search for its properties.

Similarly, the propositions and arguments with which the procedures and concepts are presented and linked are focused on the early search for a degree of knowledge about the uncertain (i.e., the rupture of the deterministic paradigm and the use of mathematics to model and analyze the unknown). Using inductive reasoning and recursive expressions as arguments, the subject constructs expressions that mathematically model the counting of cases and employs them to his or her benefit in the prediction of phenomena and practices such as divination or games of chance. For example, the product and sum rule is considered essential for the counting of cases, which allows the generation of more complex ones, such as the multiplication of the number of possible cases of two selections to obtain the number of total arrangements. This corresponds to permutations, whose argumentation is one of the main steps for the construction of combinatorics [43] and the understanding of the relationship between figured numbers, combinatorial coefficients, and Pascal's triangle [39]. In addition, we consider that a subject can use graphical representations as argumentation of propositions, as they follow non-explicit principles of inductive or deductive reasoning, as presented in Moreno [49].

Finally, the language used for this meaning is not very refined with respect to the theory of mathematics and probability, since it does not yet involve demonstrations. In addition to the common language, the use of symbolic language (mathematical-probabilistic) is presented for the counting of cases, also making possible their registration or study using tabular or graphical language (tree diagram), as shown in Table 8 and Figure 2.

Table 8. Tabular language for representing the roll of two dice (retrieved from [44]).

	1	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

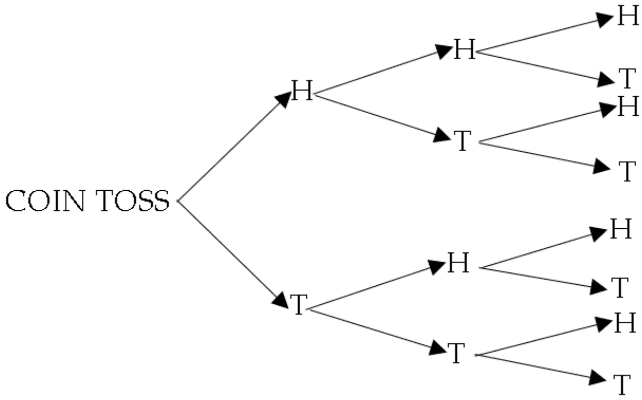

Figure 2. Graphical language for representing three coin tosses (binomial trial), where H = head and T = tails (adapted from [43]).

The elements of this meaning correspond to those explicitly or implicitly referred to by the early experts for the study of phenomena of a binomial nature, taking the empirical work to the scientific-mathematical, quantifying the observable, and constructing mathematical expressions such as permutations, which allow the counting of cases by their application.

6.2. Primary Objects of Partial Meaning 2 (Particular Binomial Situations)

To the range of definitions and concepts mentioned above, we add those belonging to the early state of probability theory, in which mathematics is used for the calculation of probabilities. These are probability as a numerical value between 0 and 1 (no longer as a proportion), the value associated with a random experiment, summation, and the extension of the development of the binomial. Knowing already the probability as a numerical value that allows the analysis of a variable (in this case, discrete), the concept of probability distribution, and its function allows us to analyze in an empirical or purely theoretical way phenomena that we identify as binomial [43] and compare those values, observing how these become increasingly similar as the number of trials increases. This also allows us to give a significant value to the probabilities (monetary, for example), facilitating their application in phenomena of uncertainty in daily life [50].

The processes associated with this meaning are homologous to those carried out by experts such as Pascal and Fermat to solve the problem of points: the generation, verification, and correction of mathematical models of phenomena of a binomial nature once identified, as well as the study of the expectation when assigning values to the probabilities of a phenomenon. For this, the subject has available the definitions and concepts mentioned so far, which allow for identifying binomial situations and constructing expressions for the calculation of the probability of binomial phenomena that follow specific rules. For example, if one wishes to obtain the probability of a part of a distribution, one uses the principles of probability (homologous to the number of cases) that can be simplified using summation [44].

