A Sylvester-Type Matrix Equation over the Hamilton Quaternions with an Application

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Abstract: We derive the solvability conditions and a formula of a general solution to a Sylvester-type matrix equation over Hamilton quaternions. As an application, we investigate the necessary and sufficient conditions for the solvability of the quaternion matrix equation, which involves $\eta$-Hermiticity. We also provide an algorithm with a numerical example to illustrate the main results of this paper.

Keywords: matrix equation; Hamilton quaternion; $\eta$-Hermitian matrix; Moore–Penrose inverse; rank

MSC: 15A03; 15A09; 15A24; 15B33; 15B57

1. Introduction

Let $\mathbb{R}$ stand for the real number field and

$$\mathbb{H} = \{ u_0 + u_1 i + u_2 j + u_3 k | i^2 = j^2 = k^2 = ijk = -1, u_0, u_1, u_2, u_3 \in \mathbb{R} \}. $$

$\mathbb{H}$ is called the Hamilton quaternion algebra, which is a non-commutative division ring. Hamilton quaternions and Hermitian quaternion matrices have been utilized in statistics of quaternion random signals [1], quaternion matrix optimization problems [2], signal and color image processing, face recognition [3,4], and so on.

Sylvester and Sylvester-type matrix equations have a large number of applications in different disciplines and fields. For example, the Sylvester matrix equation

$$A_1X + XB_1 = C_1$$

and the Sylvester-type matrix equation

$$A_1X + YB_1 = C_1$$

have been applied in singular system control [5], system design [6], perturbation theory [7], sensitivity analysis [8], $H_\alpha$-optimal control [9], linear descriptor systems [10], and control theory [11]. Roth [12] gave the Sylvester-type matrix Equation (2) for the first time over the polynomial integral domain. Baksalary and Kala [13] established the solvability conditions for Equation (2) and gave an expression of its general solution. In addition, Baksalary and Kala [14] derived the necessary and sufficient conditions for a two-sided Sylvester-type matrix equation

$$A_{11}X_1B_{11} + C_{11}X_2D_{11} = E_{11}$$

to be consistent. Özgüler [15] studied (3) over a principal ideal domain. Wang [16] investigated (3) over an arbitrary regular ring with an identity element.

Due to the wide applications of quaternions, the investigations on Sylvester-type matrix equations have been extended to $\mathbb{H}$ in the last decade (see, e.g., [17–24]). They are
applied for signal processing, color-image processing, and maximal invariant semidefinite or neutral subspaces, etc. (see, e.g., [25–28]). For instance, the general solution to Sylvester-type matrix Equation (2) can be used in color-image processing. He [29] derived the matrix Equation (2) as an essential finding. Roman [25] established the necessary and sufficient conditions for Equation (1) to have a solution. Kychei [30] investigated Cramer’s rules to drive the necessary and sufficient conditions for Equation (3) to be solvable. As an extension of Equations (2) and (3), Wang and He [31] gave the solvability conditions and the general solution to the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1$$  \hspace{0.5cm} (4)

over the complex number field $C$, which can be generalized to $\mathbb{H}$ and applicable in some Sylvester-type matrix equations over $\mathbb{H}$ (see, e.g., [29,32]).

We know that in system and control theory, the more unknown matrices that a matrix equation has, the wider its application will be. Consequently, for the sake of developing theoretical studies and the applications mentioned above of Sylvester-type matrix equation and their generalizations, in this paper, we aim to establish some necessary and sufficient conditions for the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + A_2Y_1B_2 + A_3Y_2B_3 + A_4Y_3B_4 = B$$  \hspace{0.5cm} (5)

to have a solution in terms of the rank equalities and Moore–Penrose inverses of some coefficient quaternion matrices in Equation (5) over $\mathbb{H}$. We derive a formula of its general solution when it is solvable. It is clear that Equation (5) provides a proper generalization of Equation (4), and we carry out an algorithm with a numerical example to calculate the general solution of Equation (5). As a special case of Equation (5), we also obtain the solvability conditions and the general solution to the two-sided Sylvester-type matrix equation

$$A_{11}Y_1B_{11} + A_{22}Y_2B_{22} + A_{33}Y_3B_{33} = T_1.$$  \hspace{0.5cm} (6)

To the best of our knowledge, so far, there has been little information on the solvability conditions and an expression of the general solution to Equation (6) by using generalized inverses.

As usual, we use $A^\dagger$ to denote the conjugate transpose of $A$. Recall that a quaternion matrix $A$, for $\eta \in \{i,j,k\}$, is said to be $\eta$-Hermitian if $A = A^\eta^\dagger$, where $A^\eta = -\eta A^* \eta$ [33]. For more properties and information on $\eta^\dagger$-quaternion matrices, we refer to [33]. We know that $\eta$-Hermitian matrices have some applications in linear modeling and statistics of quaternion random signals [1,33]. As an application of Equation (5), we establish some necessary and sufficient conditions for the quaternion matrix equation

$$A_1X_1 + (A_1X_1)^{\eta^\dagger} + A_2Y_1A_2^{\eta^\dagger} + A_3Y_2A_3^{\eta^\dagger} + A_4Y_3A_4^{\eta^\dagger} = B$$  \hspace{0.5cm} (7)

to be consistent. Moreover, we derive a formula of the general solution to Equation (7) where $B = B^{\eta^\dagger}$, $Y_i = Y_i^{\eta^\dagger}$ ($i = 1,3$) over $\mathbb{H}$.

The rest of this paper is organized as follows. In Section 2, we review some definitions and lemmas. In Section 3, we establish some necessary and sufficient conditions for Equation (5) to have a solution. In addition, we give an expression of its general solution to Equation (5) when it is solvable. In Section 4, as an application of Equation (5), we consider some solvability conditions and the general solution to Equation (7), where $Y_i = Y_i^{\eta^\dagger}$ ($i = 1,3$). Finally, we give a brief conclusion to the paper in Section 5.

2. Preliminaries

Throughout this paper, $\mathbb{H}^{m \times n}$ stands for the space of all $m \times n$ matrices over $\mathbb{H}$. The symbol $r(A)$ denotes the rank of $A$. $I$ and $0$ represent an identity matrix and a zero matrix of appropriate sizes, respectively. In general, $A^\dagger$ stands for the Moore–Penrose inverse of $A$. As an application of Equation (5), we establish some solvability conditions and the general solution to the Sylvester-type matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1$$  \hspace{0.5cm} (4)
of $A \in \mathbb{H}^{l \times k}$, which is defined as the solution of $AYA = A$, $YAY = Y$, $(AY)^* = AY$ and $(YA)^* = YA$. Moreover, $L_A = I - A^tA$ and $R_A = I - AA^t$ represent two projectors along $A$.

The following lemma is due to Marsaglia and Styan [34], which can be generalized to $\mathbb{H}$.

**Lemma 1** ([34]). Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times k}$, $C \in \mathbb{H}^{l \times n}$, $D \in \mathbb{H}^{l \times k}$ and $E \in \mathbb{H}^{l \times l}$ be given. Then, we have the following rank equality:

$$
 r\left(\begin{array}{c} A \\ R \end{array}\right) = r(A) - r(B) - r(E).
$$

**Lemma 2** ([35]). Let $A \in \mathbb{H}^{m \times n}$ be given. Then,

1. $(A^\eta)^t = (A^+)^\eta$, $(A^\eta)^t = (A^+)^\eta$.
2. $r(A) = r(A^\eta) = r(A^\eta A^\eta) = r(A^\eta A^\eta)$.
3. $(L_A)^\eta = -\eta(L_A) = (L_A)^\eta = L_A^\eta = R_A^\eta$.
4. $(R_A)^\eta = -\eta(R_A) = (R_A)^\eta = R_A^\eta = L_A^\eta$.
5. $(AA^t)^\eta = (A^t)^\eta A^\eta = (A^t)^\eta A^\eta$.
6. $(A^t A)^\eta = A^\eta (A^t)^\eta = (A^t)^\eta A^\eta$.

**Lemma 3** ([16]). Let $A_{ii}, B_i$ and $C_i$ ($i = 1, 2$) be given matrices with suitable sizes over $\mathbb{H}$. $A_1 = A_{22}L_{A_{11}}$, $T = R_{B_{11}}B_{22}$, $F = B_{22}L_{T}$, $G = R_AA_{22}$. Then, the following statements are equivalent:

1. The system

   $A_{11}X_1B_{11} = C_1$, $A_{22}X_1B_{22} = C_2$  \hspace{1cm} (8)

   has a solution.
2. $A_{ii}A_{ii}^tC_iB_i^tB_{ii} = C_i$ ($i = 1, 2$)

and

3. $G(A_{12}A_{22}^tB_{22}^t - A_{11}C_1B_{11}^t)F = 0$.