Because of the above, it is not strange that the propositions and arguments deal with the relationship between mathematics and probability (i.e., associating probability with the hope or value of a trial, the use of multiplicative and additive principles related to binomial coefficients, and the expansion of the binomial for the calculation of the probability of binomial phenomena. From the search for the solution to the problem of points, we can identify the proposition that the value of a binomial experiment corresponds to the sum of the products of the probability of its outcomes and the value that each one has, argued by multiplicative and additive principles (deductive reasoning). The relationship between combinatorics, figured numbers, binomial coefficients, the number of possible cases (demonstrable using induction), and the multiplicative and additive principles introduced by Cardano (1663), generated by trial and error, as well as direct inspection allow us to argue propositions such as the probability of a binomial phenomenon having the same result an "n" number of times is given by the probability of that result raised to "n", and we can obtain the probabilities associated with a binomial phenomenon by studying the expansion expression "$(p + q)^n$", in which "p" and "q" are their probabilities and "n" is the number of trials.

The language used is complemented with new mathematical expressions and probabilistic ideas, using symbolic language for the construction of new mathematical expressions and using tables and tree diagrams to record and analyze the probabilities of binomial phenomena with a specific "n" or other notions, such as the expected value (Figures 3 and 4 and Table 9), thus allowing the continued use of representations as an argumentative tool.

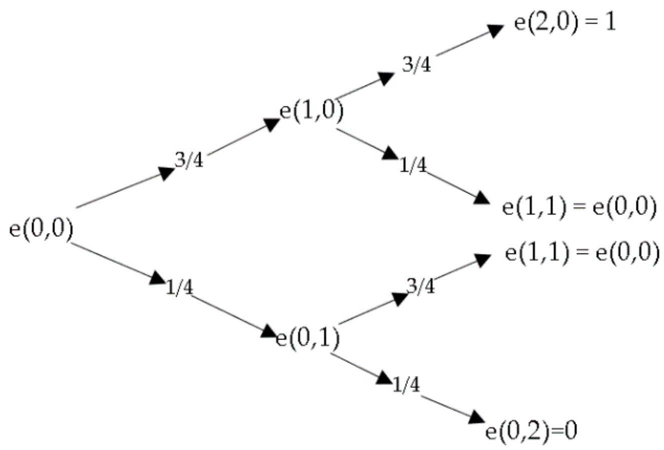

Figure 3. Tree diagram for calculating the expectation of a game that ends with two binomial trials, with a probability of success p = 3/4 and probability of failure q = 1/4. By multiplicative principles and resolving the resulting equation, the expectation or value of the situation for the player is 9/10 (adapted from [37]).

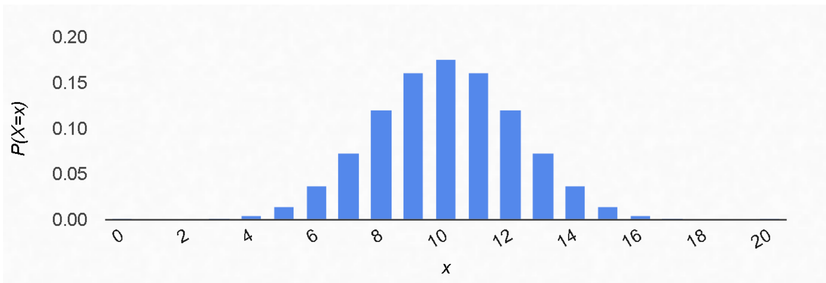

Figure 4. Graphical language element for the binomial with n = 20 and p = 1/2.

Table 9. Study of the binomial phenomenon with n = 4, with x being the number of successes (S) obtained (retrieved from [44]).