**Lemma 4** ([13]). Let $A_1$, $B_1$ and $C_1$ be given matrices with suitable sizes. Then, the Sylvester-type Equation (2) is solvable if and only if

$$
 R_{A_1}C_1B_{11} = 0.
$$

In this case, the general solution to Equation (2) can be expressed as

$$
 X = A_1^tC_1 - A_1^tU_1B_1 + L_{A_1}U_2, y = R_{A_1}C_1B_1^t + A_1A_1^tU_1 + U_3R_{B_1},
$$

where $U_1$, $U_2$, and $U_3$ are arbitrary matrices with appropriate sizes.
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In this case, the general solution to Equation (5). We begin with the following lemma, which is used to reach the main results of this paper.

Lemma 5 ([31]). Let \( A_1, B_1, C_3, D_3, C_4, D_4 \) and \( E_1 \) be given matrices over \( \mathbb{H} \). Put

\[
A = R_{A_1}C_3, \quad B = D_3L_{B_1}, \quad C = R_{A_1}C_4, \quad D = D_4L_{B_1},
\]

\[
E = R_{A_1}E_1L_{B_1}, \quad M = R_A C, \quad N = DL_B, \quad S = CL_M.
\]

Then, the following statements are equivalent:

1. Equation (4) has a solution.
2. \( R_M R_A E = 0, \quad EL_B L_N = 0, \quad R_A EL_D = 0, \quad R_E L_B = 0. \)
3. \( r\begin{pmatrix} E_1 & C_4 & C_3 & A_1 \\ B_1 & 0 & 0 & 0 \end{pmatrix} = r(B_1) + r(C_4, C_3, A_1), \)
   \( r\begin{pmatrix} E_1 & A_1 \\ D_3 & 0 \\ D_4 & 0 \\ B_1 & 0 \end{pmatrix} = r\begin{pmatrix} D_3 \\ D_4 \\ B_1 \end{pmatrix} + r(A_1), \)
   \( r\begin{pmatrix} E_1 & C_3 & A_1 \\ D_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1, C_3) + r\begin{pmatrix} D_4 \\ B_1 \end{pmatrix}, \)
   \( r\begin{pmatrix} E_1 & C_4 & A_1 \\ D_3 & 0 & 0 \\ B_1 & 0 & 0 \end{pmatrix} = r(A_1, C_4) + r\begin{pmatrix} D_3 \\ B_1 \end{pmatrix}. \)

In this case, the general solution to Equation (4) can be expressed as

\[
X_1 = A_{11}^T(E_1 - C_2X_2D_3 - C_4X_4D_4) - A_{11}^T T_7 B_1 + L_{A_1} T_6,
\]

\[
X_2 = R_{A_1}(E_1 - C_2X_2D_3 - C_4X_4D_4)B_1^T + A_1 A_{11}^T T_7 + T_8 R_{B_1},
\]

\[
X_3 = A^T E B^T - A^T C M^T E B^T - A^T S C^T E N^T D B^T - A^T S T_2 R_N D B^T + L_A T_4 + T_9 R_B,
\]

\[
X_4 = M^T E D^T + S^T S C^T E N^T + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D,
\]

where \( T_1, \ldots, T_8 \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \).

3. Some Solvability Conditions and a Formula of the General Solution

In this section, we establish the solvability conditions and a formula of the general solution to Equation (5). We begin with the following lemma, which is used to reach the main results of this paper.

Lemma 6. Let \( A_{11}, B_{11}, C_{11}, \) and \( D_{11} \) be given matrices with suitable sizes over \( \mathbb{H} \), \( A_{11} L_{A_{22}} = 0 \) and \( R_{B_{11}} B_{22} = 0 \). Set

\[
A_1 = A_{22}L_{A_{11}}, \quad C_{11} = C_2 - A_{22} A_{11}^T C_1 B_{11}^T B_{22}.
\]  \hspace{1cm} (9)

Then, the following statements are equivalent:

1. The system (8) is consistent.
2. \( R_{A_i} C_i = 0, \quad C_i L_{B_{ii}} = 0 \quad (i = 1, 2), \quad R_{A_1} C_{11} = 0. \)
3. \( A_{ii} A_{ii}^T C_i B_{ii}^T B_{ii} = C_i \quad (i = 1, 2), \quad C_1 B_{11}^T B_{22} = A_{11} A_{22}^T C_2. \)
(4) \[ r(A_{ii}, C_i) = r(A_{ii}), r\left(\frac{B_{ii}}{C_i}\right) = r(B_{ii}) \quad (i = 1, 2), \]
\[ r\begin{pmatrix} C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{22} \\ B_{11} & B_{22} & 0 \end{pmatrix} = r(A_{22}) + r(B_{11}). \]

In this case, the general solution to system (8) can be expressed as
\[ X_1 = A_{11}^t C_1 B_{11}^t + L_{A_{ii}} A_{22}^t C_2 B_{22}^t + L_{A_{ii}} V_1 + V_2 R_{B_{ii}} + L_{A_{ii}} V_3 R_{B_{ii}}, \tag{10} \]
where \(V_1, V_2,\) and \(V_3\) are arbitrary matrices with appropriate sizes over \(\mathbb{H}.

**Proof.** (1) \(\Leftrightarrow\) (2) It follows from Lemma 3 that
\[ G(A_{22}^t C_2 B_{22}^t - A_{11}^t C_1 B_{11}^t) F = 0 \]
\[ \Leftrightarrow R_{A_i} (A_1 + A_{22} A_{11}^t A_{11}) A_{22}^t C_2 B_{22} - A_{11}^t C_1 B_{11} B_{22} = 0 \]
\[ \Leftrightarrow R_{A_i} A_{22} A_{11}^t A_{22}^t C_2 B_{22} - A_{22} A_{11}^t C_1 B_{11} B_{22} = 0 \]
\[ \Leftrightarrow R_{A_i} A_{22} A_{11}^t A_{22}^t C_2 B_{22} - A_{22} A_{11}^t C_1 B_{11} B_{22} = 0 \]
\[ \Leftrightarrow R_{A_i} A_{22}^t - A_{22} A_{11}^t C_1 B_{11} B_{22} = 0 \Leftrightarrow R_{A_i} C_{11} = 0, \]
where \(G\) and \(F\) are given in Lemma 3.

(1) \(\Rightarrow\) (3) If the system (8) has a solution, then there exists a solution \(X_0\) such that
\[ A_{11} X_0 B_{11} = C_1, \]  
\[ A_{22} X_0 B_{22} = C_2. \]

It is easy to show that
\[ R_{A_i} C_{i} = 0, \quad C_{i} L_{B_{ii}} = 0 \quad (i = 1, 2). \]

Thus, \(A_{ii} A_{ii}^t C_i B_{ii}^t B_{ii} = C_i \quad (i = 1, 2).\) It follows from \(R_{B_{ii}} B_{22} = 0, A_{11} L_{A_{ii}} = 0\) that
\[ C_{i} L_{B_{ii}} B_{11} B_{22} = A_{11} X_0 B_{11} B_{11} B_{22} = A_{11} X_0 B_{22} = A_{11} A_{22} A_{22} X_0 B_{22} = A_{11} A_{22} C_2. \]

(3) \(\Rightarrow\) (2) Since \(A_{22} - A_1 = A_{22} A_{11}^t A_{11}\) and \(C_{i} L_{B_{ii}} B_{11} B_{22} = A_{11} A_{22} C_2,\) we have that
\[ R_{A_i} C_{11} = R_{A_i} C_2 - R_{A_i} A_{22} A_{11}^t C_1 B_{11} B_{22} = R_{A_i} C_2 - R_{A_i} A_{22} A_{11}^t A_{11} A_{22} C_2 \]
\[ = R_{A_i} C_2 - R_{A_i} (A_{22} - A_1) A_{22} C_2 = R_{A_i} C_2 - R_{A_i} A_{22} C_2 = 0. \]