Outcome	x	Probability	Outcome	x	Probability
SSSS	4	p^4	FSSS	3	$p^3(1-p)$
SSSF	3	$p^3(1-p)$	FSSF	2	$p^2(1-p)^2$
SSFS	3	$p^3(1-p)$	FSFS	2	$p^2(1-p)^2$
SSFF	2	$p^2(1-p)^2$	FSFF	1	$p(1-p)^3$
SFSS	3	$p^3(1-p)$	FFSS	2	$p^2(1-p)^2$
SFSF	2	$p^2(1-p)^2$	FFSF	1	$p(1-p)^3$
SFFS	2	$p^2(1-p)^2$	FFFS	1	$p(1-p)^3$
SFFF	1	$p^2(1-p)^3$	FFFF	0	$(1-p)^4$

The elements of this meaning are related to the first uses of probability as a numerical value that follows a set of mathematical rules and can be analyzed by means of these rules for the search of solutions to specific situations of uncertainty, though not necessarily in games of chance.

6.3. Primary Objects of Partial Meaning 3 (General or Complex Situations)

The problem situations characteristic of this meaning are those whose adequate resolution involves the direct use of the binomial distribution formula. These situations consist of binomial phenomena in which a high, multiple, or infinite number of trials or variable values of the probabilities are presented, reaching different levels of complexity, one of the simplest being that which is answered directly with the formula, such as what the probability of obtaining "k" successes in 30 trials is with p = 60 [44]. On the other hand, one of the complex situations would be how many trials are necessary to have at least "c" successes and, when letting "a" be the number of successes and "b" be the number of failures obtained so far, what the expectation is that the number of successes is greater than a specific value "m".

The definitions and concepts that are added to this meaning are related to the theory of probability applied to the binomial distribution once it is formalized: the binomial distribution and its formula, probability function, parameter, probabilistic inference, degree of knowledge (subjective probability), mean, variance, and expected value (hope). Wackerly et al. [43] also introduced as a concept of this meaning the derivative and one of its partial meanings (i.e., to maximize the probability of obtaining a certain number of hits). Walpole et al. [41] mentioned the concept of binomial summation, referring to the notion of summation applied to the various probabilities of the distribution. Devore [45] referred to the expected values of successes and failures, considering as a base notion the test of hypotheses, as well as the use of the binomial distribution to calculate the number of trials needed to have a certain number of successes (i.e., the negative binomial). Finally, the binomial distribution becomes part of probability theory, and it is possible to associate it with the axioms of probability, such as indicating that the binomial summation must give one and that the probability of every event is equal to or greater than zero [13].

The processes of this meaning address the use of binomial distribution for the analysis of different probabilistic phenomena, such as by directly applying the formula to calculate the probability of an event once its parameters have been identified and using it to estimate values such as the mean and variance, as well as compare the theoretical values with the empirical ones. In the same way, the propositions and arguments deal with this binomial distribution as an object for the analysis of situations. Thus, propositions are presented such as that in a binomial trial, with the probability of obtaining "m" hits in "n" trials with a probability 'p' of success given by the binomial formula, the expected value in a binomial phenomenon is given by its mean, and the probability of having a number greater than or equal to "x" number of hits or misses is given by the first or last "x" elements of the binomial expansion [42]. In addition, propositions related to the identification of a trial as binomial or Bernoulli [44] are considered essential. These types of propositions are argued using the combinatorial and probabilistic principles that gave rise to the binomial distribution formula.

Finally, regarding the language, as the maximum generality of the mathematical concept is reached, the use of symbolic language predominates, approaching the binomial distribution from its general formula and other mathematical concepts, such as functions and mathematical sets [40,41]. However, tabular or graphical representations are still used, for example, to represent particular phenomena and compare them with the binomial distribution or deliver specific values of interest, such as cumulative binomial distributions (see Figure 5 and Tables 10 and 11). The automatic generation of these representations, using probabilistic software or previously constructed resources, allows them to continue to be used for argumentation, for example, by using the graph of the binomial distribution to visually approximate its mean.