(2) \(\Leftrightarrow\) (4) It follows from \(R_{B_{ii}} B_{22} = 0\) and \(A_{11} L_{A_{ii}} = 0\) that
\[ r(B_{22}, B_{11}) = r(B_{11}), \quad \begin{pmatrix} A_{11} \\ A_{22} \end{pmatrix} = r(A_{22}). \]
By Lemma 1,

\[
R_{A_i} C_i = 0 \iff r(R_{A_i} C_i) = 0 \iff r(A_{ii}, C_i) = r(A_{ii}) \quad (i = 1, 2),
\]

\[
C_i L_{B_{ii}} = 0 \iff r(C_i L_{B_{ii}}) = 0 \iff r(B_{ii}) \quad (i = 1, 2),
\]

\[
R_{A_i} C_{11} = 0 \iff r(R_{A_i} C_{11}) = 0 \iff r(C_{11}, A_1) = r(A_1)
\]

\[
\iff r\left(\begin{array}{cc}
C_{11} \\
R_{B_{ii}} B_{22}
\end{array}\right) = r(A_{22} L_{A_{11}}) + r(R_{B_{ii}} B_{22})
\]

\[
\iff r\left(\begin{array}{ccc}
A_{22} A_{11} & 0 & A_{22} \\
0 & 0 & A_{11} \\
B_{22} & B_{11} & 0
\end{array}\right) = r\left(\begin{array}{cc}
A_{11} \\
A_{22}
\end{array}\right) + r(B_{11}, B_{22})
\]

\[
\iff r\left(\begin{array}{cc}
C_1 & 0 \\
B_{22} & B_{11}
\end{array}\right) + r(B_{11}, B_{22}) = r(A_{22}) + r(B_{11}).
\]

We now prove that \( X_1 \) in (10) is the general solution of the system (8). We prove it in two steps. We show that \( X_1 \) is a solution of system (8) in Step 1. In Step 2, if the system (8) is consistent, then the general solution to system (8) can be expressed as (10).

Step 1. In this step, we show that \( X_1 \) is a solution of system (8). Substituting \( X_1 \) in (10) into the system (8) yields

\[
A_{11} X_1 B_{11} = A_{11} X_0 B_{11}, \quad A_{22} X_1 B_{22} = A_{22} X_0 B_{22}, \tag{11}
\]

where \( X_0 = A_{11}^t C_1 B_{11}^t + L A_{11} A_{22}^t C_2 B_{22}^t \). Since \( R_{A_{11}} C_1 = 0 \) and \( C_1 L_{B_{11}} = 0 \), we have that

\[
A_{11} X_0 B_{11} = A_{11} A_{11}^t C_1 B_{11}^t + L A_{11} A_{22} A_{22}^t C_1 B_{22}^t B_{11}
\]

\[
= A_{11} A_{11}^t C_1 B_{11}^t B_{11} + A_{11} L_{A_{11}} A_{11}^t C_1 B_{11} - R_{A_{22}} C_1 B_{22}^t B_{11} = A_{11} A_{11}^t C_1 B_{11}^t B_{11}
\]

\[
= -R_{A_{11}} C_1 B_{11}^t B_{11} - C_1 L_{B_{11}} + C_1 - C_1.
\]

By

\[
R_{B_{ii}} B_{22} = 0, \quad R_{A_{22}} C_{22} = 0, \quad C_{2 L_{B_{22}} = 0} \quad \text{and} \quad C_1 B_{11}^t B_{22} = 0, \quad A_{11} A_{11}^t C_2,\]

we have that

\[
A_{22} X_0 B_{22} = A_{22} (A_{11}^t C_1 B_{11}^t + L A_{11} A_{22} C_2 B_{22}^t) B_{22}
\]

\[
= A_{22} A_{11}^t C_1 B_{11} B_{22} + A_{22} A_{22} C_2 B_{22}^t B_{22} - A_{22} A_{11}^t A_{11} A_{22} C_2 B_{22}^t B_{22}
\]

\[
= C_2 + A_{22} A_{11}^t C_1 B_{11} B_{22} - A_{22} A_{11}^t A_{22} C_1 B_{22}^t B_{22} = C_2.
\]

Thus, \( A_{11} X_1 B_{11} = C_1, \quad A_{22} X_1 B_{22} = C_2, \quad X_1 \) is a solution of system (8).

Step 2. In this step, we show that the general solution to the system (8) can be expressed as (10). It is sufficient to show that for an arbitrary solution, say, \( X_0 \) of (8), \( X_0 \) can be expressed in form (10). Put

\[
V_1 = X_0 B_{22} B_{12}^t, \quad V_2 = X_0, \quad V_3 = X_0 B_{11} B_{11}^t.
\]
It follows from $B_{22} = B_{11}B_{11}^tB_{22}$ and $A_{11} = A_{11}A_{22}^tA_{22}$ that

$$X_1 = A_{11}^tC_1B_{11}^t + L_{A_{11}}A_{22}^tC_2B_{22} + L_{A_{22}}V_1 + V_2R_{B_{11}} + L_{A_{11}}V_3R_{B_{22}}$$

$$= A_{11}^tC_1B_{11}^t + L_{A_{11}}A_{22}^tC_2B_{22} + L_{A_{22}}X_{01}B_{22}B_{22}^t + X_{01}R_{B_{11}} + L_{A_{11}}X_{01}B_{11}B_{11}^tR_{B_{22}}$$

$$= A_{11}^tC_1B_{11}^t + L_{A_{11}}A_{22}^tC_2B_{22} + X_{01}B_{22}B_{22}^t - A_{22}^tA_{22}X_{01}B_{22}B_{22}^t + X_{01} - X_{01}B_{11}B_{11}^t$$

$$+ X_{01}B_{11}B_{11}^tR_{B_{22}} - A_{11}^tA_{11}X_{01}B_{11}B_{11}^tR_{B_{22}}$$

$$= A_{11}^tC_1B_{11}^t + L_{A_{11}}A_{22}^tC_2B_{22} - X_{01}R_{B_{11}}B_{22}B_{22}^t + X_{01} - A_{22}^tA_{22}X_{01}B_{22}B_{22}^t$$

$$- A_{11}^tA_{11}X_{01}B_{11}B_{11}^t - A_{11}^tA_{11}X_{01}B_{11}B_{11}^tB_{22}B_{22}^t$$

$$= X_{01}^tA_{11}^tX_{01}B_{11}B_{11}^t + A_{11}^tA_{11}X_{01}B_{22}B_{22}^t - A_{11}^tA_{11}A_{22}X_{01}B_{22}B_{22}^t$$

$$= X_{01}B_{11}X_{01}B_{22}B_{22}^t - A_{11}X_{01}B_{22}B_{22}^t = X_{01}.$$

Hence, $X_{01}$ can be expressed as (10). To sum up, (10) is the general solution of the system (8).

Now, we give the fundamental theorem of this paper.

**Theorem 1.** Let $A_i$, $B_i$, and $B (i = \overline{1,4})$ be given quaternion matrices with appropriate sizes over $\mathbb{H}$. Set

$$R_{A_{11}}A_{22} = A_{11}, \quad R_{A_1}A_{33} = A_{22}, \quad R_{A_1}A_4 = A_{33}, \quad B_1L_{B_1} = B_{11}, \quad B_{22}L_{B_{22}} = N_{11},$$

$$B_1L_{B_1} = B_{12}, \quad B_2L_{B_2} = B_{23}, \quad R_{A_1}A_{22} = M_1, \quad S_1 = A_{22}L_{M_1}, \quad R_{A_1}B_{11}L_{B_1} = T_1,$$

$$C = R_{M_1}R_{A_{11}}, \quad C_1 = CA_{33}, \quad C_2 = R_{A_{11}}A_{33}, \quad C_3 = R_{A_{22}}A_{33}, \quad C_4 = A_{33},$$

$$D = L_{B_{11}}L_{N_{11}}, \quad D_1 = B_{33}, \quad D_2 = B_{33}L_{B_{22}}, \quad D_3 = B_{33}L_{B_{11}}, \quad D_4 = B_{33}D,$$

$$E_1 = CT_1, \quad E_2 = R_{A_{11}}T_1L_{B_{22}}, \quad E_3 = R_{A_{22}}T_1L_{B_{11}}, \quad E_4 = T_1D,$$

$$C_{11} = (L_{C_2}, L_{C_4}), \quad D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, \quad C_{22} = L_{C_1}, \quad D_{22} = R_{D_2}, \quad C_{33} = L_{C_3},$$

$$D_{33} = R_{D_4}, \quad E_{11} = R_{C_1}C_1, \quad E_{22} = R_{C_1}C_2, \quad E_{33} = D_{22}L_{D_{11}}, \quad E_{44} = D_{33}L_{D_{11}},$$

$$M = E_{44}L_{E_{33}}, \quad N = E_{33}, \quad F_1 = F_2, \quad E = R_{C_1}F_{L_{D_{11}}}, \quad S = E_{22}L_{M_1},$$

$$F_{11} = C_2L_{C_1}, \quad G_1 = E_2 - C_2C_1^tE_1D_1^tD_2, \quad F_{22} = C_4L_{C_3}, \quad G_2 = E_4 - C_4C_3^tE_3D_3^tD_4,$$

$$F_1 = C_1^tE_1D_1^t + L_{C_2}C_2^tE_2D_2^t, \quad F_2 = C_3^tE_3D_3^t + L_{C_3}C_3^tE_4D_4^t.$$