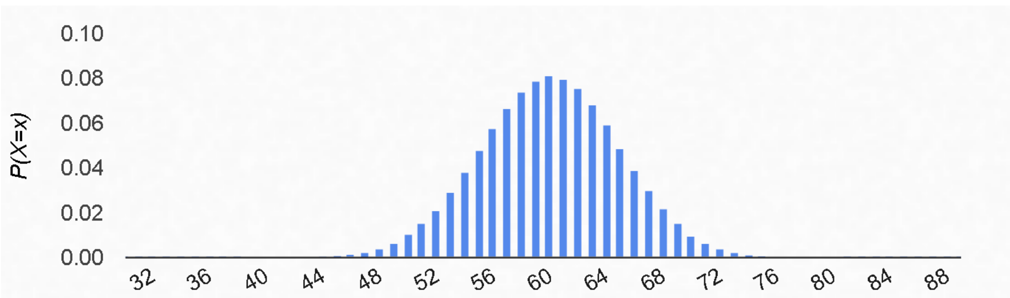

Figure 5. Histogram of 100 binomial recounts (N = 100, p = 0.6) resembling the probability density of the normal distribution.

Table 10. Comparison table of theoretical probability and observed ratios in a binomial situation ending with 4 successes or 4 failures, considering p = q (retrieved from [44]).

Series Length	Observed Proportion	Theoretical Probability
4	17/68 = 0.250	0.125
5	16/68 = 0.235	0.250
6	21/68 = 0.309	0.3125

Table 11. Table of cumulative binomial distributions for n = 3 (retrieved from [45]).

		P														
		0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
x	0	0.951	0.774	0.590	0.328	0.237	0.168	0.078	0.031	0.010	0.002	0.001	0.000	0.000	0.000	0.000
	1	0.999	0.977	0.919	0.737	0.633	0.528	0.337	0.188	0.087	0.031	0.016	0.007	0.000	0.000	0.000
	2	1.000	0.999	0.991	0.942	0.896	0.837	0.683	0.500	0.317	0.163	0.104	0.058	0.009	0.001	0.000
	3	1.000	1.000	1.000	0.993	0.984	0.969	0.913	0.812	0.663	0.472	0.367	0.263	0.081	0.023	0.001

The primary objects mentioned above give meaning to the binomial distribution as a structural whole, which can be identified from a set of mathematical and probabilistic properties and allows for analyzing different cases with ease and studying what happens with infinite cases, losing some numerical substances and algorithms such as the use of the tree diagram.

A synthesis of components of the epistemic configurations of the three meanings of the binomial distribution is presented in Tables 12–14.

Table 12. Primary objects of the first meaning of binomial distribution.

Primary Object	Description
Problem Situation	S1. Study the cases that a binomial phenomenon can give. S1.1 In how many and what ways can two things be selected from "X" different things? Example: How many ways can two things be selected from 5 different things? S1.2 How many or which cases (favorable and unfavorable) do I have when throwing a die or flipping a coin if I want a specific amount of successes? Examples: What are the possible cases in throwing 2 or 3 dice? What are the favorable cases in flipping 2 coins if I want one side?
Definitions-Concepts	Chance, case, event (both excluding and non-excluding events), numerical pattern, power, inequality, unknown, binomial coefficient, combinatorial, variable value, order, permutation, variation, probability in the form of ratios, figured numbers, Pascal's triangle, repetition, sample space, discrete random variable, and factorial.
Processes	Exploration and modeling Direct counting of cases, exploration of possible outcomes (construction of the sample space), pattern recognition and construction as well as validation of algorithms based on them, search for properties, and use of representations such as tree diagrams.
Properties-Propositions	Linking mathematics with the uncertainty of chance Games of chance, as well as other phenomena, follow a behavior that allows them to be analyzed among different representations. A degree of knowledge of chance can be obtained by using mathematical expressions. Adding the possible cases of two or more events gives the number of total arrangements (additive principle for counting cases). Multiplying the possible cases of two or more events gives the number of total arrangements (multiplicative principle for counting cases).
Arguments	Inductive and recursive principles for constructing expressions and discovering properties. Use of representations.
Language	Unrefined mathematically and probabilistically (arithmetic expressions), focused on case counting. Tabular and graphical.