Then, the following statements are equivalent:

1. Equation (5) is consistent.
2. $R_{C_i}E_i = 0$, $E_iL_{D_i} = 0 (i = \overline{1,4})$, $R_{E_{11}}LE_{44} = 0$.

$$r\begin{pmatrix} B \\ A_2 \\ A_3 \\ A_4 \\ A_1 \\ B_1 \end{pmatrix} = r(B_1) + r(A_2, A_3, A_4, A_1),$$

$$r\begin{pmatrix} B \\ A_2 \\ A_4 \\ A_1 \\ B_3 \end{pmatrix} = r(A_2, A_4, A_1) + r(B_3),$$

$$r\begin{pmatrix} B \\ A_3 \\ A_4 \\ A_1 \\ B_2 \end{pmatrix} = r(A_3, A_4, A_1) + r(B_2).$$
where A is solvable. According to Equation (28), we have that has a solution if and only if there exist Y solution, we get the following:

Proof. (1) ⇔ (2) Equation (5) can be written as

\[ A_1 X_1 + X_2 B_1 = B - (A_2 Y_1 B_2 + A_3 Y_2 B_3 + A_4 Y_3 B_4). \]  

(26)

Clearly, Equation (5) is solvable if and only if Equation (26) has a solution. By Lemma 4, Equation (26) is consistent if and only if there exist \( Y_i (i = 1, 3) \) in Equation (26) such that

\[ R_{A_i} [B - (A_2 Y_1 B_2 + A_3 Y_2 B_3 + A_4 Y_3 B_4)] L_{B_i} = 0, \]  

(27)
i.e.,

\[ A_{11} Y_1 B_{11} + A_{22} Y_2 B_{22} + A_{33} Y_3 B_{33} = T_1, \]  

(28)

where \( A_{ii}, B_i (i = 1, 3) \), and \( T_1 \) are defined by (12). In addition, when Equation (26) has a solution, we get the following:

\[ X_1 = A_1^{-1} (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) - A_1^{-1} U_1 B_1 + L_{A_i} U_2, \]

\[ X_2 = R_{A_i} (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_1^t + A_1 A_1^t U_1 + U_3 R_{B_i}, \]

where \( U_i (i = 1, 3) \) are any matrices with appropriate dimensions over \( \mathbb{H} \). Hence, Equation (26) has a solution if and only if there exist \( Y_i (i = 1, 3) \) in Equation (26) such that Equation (28) is solvable. According to Equation (28), we have that

\[ A_{11} Y_1 B_{11} + A_{22} Y_2 B_{22} = T_1 - A_{33} Y_3 B_{33}. \]  

(29)
Hence, Equation (28) is consistent if and only if Equation (29) is solvable. It follows from Lemma 5 that Equation (29) has a solution if and only if there exists $Y_3$ in Equation (29) such that

$$
\begin{align*}
R_{M1}R_{A11}(A33Y_3B_{33} - T_1) &= 0, \\
R_{A11}(T_1 - A33Y_3B_{33})L_{B22} &= 0, \\
R_{A22}(T_1 - A33Y_3B_{33})L_{B11} &= 0, \\
(T_1 - A33Y_3B_{33})L_{B11}L_{N1} &= 0,
\end{align*}
$$

i.e.,

$$
C_1Y_3D_1 = E_1, \ C_2Y_3D_2 = E_2, \ C_3Y_3D_3 = E_3, \ C_4Y_3D_4 = E_4,
$$

where $C_i, D_i, E_i (i = 1, 4)$ are defined by (13). Thus, according to Lemma 6, we have that

$$
Y_1 = A_{11}TB_{11} - A_{11}A_{22}M_{11}TB_{11} - A_{11}S_1A_{22}TN_1B_{22}B_{11}^T
- A_{11}U_4R_{N1}B_{22}B_{11}^T + L_{A11}U_5 + U_6R_{B11},
$$

$$
Y_2 = M_{11}TB_{22} + S_1A_{22}TN_1B_{22}B_{11}^T + L_{M11}U_5 + U_6R_{B22} + L_{M11}U_4R_{N1},
$$

where $A_{ii}, B_{ii} (i = 1, 3), M_{ii}, N_1, S_1, T_1$ are defined by (12), $T = T_1 - A33Y_3B_{33}$ and $U_j (j = 4, 8)$ are any matrices with the appropriate dimensions over $\mathbb{R}$.

It is easy to infer that

$$
C_1L_{C2} = 0, \ R_{D1}D_2 = 0, \ C_3L_{C4} = 0, \ R_{D3}D_4 = 0.
$$

Thus, according to Lemma 6, we have that the system (31) is consistent if and only if

$$
R_{C1}E_i = 0, \ E_iL_{D_i} = 0 (i = 1, 2, 3, 4), \ R_{F1}G_1 = 0, \ R_{F2}G_2 = 0.
$$

In this case, the general solution to system (31) can be expressed as

$$
Y_3 = F_1 + L_{C2}V_1 + V_2R_{D1} + L_{C1}V_3R_{D2},
$$

$$
Y_3 = F_2 - L_{C4}W_1 - W_2R_{D3} - L_{C3}W_3R_{D4},
$$

where $F_1, F_2$ are defined by (15) and $V_i, W_i (i = 1, 3)$ are any matrices with the appropriate dimensions over $\mathbb{R}$. Thus, system (31) has a solution if and only if (33) holds and there exist $V_i, W_i (i = 1, 3)$ such that (34) equals to (35), namely

$$
(L_{C2}, L_{C4})\begin{bmatrix} V_1 \\ W_1 \end{bmatrix} + (V_2, W_2)\begin{bmatrix} R_{D1} \\ R_{D3} \end{bmatrix} + L_{C1}V_3R_{D2} + L_{C3}W_3R_{D4} = F,
$$

i.e.,

$$
C_{11}\begin{bmatrix} V_1 \\ W_1 \end{bmatrix} + (V_2, W_2)D_{11} + C_{22}V_3D_{22} + C_{33}W_3D_{33} = F,
$$

where $F, C_{ii}$ and $D_{ii} (i = 1, 3)$ are defined by (14). It follows from Lemma 5 that Equation (36) has a solution if and only if

$$
R_{M1}R_{E11}E = 0, \ ELE_{33}L_N = 0, \ R_{E11}E_{44} = 0, \ R_{E22}E_{33} = 0.
$$

In this case, the general solution to Equation (36) can be expressed as

$$
V_1 = (I_m, 0)\begin{bmatrix} C_{11}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}U_{11}D_{11} + L_{C1}U_{12} \\ 0 \end{bmatrix},
$$

$$
W_1 = (0, I_m)\begin{bmatrix} C_{11}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33}) - C_{11}U_{11}D_{11} + L_{C1}U_{12} \\ 0 \end{bmatrix},
$$

where $I_m$ is an $m 	imes m$ identity matrix.
\[
W_2 = \left[R_{C_{11}} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^+ + C_{11} C_{11}^t U_{11} + U_{21} R_{D_{11}} \right] \begin{pmatrix} 0 \\ I_n \end{pmatrix},
\]

\[
V_2 = \left[R_{C_{11}} (F - C_{22} V_3 D_{22} - C_{33} W_3 D_{33}) D_{11}^+ + C_{11} (C_{11}^t U_{11} + U_{21} R_{D_{11}} \right] \begin{pmatrix} I_n \\ 0 \end{pmatrix},
\]

\[
V_3 = E_1^t F E_3^t - E_1^t E_2^t M^t E_3^t - E_1^t S E_2^t F N^t E_{44}^t E_3^t - E_1^t S U_{31} R_N E_{44}^t E_3^t + L E_{31} U_{32} + U_{33} R_{E_3^t},
\]

\[
W_3 = M^t F E_{44}^t + S^t S E_2^t F N^t + L_M L_S U_{41} + L_M U_{31} R_N - U_{42} R_{E_4},
\]

where \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \) and \( U_{42} \) are any matrices with the suitable dimensions over \( \mathbb{F} \). \( M, E, N, S, C_{11}, D_{11}, \) and \( E_{ii} \) \( (i = 1, 4) \) are defined by (14), \( m \) is the column number of \( A_4 \) and \( n \) is the row number of \( B_4 \). We summarize up that (28) has a solution if and only if (33) and (37) hold. Hence, Equation (5) is solvable if and only if (33) and (37) hold.