Table 13. Primary objects of the second meaning of the binomial distribution.

Primary Object	Description
Problem-Situation	S.2 Calculate the probability in particular binomial situations with specific data. S2.1 What is the value of a binomial trial with p probability of success and q probability of failure? Example: How much is a two-coin toss worth if USD 1000 is won if only one side comes up? S2.2 Does an event exhibit random behavior? Example: If heads comes up 5 times in a row, what can be said about the coin? S.2.3 What is the probability of getting the same result "n" times in a binomial trial? Example: What is the probability of getting heads 5 times in a row? S2.4 What is the probability of getting "X" hits in "n" tosses (without using the formula)? Example: What is the probability of getting heads in a toss of 3 coins? S2.5 What is the probability of getting at least "X" hits in "n" throws (defined probability) or another interval of the probability distribution? Example: What is the probability of getting at least 1 head in a toss of 3 coins? S2.6 Study the expectation of A (against B) if a series of trials stops, where A needs "a" successful trials and B needs "b" successful trials (with equal probability). Example: Considering that both players have an equal probability of winning, what is the hope of player A if he or she must win 3 times against player B, who wins by winning 2 times?
Definitions-Concepts	Probability as a numerical value between 0 and 1, the value associated with a random experiment, probability distribution, summation, and the extension of binomial development.
Processes	Modeling probabilistic phenomenon Identify characteristics of probabilistic phenomena. Generation, verification, or correction of mathematical models of phenomena of a binomial nature from the definitions and concepts mentioned above. Calculation of the expected value when assigning values to the probabilities of a phenomenon.
Properties-Propositions	Linking probability with mathematics The expected value of a binomial experiment corresponds to the summation of the products of the value of each variable by its respective probability of occurrence. We can obtain the probabilities associated with a binomial phenomenon by studying the expansion expression "$(p + q)^n$", in which "p" and "q" are their probabilities and "n" is the number of repetitions of the trials. The additive and multiplicative principles of probability.
Arguments	Deductive. Inductive. Use of representations.
Language	Common, symbolic, tabular, and graphical language that relates the definitions and concepts of meanings 1 and 2 to the calculation of specific probabilities.

Table 14. Primary objects of the third meaning of the binomial distribution.

Primary Object	Description
Problem-Situation	S.3 Study probability in binomial variable situations. S3.1 What is the probability of having "m" successes in "n" attempts (using the formula)? Example: What is the probability of getting "k" successes in 30 trials with p = 0.6? S3.2 What is the expectation (expected value) of a binomial variable phenomenon? Example: What is the expected outcome of a toss of 20 coins? S.3.3 What is the most probable value of the random variable number of successes? Example: What is the most probable number of heads when tossing 40 coins? S.3.4 What is the degree of dispersion of the number of successes obtained in a binomial phenomenon? Example: How dispersed are the results of 5 binomial trials with p = 3/4? S3.5 Probability of obtaining at least "X" hits in "n" trials (indefinite or variable probability) or another interval of the probability distribution. Example: What is the probability of getting at least 20 heads in a series of coin tosses? S3.6 Let "x" be the number of successes and "y" be the number of failures ($x + y = n$), and let A win if $x \geq m$. What is the expectation that B wins? Example: Considering that both players have different probabilities of winning, what is the hope of player A if he or she has to win 3 times versus player B, who wins by winning 2 times?S.3.7 How many trials are necessary to have at least "c" successes? Example: After how many binomial trials with p = 7/8 can one be sure that there is at least one failure?