In fact, \( R_{C_{2}} E_2 = 0, E_1 L_{D_1} = 0 \Rightarrow R_{F_{11}} G_1 = 0; R_{C_{2}} E_4 = 0, E_3 L_{D_3} = 0 \Rightarrow R_{F_{22}} G_2 = 0; R_{C_{3}} E_3 = 0, E_1 L_{D_1} = 0 \Rightarrow R_{M} R_{F_{11}} E_1 = 0; R_{C_{4}} E_4 = 0, E_1 L_{D_1} = 0 \Rightarrow E L_{E_{33}} L_N = 0; R_{C_{4}} E_4 = 0, E_2 L_{D_2} = 0 \Rightarrow R_{F_{22}} E L_{E_{4}} = 0. \) The specific proof is as follows. Firstly, we prove that \( R_{C_{2}} E_2 = 0, E_1 L_{D_1} = 0 \Rightarrow R_{F_{11}} G_1 = 0; R_{C_{2}} E_4 = 0, E_3 L_{D_3} = 0 \Rightarrow R_{F_{22}} G_2 = 0. \) It follows from Lemma 1 and elementary transformations that

\[
R_{C_{1}} E_1 = 0 \Leftrightarrow r(E_1, C_1) = r(C_1) = r(C_1^t, CA_{33}) = r(C A_{33}) \Leftrightarrow r(T_1, A_{33}, A_{11}, A_{22}) = r(A_{33}, A_{11}, A_{22}),
\]

(38)

\[
R_{C_{2}} E_2 = 0 \Leftrightarrow r(E_2, C_2) = r(C_2) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{22} \end{pmatrix} \begin{pmatrix} A_{33} \\ 0 \\ A_{11} \\ 0 \end{pmatrix} = r(A_{33}, A_{11}) + r(B_{22}),
\]

(39)

\[
R_{C_{3}} E_3 = 0 \Leftrightarrow r(E_3, C_3) = r(C_3) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{11} \end{pmatrix} \begin{pmatrix} A_{33} \\ 0 \\ A_{22} \\ 0 \end{pmatrix} = r(A_{33}, A_{22}) + r(B_{11}),
\]

(40)

\[
R_{C_{4}} E_4 = 0 \Leftrightarrow r(E_4, C_4) = r(C_4) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{11} \\ B_{22} \end{pmatrix} \begin{pmatrix} A_{33} \\ 0 \\ 0 \\ 0 \end{pmatrix} = r(A_{33}) + r \begin{pmatrix} B_{11} \\ B_{22} \end{pmatrix},
\]

(41)

\[
E_1 L_{D_1} = 0 \Leftrightarrow r \begin{pmatrix} E_1 \\ D_1 \end{pmatrix} \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{33} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{22} \\ 0 \\ 0 \end{pmatrix} = r(A_{11}, A_{22}) + r(B_{33}),
\]

(42)

\[
E_2 L_{D_2} = 0 \Leftrightarrow r \begin{pmatrix} E_2 \\ D_2 \end{pmatrix} = r(D_2) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{33} \\ B_{22} \end{pmatrix} \begin{pmatrix} A_{11} \\ 0 \\ 0 \end{pmatrix} = r(B_{33}) + r(A_{11}),
\]

(43)

\[
E_3 L_{D_3} = 0 \Leftrightarrow r \begin{pmatrix} E_3 \\ D_3 \end{pmatrix} = r(D_3) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{33} \\ B_{11} \end{pmatrix} \begin{pmatrix} A_{22} \\ 0 \\ 0 \end{pmatrix} = r(B_{33}) + r(A_{22}),
\]

(44)

\[
E_4 L_{D_4} = 0 \Leftrightarrow r \begin{pmatrix} E_4 \\ D_4 \end{pmatrix} = r(D_4) \Leftrightarrow r \begin{pmatrix} T_1 \\ B_{33} \\ B_{11} \\ B_{22} \end{pmatrix} = r \begin{pmatrix} B_{33} \\ B_{11} \\ B_{22} \end{pmatrix}.
\]

(45)

It follows from Lemma 6 and (32) that \( R_{F_{11}} G_1 = 0 \) and \( R_{F_{22}} G_2 = 0 \) are equivalent to

\[
r \begin{pmatrix} E_1 \\ D_1 \end{pmatrix} = r \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + r(D_1, D_2),
\]

(46)

\[
r \begin{pmatrix} E_3 \\ D_3 \end{pmatrix} = r \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} + r(D_3, D_4).
\]

(47)
According to Lemma 1, we have that

\[ (46) \leftrightarrow r \begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & A_{33} & 0 & 0 & A_{11} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & A_{11} & A_{22} & 0 \\ 0 & 0 & 0 & A_{11} \end{pmatrix} + r \begin{pmatrix} B_{33} & 0 \\ 0 & B_{22} \end{pmatrix} \]

\[ (48) \]

Thus, it follows from (48) that (46) holds when (39) and (42) hold. Similarly, if (41) and (44) hold, then (47) holds.

Secondly, we prove that \( R_{C_3} E_3 = 0, E_1 L D_1 = 0 \Rightarrow R_M R_{E_3} E = 0; R_{C_4} E_4 = 0, E_2 L D_2 = 0 \Rightarrow R_{E_2} E L E_{33} = 0. \) According to Lemma 5 and (32), we have that (37) are equivalent to

\[ r \begin{pmatrix} F & L_{C_1} & L_{C_3} \\ R_{D_1} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} = r(L_{C_1}, L_{C_3}) + r \left( \begin{array}{c} R_{D_1} \\ R_{D_3} \end{array} \right), \]

\[ (49) \]

\[ r \begin{pmatrix} F & L_{C_2} & L_{C_4} \\ R_{D_2} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} = r(L_{C_2}, L_{C_4}) + r \left( \begin{array}{c} R_{D_2} \\ R_{D_4} \end{array} \right), \]

\[ (50) \]

\[ r \begin{pmatrix} F & L_{C_1} & L_{C_4} \\ R_{D_1} & 0 & 0 \\ R_{D_4} & 0 & 0 \end{pmatrix} = r(L_{C_1}, L_{C_4}) + r \left( \begin{array}{c} R_{D_1} \\ R_{D_4} \end{array} \right), \]

\[ (51) \]

\[ r \begin{pmatrix} F & L_{C_2} & L_{C_3} \\ R_{D_2} & 0 & 0 \\ R_{D_3} & 0 & 0 \end{pmatrix} = r(L_{C_2}, L_{C_3}) + r \left( \begin{array}{c} R_{D_2} \\ R_{D_3} \end{array} \right), \]

\[ (52) \]

respectively. By Lemma 1, we have that

\[ (49) \leftrightarrow r \begin{pmatrix} F & I & I & 0 & 0 \\ I & 0 & 0 & D_1 & 0 \\ I & 0 & 0 & 0 & D_3 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} I & D_1 & 0 \\ I & 0 & D_3 \end{pmatrix} + r \begin{pmatrix} I & I \\ C_1 & 0 \\ 0 & C_3 \end{pmatrix} \]

\[ (53) \]

\[ r \begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_3 & C_3 \\ D_1 & D_3 & 0 \end{pmatrix} = r \left( \begin{array}{c} C_1 \\ C_3 \end{array} \right) + r(D_1, D_3). \]

Similarly, we can show that (50)–(52) are equivalent to

\[ r \begin{pmatrix} E_1 & 0 & C_1 \\ 0 & -E_4 & C_4 \\ D_1 & D_4 & 0 \end{pmatrix} = r \left( \begin{array}{c} C_1 \\ C_4 \end{array} \right) + r(D_1, D_4), \]

\[ (54) \]

\[ r \begin{pmatrix} E_2 & 0 & C_2 \\ 0 & -E_3 & C_3 \\ D_2 & D_3 & 0 \end{pmatrix} = r \left( \begin{array}{c} C_2 \\ C_3 \end{array} \right) + r(D_2, D_3), \]

\[ (55) \]
Substituting $C_i, D_i,$ and $E_i$ ($i = 1, 3$) in (13) into the rank equality (53) and by Lemma 1, we have that

\[(33) \iff r \begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & 0 \\ 0 & -T_1 & A_{13} & 0 & 0 & A_{22} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & A_{11} & A_{22} & 0 & 0 & A_{22} \\ A_{33} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} B_{33} & 0 \\ 0 & B_{11} \end{pmatrix} \tag{57}
\]

Hence, it follows from (40) and (42) that (57) holds. Similarly, we can prove that when (41), (42) hold and (41), (43) hold, we can get that (54) and (56) hold, respectively. Thus, Equation (28) has a solution if and only if (16) holds. That is to say, Equation (5) has a solution if and only if (16) holds.