Table 14. *Cont.*

Primary Object	Description
Problem-Situation	S.3 Study probability in binomial variable situations. S3.1 What is the probability of having "m" successes in "n" attempts (using the formula)? Example: What is the probability of getting "k" successes in 30 trials with p = 0.6? S3.2 What is the expectation (expected value) of a binomial variable phenomenon? Example: What is the expected outcome of a toss of 20 coins? S.3.3 What is the most probable value of the random variable number of successes? Example: What is the most probable number of heads when tossing 40 coins? S.3.4 What is the degree of dispersion of the number of successes obtained in a binomial phenomenon? Example: How dispersed are the results of 5 binomial trials with p = 3/4? S3.5 Probability of obtaining at least "X" hits in "n" trials (indefinite or variable probability) or another interval of the probability distribution. Example: What is the probability of getting at least 20 heads in a series of coin tosses? S3.6 Let "x" be the number of successes and "y" be the number of failures ($x + y = n$), and let A win if $x \geq m$. What is the expectation that B wins? Example: Considering that both players have different probabilities of winning, what is the hope of player A if he or she has to win 3 times versus player B, who wins by winning 2 times?S.3.7 How many trials are necessary to have at least "c" successes? Example: After how many binomial trials with p = 7/8 can one be sure that there is at least one failure?
Definitions-concepts	Binomial distribution, binomial distribution formula, probability function, parameter, probabilistic inference, degree of knowledge (subjective probability), mean, variance, and expected value (hope).
Processes	Application for analysis Determination of parameters of the binomial distribution. Calculate the probability by directly applying the formula. Compare theoretical values of a binomial phenomenon and observed ratios. Varying parameters in simulations. Approximating or calculating measures such as mean and variance.
Properties-Propositions	Link the mathematical binomial distribution with the rest of mathematics and probability. In a binomial phenomenon, there are "n" observations. The observations or trials in a binomial phenomenon are independent. In a binomial phenomenon, there are only 2 outcomes (success and failure). The probabilities of success and failure are constant. In a binomial trial, the probability of getting "m" hits in "n" trials with a probability of success "p" is given by the binomial formula. The expected value in a binomial phenomenon is given by its mean. Considering a high series of experiments, if the observed proportion does not agree with the theoretical probability, we can assume that the phenomenon is not binomial.
Arguments	Deductive. Use of representations.
Language	The use of symbolic language is predominant, while the use of tables and graphs is used to study particular cases.

7. Conclusions

In this work, by means of a historical epistemological study (HES), and with its analysis based on the notions of the Ontosemiotic Approach (OSA) to Mathematical Knowledge and Instruction, we reconstructed the holistic meaning of the binomial distribution, identifying the problem situations that gave rise to it, its resolution practices, and its main components. With this, we reaffirm this work methodology as adequate to identify essential elements in the learning of mathematical and probabilistic objects, especially those whose weaknesses and learning conflicts are associated with the lack of consideration of its historical and epistemic aspects, allowing their investigation with educational purposes based on the diverse notions and perspectives of the OSA, such as proposing the elements and relations identified as essential for its learning.

Regarding the binomial distribution as a concept, it was possible to identify that its origin came from the analysis of games of chance which, with the passage of time, was extended to the study of other phenomena that met identical or similar characteristics. In this way, it was possible to reaffirm the historical nature of the binomial. Thus, the study resulted in the identification of four families of problems: (1) problems related to counting cases, (2) problems related to calculating the probabilities of particular binomial phenomena, (3) problems related to general or complex binomial phenomena, and (4) problems that extend the binomial distribution to other elements of mathematics or probability. Each of these gives rise to a system of practices corresponding to one of the partial meanings of the binomial distribution: (1) a binomial phenomenon for studying cases, (2) the binomial distribution for particular binomial situations, (3) the binomial distribution as a tool for general or complex binomial situations, and (4) the extended meaning of the binomial distribution. What was also observed in the HES indicates that the main differences between these families are the degree of complexity and the number of situations to which their solution methods can be effectively applied, which increase over time, suggesting that the different meanings of the binomial distribution present a linear configuration if considered from a historical perspective, meaning that the resolution practices of one meaning are also applicable to the problem situations associated with the previous meaning when making the appropriate adjustments. When contrasting our results with the meanings identified in works with the OSA that do not address their historical aspects [20,22], we can conclude that the use of the HES is beneficial, since it allows identifying the development of the different meanings as a response to the apparition of more complex problem situations and how their essential elements are related, which could help to propose didactic trajectories that address the historical aspect of the respectively mathematical object. Similarly, there are also differences in comparison with the structure of holistic meanings reconstructed on basis of the OSA and HES [18,29]; that is, because the holistic meaning of the binomial distribution seems to follow a linear development defined by an increase of complexity of the problem situations addressed, the holistic meanings of objects such as the antiderivative and derivative are proposed in a more ramified form in the meantime, where some of their meanings are related and developed at the same time, addressing different families of problem situations. This can be attributed to the difference in nature of the mathematical objects studied, since the binomial distribution was developed in the same way as the meaning of probability and its theory [16] to model random phenomena with increasing complexity.

On a second level of analysis, supported by the use of a specialized bibliography in statistics and probability, we identified the primary objects of each of the exclusive meanings of the binomial distribution (one, two, and three). For the first partial meaning, we highlighted the definitions-concepts of chance, case, probability as a ratio, discrete random variables, combinatorials, variable values, sample space, and factorials, procedures focused on the exploration and generation of mathematical models, propositions whose function is to mathematize uncertainty and the counting of cases, inductive and recursive arguments, and an unrefined symbolic language and graphic language used for the registration and counting of possible cases. Regarding the second partial meaning, we highlighted the definitions-concepts of probability as a numerical value between 0 and 1, the use of modeling procedures, propositions that link probability with mathematics (associated with the development of probability theory), the use of arguments based on mathematical properties and the characteristics of the binomial phenomenon, and the articulation between symbolic, tabular, and graphic languages. Finally, for the third partial meaning, we highlighted the definitions-concepts of the binomial distribution and its formula, the probability function, parameters, probabilistic inference, mean, variance, and expected value (hope), procedures focused on the application of the formula and its properties for the analysis of situations, propositions and arguments that link the binomial distribution (already formalized) with the rest of mathematics and probability, and the predominance of algebraic symbolic language. This also allowed us to identify that the development of the meaning of the binomial distribution is related to the evolution of other mathematical-probabilistic concepts such as probability (Laplacian meaning, frequency meaning, and mathematical meaning relations) [16] and that of hope (expected case–trial and value–hope relations).

The meanings and elements identified are key when planning learning processes of the concept and proposing suitability criteria both for students and teachers, such as suggesting that for a person to be able to answer problem situations of case counting, it is necessary to handle the primitive concepts of probability such as chance, case, and probability as ratios, in addition to handling inductive reasoning and the recursive use of expressions and representations such as the tree diagram. In the same way, we can conclude that for the understanding of the holistic meaning of the binomial distribution, it is necessary to understand its use to solve the three families of problems (case counting, particular situations, and variable situations), preferably starting with the reconstruction of the combinatorial and probabilistic principles (from logical reasoning, graphical reasoning, and numerical patterns) applied to case counting, followed by the calculation of probabilities for specific binomial situations, building expressions from probabilistic principles, and their relationship with numerical patterns, fostering inductive reasoning and strengthening the understanding of probabilistic or mathematical ideas (the difference between probability and possibility or between permutation and combinatorics, for example), thus ending with the construction of general expressions (binomial distribution formula) from the principles and specific situations analyzed above to analyze the same binomial phenomenon or apply it in conjunction with other probability concepts such as mean and variance. These results are in agreement with the essential elements identified in other research regarding the understanding of the binomial distribution [8,9,51–53] and, as a whole, facilitate the identification of learning conflicts, the generation of experiences, and resources for learning, as well as the comparative analysis with the elements identified with respect to what is presented in textbooks, didactic proposals, the curriculum, and the knowledge of the teacher in practice and in training, strengthening the binomial distribution as an object of study in probability and its didactics. For this last reason, we consider as a possible research area the exploration of the cognitive configurations that are manifested in students and teachers at different levels, as well as in educational resources, namely study programs, textbooks, and training programs.

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