(2) $\iff$ (3) We prove the equivalence in two parts. In the first part, we want to show that (38) to (45) are equivalent to (17) to (24), respectively. In the second part, we want to show that (55) is equivalent to (25).

Part 1. We want to show that (38) to (45) are equivalent to (17) to (24), respectively. It follows from Lemma 1 and elementary operations to (38) that

\[(38) \iff r \begin{pmatrix} R_{A_1}B_1L_{B_{11}} & R_{A_1}A_4 & R_{A_1}A_2 & R_{A_1}A_3 \\ B_4L_{B_{11}} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} R_{A_1}A_4 & R_{A_1}A_2 & R_{A_1}A_3 \\ 0 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} B_4L_{B_{11}} \end{pmatrix} \iff (17).
\]

Similarly, we can show that (39) to (41) are equivalent to (18) to (20), respectively. Now, we turn to prove that (42) is equivalent to (19). It follows from the Lemma 1 and elementary transformations that

\[(42) \iff r \begin{pmatrix} R_{A_1}B_1L_{B_{11}} & R_{A_1}A_2 & R_{A_1}A_3 \\ B_4L_{B_{11}} & 0 & 0 \end{pmatrix} = r \begin{pmatrix} R_{A_1}A_2 & R_{A_1}A_3 \\ 0 & 0 \end{pmatrix} + r \begin{pmatrix} B_4L_{B_{11}} \end{pmatrix} \iff (21).
\]

Similarly, we can show that (43) to (45) are equivalent to (22) to (24). Hence, (38) to (45) are equivalent to (17) to (24), respectively.

Part 2. We want to show that (55) $\iff$ (25). It follows from Lemma 1 and elementary operations to (55) that

\[(55) \iff r \begin{pmatrix} R_{A_1}T_1L_{B_{32}} & R_{A_1}A_3 \\ B_{33}L_{B_{32}} & 0 \end{pmatrix} = r \begin{pmatrix} R_{A_1}A_3 \\ 0 \end{pmatrix} + r \begin{pmatrix} B_{33}L_{B_{32}} \end{pmatrix} \]

\[
\begin{pmatrix} T_1 & 0 & 0 & A_{11} & A_{22} & A_{33} \\ 0 & -T_1 & 0 & 0 & A_{22} & A_{33} \\ B_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & 0 & 0 & 0 \\ B_{33} & B_{33} & 0 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} B_{22} & 0 & 0 & A_{11} & 0 & A_{33} \\ B_{33} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + r \begin{pmatrix} A_{11} & 0 & A_{33} \\ B_{33} & B_{33} & 0 \end{pmatrix}.
\]
\[
\begin{pmatrix}
B_3 & 0 \\
0 & B_2 \\
B_4 & B_1 \\
B_1 & 0
\end{pmatrix}
+ r
\begin{pmatrix}
A_2 & 0 & A_4 & A_1 & 0 \\
0 & A_3 & A_4 & 0 & A_1 \\
A_3 & 0 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 & 0 \\
0 & B_1 & 0 & 0 & 0 \\
0 & B_1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
= r
\begin{pmatrix}
B & 0 & A_2 & 0 & A_4 & A_1 & 0 \\
0 & -B & 0 & A_3 & A_4 & 0 & A_1 \\
B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 & 0 & 0 & 0 \\
B_4 & B_1 & 0 & 0 & 0 & 0 & 0 \\
B_1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence, (38) to (45) and (55) are equivalent to (17) to (25), respectively. 

Next, we give the formula of general solution to matrix Equation (5) by using Moore–Penrose. According to Theorem 1, we get the following theorem:

**Theorem 2.** Let matrix Equation (5) be solvable. Then, the general solution to matrix Equation (5) can be expressed as

\[
X_1 = A_1^t (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) - A_1^t U_1 B_1 + L A_1 U_2,
\]

\[
X_2 = R A_1 (B - A_2 Y_1 B_2 - A_3 Y_2 B_3 - A_4 Y_3 B_4) B_1^t + A_1 A_1^t U_1 + U_3 R B_1,
\]

\[
Y_1 = A_1^t T B_1^t + A_1^t A_2^t M_1^t T B_1^t + A_1^t S_1^t A_2^t T N_1^t B_2^t B_1^t - A_1^t S_1^t U_1 R N_1 B_2^t B_1^t + L A_1 U_5 + U_6 R B_1,
\]

\[
Y_2 = M_1^t T B_2^t + S_1^t S_2^t A_2^t T N_1^t + M A_1^t U_5 + U_8 R B_2 + L A_1 U_4 R N_1,
\]

\[
Y_3 = F_1 + L C_1 V_1 + V_2 R D_1 + L C_1 V_3 D_2, \text{ or } Y_3 = F_2 - L C_1 W_1 - W_2 R D_3 - L C_3 W_3 D_4,
\]

where \( T = T_1 - A_{33} Y_3 B_{33}, U_4(i = \bar{1}, 3) \) are arbitrary matrices with appropriate sizes over \( \mathbb{H} \).

**Algorithm with a Numerical Example**

In this section, we give Algorithm 1 with a numerical example to illustrate the main results.

**Algorithm 1** Algorithm for computing the general solution of Equation (5)

1. Input the quaternion matrices \( A_i, B_i \) \((i = \bar{1}, 4)\) and \( B \) with conformable shapes.
2. Compute all matrices given by (12)–(15).
3. Check equalities in (16) or (17)–(25). If not, it returns inconsistent.
4. Else, compute \( X_i, Y_j(i = \bar{1}, 2, j = \bar{1}, 3) \).
Example 1. Consider the matrix Equation (5). Put
\[
A_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 \\ k & i \end{pmatrix}, \quad B = \begin{pmatrix} 3i & i - 1 \\ 0 & j \end{pmatrix}.
\]

Computation directly yields
\[
\begin{align*}
r \begin{pmatrix} B & A_2 & A_3 & A_4 & A_1 \\ B_1 & 0 & 0 & 0 & 0 \end{pmatrix} &= r(B_1) + r(A_2, A_3, A_4, A_1) = 3, \\
r \begin{pmatrix} B & A_2 & A_4 & A_1 \\ B_3 & 0 & 0 & 0 \end{pmatrix} &= r(A_2, A_4, A_1) + r \left( \frac{B_3}{B_1} \right) = 4, \\
r \begin{pmatrix} B & A_3 & A_4 & A_1 \\ B_2 & 0 & 0 & 0 \end{pmatrix} &= r(A_3, A_4, A_1) + r \left( \frac{B_2}{B_1} \right) = 4, \\
r \begin{pmatrix} B & A_4 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} &= r(B_2) + r(A_4, A_1) = 3, \\
r \begin{pmatrix} B & A_2 & A_3 & A_1 \\ B_4 & 0 & 0 & 0 \end{pmatrix} &= r(B_2, A_3, A_1) + r \left( \frac{B_4}{B_1} \right) = 4, \\
r \begin{pmatrix} B & A_2 & A_1 \\ B_3 & 0 & 0 \end{pmatrix} &= r(B_3) + r(A_2, A_1) = 3, \\
r \begin{pmatrix} B & A_3 & A_1 \\ B_2 & 0 & 0 \end{pmatrix} &= r(B_2) + r(A_3, A_1) = 3, \\
r \begin{pmatrix} B & A_1 \\ B_2 & 0 \end{pmatrix} &= r(B_2) + r(A_1) = 3, \\
r \begin{pmatrix} B & A_2 & A_1 & 0 & 0 & 0 & A_4 \\ B_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & A_3 & A_1 & A_4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & B_4 & 0 & 0 \end{pmatrix} &= r \left( \frac{B_3}{B_1} \right) + r \left( \frac{A_2}{0} \right) = 7.
\end{align*}
\]

All rank equalities in (17) to (25) hold. Hence, according to Theorem 1, Equation (5) has a solution. Moreover, by Theorem 2, we have that
\[
X_1 = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & j \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} i & j \\ 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} i & k \\ 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} i & j \\ k & 0 \end{pmatrix}.
\]
Remark 1. Chu et al. gave potential applications of the maximal and minimal ranks in the discipline of control theory (e.g., [36–38]). We may consider the rank bounds of the general solution of Equation (5).

4. The General Solution to Equation with $\eta$-Hermicity

In this section, as an application of (5), we establish some necessary and sufficient conditions for quaternion matrix Equation (7) to have a solution and derive a formula of its general solution involving $\eta$-Hermicity.

Theorem 3. Let $A_i$ ($i=\overline{1,4}$) and $B$ be given matrices with suitable sizes over $\mathbb{H}$, $B = B^\eta$. Set

$$
R_{A_i}A_2 = A_{11}, \quad R_{A_i}A_3 = A_{22}, \quad R_{A_i}A_4 = A_{33}, \quad R_{A_i}A_2 = M_1, \quad S_1 = A_{22}L_{M_1},
$$

$$
R_{A_i}B(R_{A_i})^\eta = T_1, \quad C = R_{M_1}R_{A_i}, \quad C_1 = CA_{33}, \quad C_2 = R_{A_i}A_{33},
$$

$$
C_3 = R_{A_2}A_{33}, \quad C_4 = A_{33}, \quad E_1 = CT_1, \quad E_2 = R_{A_i}T_1(R_{A_i})^\eta, \quad E_3 = R_{A_2}T_1(R_{A_i})^\eta, \quad E_4 = T_1C^\eta,
$$

$$
C_{11} = (L_{C_1}, L_{C_2}), \quad C_{22} = L_{C_1}, \quad C_{33} = L_{C_1}, \quad E_{11} = R_{C_1}C_{22}, \quad E_{22} = R_{C_1}C_{33},
$$

$$
M = R_{E_1}E_{22}, \quad N = (R_{E_2}E_{11})^\eta, \quad F = F_2 - F_1, \quad E = R_{C_1}F(R_{C_1})^\eta, \quad S = E_{22}L_M,
$$

$$
F_{11} = C_2L_{C_1}, \quad G_1 = E_2 - C_2C_1^\eta E_1(C_1^\eta)^\dagger C_3^\eta, \quad F_{22} = C_4L_{C_1}, \quad G_2 = E_4 - C_4C_3^\eta E_3(C_3^\eta)^\dagger C_4^\eta,
$$

$$
F_1 = C_1^\eta E_1(C_1^\eta)^\dagger + L_{C_1}C_1^\eta E_2(C_1^\eta)^\dagger, \quad F_2 = C_4^\eta E_3(C_3^\eta)^\dagger + L_{C_4}C_4^\eta E_4(C_4^\eta)^\dagger.
$$

Then, the following statements are equivalent:

1. Equation (7) is consistent.
2. $R_{C_i}E_i = 0$ ($i=\overline{1,4}$), $R_{E_{22}}E(R_{E_{22}})^\eta = 0$.
3. \[
\begin{pmatrix}
B & A_2 & A_3 & A_4 & A_1 \\
A_1^* & 0 & 0 & 0 & 0
\end{pmatrix} = r(A_1) + r(A_2, A_3, A_4, A_1),
\]

\[
\begin{pmatrix}
B & A_2 & A_3 & A_1 \\
A_1^* & 0 & 0 & 0
\end{pmatrix} = r(A_2, A_3, A_1) + r(A_4, A_1),
\]

\[
\begin{pmatrix}
B & A_2 & A_4 & A_1 \\
A_1^* & 0 & 0 & 0
\end{pmatrix} = r(A_2, A_4, A_1) + r(A_3, A_1),
\]

\[
\begin{pmatrix}
B & A_3 & A_4 & A_1 \\
A_1^* & 0 & 0 & 0
\end{pmatrix} = r(A_3, A_4, A_1) + r(A_2, A_1),
\]

\[
\begin{pmatrix}
B & 0 & A_2 & 0 & A_1 & 0 \\
0 & -B & 0 & A_4 & A_4 & 0 \\
A_2^* & 0 & 0 & 0 & 0 & 0 \\
A_4^* & 0 & 0 & 0 & 0 & 0 \\
A_4^* & 0 & 0 & 0 & 0 & 0 \\
0 & A_1^* & 0 & 0 & 0 & 0
\end{pmatrix} = 2r
\begin{pmatrix}
A_2 & 0 & A_4 & A_1 & 0 \\
0 & A_3 & A_4 & 0 & A_1
\end{pmatrix}.
\]
In this case, the general solution to Equation (7) can be expressed as

\[
X_1 = \frac{X_1 + (X_2)\eta^*}{2}, \quad Y_1 = \frac{Y_1 + (Y_2)\eta^*}{2}, \quad Y_2 = \frac{Y_2 + (Y_3)\eta^*}{2}, \quad Y_3 = \frac{Y_3 + (Y_3)\eta^*}{2},
\]

\[
X_1 = A_1^T(C_1 - A_2Y_1A_2^T - A_3Y_2A_3^T - A_4Y_3A_4^T) + L_{A_1}, \quad U_2,
\]

\[
X_2 = R_{A_1}(C_1 - A_2Y_1A_2^T - A_3Y_2A_3^T - A_4Y_3A_4^T)(A_1^T)\eta^* + A_1^T(U_1 + U_3R_{A_1}^T),
\]

\[
Y_1 = A_1^T(T(A_1^T)\eta^* - A_1^TA_{22}M_1^TT(A_1^T)\eta^* - A_1^TU_1A_{12}M_1^TT(A_1^T)\eta^* + L_{A_1}U_5 + U_8R_{A_1}^T),
\]

\[
Y_2 = M_1^TT(A_{22}^T)\eta^* + S^T_{12}S_{12}^TT(M_1^T)\eta^* + L_{M_1}L_{21}U_2 + U_6R_{A_1}^T + L_{M_1}U_4R_{M_1}^T,
\]

\[
\bar{Y}_3 = f_1 + L_{C_1}V_1 + V_2R_{C_1^T} + L_{C_1}V_3R_{C_1^T}, \text{ or } \bar{Y}_3 = f_2 - L_{C_1}W_1 - W_2R_{C_1^T} - L_{C_1}W_3R_{C_1^T},
\]

where \( T = T_1 - A_{33}Y_3(A_{33})\eta^* \).

\[
V_1 = (I_m, 0) \left[ C_{11}^T(F - C_{22}V_3C_{33}^T - C_{33}W_3C_{22}^T) - C_{11}^TU_{11}C_{11}^T + L_{C_1}U_{12} \right],
\]

\[
W_1 = (0, I_m) \left[ C_{11}^T(F - C_{22}V_3C_{33}^T - C_{33}W_3C_{22}^T) - C_{11}^TU_{11}C_{11}^T + L_{C_1}U_{12} \right],
\]

\[
W_2 = \begin{bmatrix} R_{C_1} (F - C_{22}V_3C_{33}^T - C_{33}W_3C_{22}^T)(C_{11}^T)\eta^* + C_{11}^TU_{11}U_{11} + U_{21}L_{C_1}^T \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix},
\]

\[
V_2 = \begin{bmatrix} R_{C_1} (F - C_{22}V_3C_{33}^T - C_{33}W_3C_{22}^T)(C_{11}^T)\eta^* + C_{11}^TU_{11}U_{11} + U_{21}L_{C_1}^T \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix},
\]

\[
V_3 = E_{11}^T(F(E_{22}^T)^T - E_{11}^TE_{22}M_1^TF(E_{22}^T)^T - E_{11}^TSE_{12}^TFN^TE_{11}^T(E_{22}^T)^T
\]

\[
- E_{11}^TLS_{31}R_N E_{11}^T(E_{22}^T)^T + L_{E_{11}}U_{32} + U_{33}L_{E_{22}},
\]

\[
W_3 = M_1^TF(E_{11}^T)^T + S^TSE_{12}^TFN^T + L_{M_1}L_{21}U_2 + L_{M_1}U_{31}R_N - U_{42}L_{E_{11}},
\]

\[
U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \text{ and } U_{42} \text{ are any matrices with suitable dimensions over } \mathbb{H}.
\]

**Proof.** It is easy to show that (7) has a solution if and only if the following matrix equation has a solution:

\[
A_1 \bar{X}_1 + \bar{X}_2A_1^T + A_2 \bar{Y}_1A_2^T + A_3 \bar{Y}_2A_3^T + A_4 \bar{Y}_3A_4^T = B.
\]

(58)

If (7) has a solution, say, \((X_1, Y_1, Y_2, Y_3)\), then

\[
(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Y}_3) := (X_1, X_2, Y_1, Y_2, Y_3)
\]

is a solution of (58). Conversely, if (58) has a solution, say

\[
(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Y}_3).
\]

It is easy to show that (7) has a solution

\[
(X_1, Y_1, Y_2, Y_3) := \left( \frac{X_1 + (X_2)\eta^*}{2}, \frac{Y_1 + (Y_2)\eta^*}{2}, \frac{Y_2 + (Y_2)\eta^*}{2}, \frac{Y_3 + (Y_3)\eta^*}{2} \right).
\]

\(\Box\)

Letting \(A_1\) and \(B_1\) vanish in Theorem 1, it yields to the following result.
Corollary 1. Let $A_{ij}, B_{ij}$ $(i = 1 \ldots 3)$, and $T_1$ be given matrices with appropriate sizes over $\mathbb{H}$. Set

\[ M_1 = R_{A_{11}}A_{22}, \quad N_1 = B_{22}L_{B_{11}}, \quad S_1 = A_{22}L_{M_1}, \]
\[ C = R_{M_1}R_{A_{11}}, \quad C_1 = CA_{33}, \quad C_2 = R_{A_{11}}A_{33}, \quad C_3 = R_{A_{22}}A_{33}, \quad C_4 = A_{33}, \]
\[ D = L_{B_{11}}L_{N_1}, \quad D_1 = B_{33}, \quad D_2 = B_{33}L_{B_{22}}, \quad D_3 = B_{33}L_{B_{11}}, \quad D_4 = B_{33}D, \]
\[ E_1 = CT_1, \quad E_2 = R_{A_{11}}T_1L_{B_{22}}, \quad E_3 = R_{A_{22}}T_1L_{B_{11}}, \quad E_4 = T_1D, \]
\[ C_{11} = (L_{C_2}, L_{C_4}), \quad D_{11} = \begin{pmatrix} R_{D_1} \\ R_{D_3} \end{pmatrix}, \quad C_{22} = L_{C_1}, \quad D_{22} = R_{D_2}, \quad C_{33} = L_{C_3}, \]
\[ D_{33} = R_{D_4}, \quad E_{11} = R_{C_1}C_{22}, \quad E_{22} = R_{C_1}C_{33}, \quad E_{33} = D_{33}L_{D_{11}}, \quad E_{44} = D_{33}L_{D_{11}}, \]
\[ M = R_{E_{11}}E_{22}, \quad N = E_{44}E_{33}, \quad F = F_2 - F_1, \quad E = R_{C_1}F_{1L_{D_{11}}}, \quad S = E_{22}L_{M_1}, \]
\[ F_{11} = C_2L_{C_1}, \quad G_1 = E_2 - C_2C_1^T F_1 L_{B_{22}} D_2, \quad F_{22} = C_4 L_{C_3}, \quad G_2 = E_4 - C_4 C_3^T F_1 L_{B_{22}} D_4, \]
\[ F_1 = C_1^T F_1 L_{B_{22}}^T + L_{C_2} C_{4}^T E_2 D_{22}^T, \quad F_2 = C_3^T E_3 D_{22}^T + L_{C_3} C_4^T E_4 D_{44}. \]

Then, the following statements are equivalent:

1. Equation (6) is consistent.
2. $R_{C_1} E_i = 0, E_i L_{D_1} = 0$ $(i = 1 \ldots 4)$, $R_{E_{32}} E L_{E_{33}} = 0$.
3. 

\[
\begin{align*}
    r(T_1, A_{11}, A_{22}, A_{33}) &= r(A_{11}, A_{22}, A_{33}), \\
    r\left(\begin{array}{ccc}
    T_1 & A_{11} & A_{33} \\
    B_{11} & 0 & 0 \\
    B_{22} & 0 & 0 \\
    B_{33} & 0 & 0 \\
    \end{array}\right) &= r(A_{11}, A_{22}) + r(B_{33}), \\
    r\left(\begin{array}{ccc}
    T_1 & A_{22} & A_{33} \\
    B_{11} & 0 & 0 \\
    B_{22} & 0 & 0 \\
    B_{33} & 0 & 0 \\
    \end{array}\right) &= r(A_{33}, A_{22}) + r(B_{11}), \\
    r\left(\begin{array}{ccc}
    T_1 & A_{11} & A_{22} \\
    B_{11} & 0 & 0 \\
    B_{22} & 0 & 0 \\
    B_{33} & 0 & 0 \\
    \end{array}\right) &= r\left(\begin{array}{ccc}
    T_1 & 0 & 0 \\
    B_{11} & 0 & 0 \\
    B_{22} & 0 & 0 \\
    B_{33} & 0 & 0 \\
    \end{array}\right) + r(A_{33}), \\
    r\left(\begin{array}{ccc}
    T_1 & 0 & 0 \\
    0 & T_1 & 0 \\
    0 & 0 & T_1 \\
    0 & 0 & 0 \\
    \end{array}\right) &= \begin{pmatrix} B_{22} & 0 \\ B_{33} & 0 \end{pmatrix} + r(A_{11}, A_{22}, A_{33}).
\end{align*}
\]

In this case, the general solution to Equation (6) can be expressed as

\[ Y_1 = A_{11}^T T B_{11}^T - A_{11}^A A_{22} M_{11}^T T B_{11}^T - A_{11}^A S_1 A_{22}^T T N_{11}^T B_{22} B_{11}^T \]
\[ - A_{11}^A S_1 U_{11} R_{N_{11}} B_{22} B_{11}^T + L_{A_{11}} U_5 + U_6 R_{B_{11}}, \]
\[ Y_2 = M_{11}^T T B_{11}^T + S_1^A S_1 A_{22}^A T N_{11}^T + L_{M_{11}} U_7 + U_8 R_{B_{11}} + L_{M_{11}} U_4 R_{N_{11}}, \]
\[ Y_3 = F_1 + L_{C_1} V_1 + V_2 R_{D_1} + L_{C_1} V_3 R_{D_2}, \quad \text{or} \quad Y_3 = F_2 - L_{C_1} W_1 - W_2 R_{D_3} - L_{C_1} W_3 R_{D_4}, \]

where $T = T_1 - A_{33} Y_3 B_{33}, U_i(i = 1, ..., 8)$ are any matrices with suitable dimensions over $\mathbb{H},$

\[ V_1 = (I_m, 0) \begin{pmatrix} C_{11}^T (F - C_{22} V_{32} D_{22} - C_{33} W_{53} D_{33}) - C_{11}^T U_{11} D_{11} + L_{C_1} U_{12} \end{pmatrix}, \]
\[ W_1 = (0, I_m) \begin{pmatrix} C_{11}^T (F - C_{22} V_{32} D_{22} - C_{33} W_{53} D_{33}) - C_{11}^T U_{11} D_{11} + L_{C_1} U_{12} \end{pmatrix}. \]
\[
W_2 = \begin{bmatrix} R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^t + C_{11}C_{11}^tU_{11} + U_{21}R_{D_{11}} \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix},
\]
\[
V_2 = \begin{bmatrix} R_{C_{11}}(F - C_{22}V_3D_{22} - C_{33}W_3D_{33})D_{11}^t + C_{11}C_{11}^tU_{11} + U_{21}R_{D_{11}} \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix},
\]
\[
V_3 = E_{11}^tF_{E_{33}} - F_{E_{23}}^tM^tF_{E_{33}} - E_{11}^tS_{E_{22}}F_{N_{E_{44}}}^tE_{33} - E_{11}^tS_{U_{31}}R_{N}E_{44}^tE_{33} + L_{E_{11}}U_{32} + U_{33}R_{E_{33}},
\]
\[
W_3 = M^tF_{E_{44}} + S_{E_{22}}^tF_{N_{E_{44}}}^t + L_ML_SU_{41} + L_MU_{31}R_N - U_{42}R_{E_{44}},
\]

\[U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41}, \text{ and } U_{42} \text{ are any matrices with suitable dimensions over } \mathbb{H}.\]

5. Conclusions

We have established the solvability conditions and an exact formula of a general solution to quaternion matrix Equation (5). As an application of Equation (5), we also have established some necessary and sufficient conditions for Equation (7) to have a solution and derived a formula of its general solution involving \(\eta\)-Hermicity. The quaternion matrix Equation (5) plays a key role in studying the solvability conditions and general solutions of other types of matrix equations. For example, we can use the results on Equation (5) to investigate the solvability conditions and the general solution of the following system of quaternion matrix equations

\[
A_2Y_1 = C_2, \quad Y_1B_2 = D_2, \\
A_3Y_2 = C_3, \quad Y_2B_3 = D_3, \\
A_4Y_3 = C_4, \quad Y_3B_4 = D_4, \\
G_1Y_1H_1 + G_2Y_2H_2 + G_3Y_3H_3 = G
\]

where \(Y_1, Y_2\) and \(Y_3\) are unknown quaternion matrices and the others are given.

It is worth mentioning that the main results of (5) are available over not only \(\mathbb{R}\) and \(\mathbb{C}\) but also any division ring. Moreover, inspired by [39], we can investigate Equation (5) in tensor form.

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